

INTEGRAL EQUATIONS AND BOUNDARY VALUE PROBLEMS

[with Green's function technique and its applications]

[For M.A./M.Sc. (Mathematics) and M.Sc. (Physics) students of all Indian Universities/Institutions according to latest U.G.C model curriculum and various engineering and professional examinations such as GATE, C.S.I.R NET/JRF and SLET etc.]

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PREFACE TO THE SIXTH EDITION

Reference to the latest papers of GATE and various universities have been inserted at proper places. Solutions of some new problems are also given.

Suggestions for further improvement of the book will be gratefully received.

M.D. Raisinghania

PREFACE TO THE FOURTH EDITION

New matter and latest questions of various universities have been added at appropriate places. In addition to this, the following new useful topics have been added.

Appendix A: Boundary value problems and Green's identities.

Appendix B: Two and three dimensional Dirac delta functions

Appendix C: Additional topics and problems based on Green's functions

I hope that these changes will make the material of this book more useful to the reader.

Suggestions for further improvement of the book will be gratefully received.

M.D. Raisinghania

PREFACE TO THE THIRD EDITION

Reference to the latest papers of various universities and GATE have been inserted at proper places. More additional problems have been inserted in the miscellaneous set of problems given at the end of the book.

I hope that these changes will make the material more accessible and attractive to the reader.

All valuable suggestions for further improvement of the book will be highly appreciated.

M.D. Raisinghania

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PREFACE

This book on “Linear integral equations and boundary value problems” has been specially written as per latest UGC model curriculum for MA/M.Sc. students of all Indian universities/institutions. In addition, this book will prove very useful for students preparing for various engineering and professional examinations such as GATE, C.S.I.R. NET/JRF and SLET etc.

The author possesses a very long and rich experience of teaching mathematics and has first hand experience of the problems and difficulties that students generally face.

The silent features of this book are :

- * The matter has been presented in a simple and lucid language, so that students themselves shall be able to understand the solutions of the problems.
- * Each chapter opens with necessary definitions and complete proofs of the standard results and theorems. These in turn are followed by solved examples which have been classified in various types and methods. This classification will help the students to revise the subject matter at the time of examination without losing any confidence.
- * Care has been taken not to omit important steps so that the students can understand every thing without the guidance of a teacher. Furthermore, a set of unsolved exercises is given in each chapter to instill confidence in the students.

In view of these special features, it is sincerely hoped that the book will surely serve its purpose.

I am grateful to Shri Ravindra Kumar Gupta, Managing Director, Shri Navin Joshi, General Manager and Shri R.S. Saxena (Adviser, Publishing) for showing keen interest throughout the preparation of the book. My sincere thanks are due to Shri Shishir Bhatnagar for bringing the book in an excellent form.

All valuable suggestions for further improvement of the book will be highly appreciated.

M.D. Raisinghania

**COMPLETE SYLLABUS FOR
INTEGRAL EQUATIONS AND BOUNDARY VALUE PROBLEMS
AS PER LATEST U.G.C. MODEL CURRICULUM
FOR M.A./M.SC MATHEMATICS OF ALL INDIAN UNIVERSITIES/INSTITUTIONS**

Definitions of integral equations and their classification. Eigenvalues and eigenfunctions. Fredholm integral equations of second kind with separable kernels. Reduction to a system of algebraic equations. An approximate method.

Method of successive approximations. Iterative schemes for Fredholm integral equations of the second kind. Conditions of uniform convergence and uniqueness of series solution. Resolvent kernel and its results. Application of iterative scheme to Volterra integral equations of the second kind.

Classical Fredholm theory. Fredholm theorems.

Integral transform methods. Fourier transform. Convolution integral. Application to Volterra integral equations with convolution-type kernels.

Abel's equations. Inversion formula for singular integral equation with kernel of the type $h(s) - h(t)$, $0 < a < 1$. Cauchy's principal value of singular integrals. Solution of Cauchy-type integral equation. The Hilbert kernel. Solution of the Hilbert-type singular integral equation.

Symmetric kernels. Complex Hilbert space. Orthonormal system of functions. Fundamental properties of eigenvalues and eigenfunctions for symmetric kernels. Expansion in eigenfunction and bilinear form. Hilbert Schmidt theorem and some immediate consequences. Solutions of integral equations with symmetric kernels.

Definition of a boundary value problem for an ordinary differential equation of the second order and its reduction to a Fredholm integral equation of the second kind. Dirac delta function. Green's function approach to reduce boundary value problems of a self-adjoint differential equation with homogeneous boundary conditions to integral equation forms. Auxiliary problem satisfied by Green's function. Integral equation formulations of boundary value problems with more general and inhomogeneous boundary conditions. Modified Green's function.

Integral representation for the solution of the Laplace's and Poisson's equations. Newtonian single-layer and double layer potentials. Interior and exterior Dirichlet and Neumann boundary value problems for Laplace's equation. Green's function for Laplace's equation in a space as well as in a space bounded by a ground vessel. Integral equation formulation of boundary value problems for Laplace's equation. Poisson's integral formula. Green's function for the space bounded by grounded two parallel plates or an infinite circular cylinder.

Perturbation techniques and its applications to mixed boundary value problems. Two part and three part boundary value problems.

Solutions of electrostatic problems involving a charged circular and annular disc, a spherical cap, an annular spherical cap in a free space or a bounded space.

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*Dedicated to the memory of
my parents*

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GENERAL NOTATIONS

[Numbers refer the page on which the explanation first appeared]

A^T	Transpose of matrix A	4.23
$B(x, y)$	Beta function	8.1
	Fredholm determinant	6.7
	Fredholm minor	6.7
	divergence of vector A	12.1
$\exp a$	exponential of a , i.e., e^a	12.18
$E(x, t)$	fundamental solution or free space solution	12.2
	Green's function for Laplace's equation	10.11
F	Fourier transform	9.17
F^{-1}	inverse Fourier transform	9.17
F_c	Fourier cosine transform	9.17
F_c^{-1}	inverse Fourier cosine transform	9.18
F_s	Fourier sine transform	9.17
F_s^{-1}	inverse Fourier sine transform	9.17
$ f $	norm of the function f	1.8
(f, g)	inner (or scalar) product of f and g	1.8
or grad u	gradient of scalar point function u	12.1
$G(x, t)$	Green's function	11.1
$G_M(x, t)$	modified Green's function	11.49
$H(x - a)$	Heaviside (step or unit) function	9.2
	Hankel functions or Bessel functions of the third kind	10.14
I	unit or identity matrix	4.21
$I_\alpha(z)$	modified Bessel function	10.14
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THE GREEK ALPHABET				
alpha	A	nu		N
beta	B	xi		
gamma		omicron	o	O
delta		pi		
epsilon	E	rho		P
zeta	Z	sigma		
eta	H	tau		T
theta		upsilon		
iota	I	phi		
kappa	K	chi		X
lambda		psi		
mu	M	omega		

CHAPTER 1

Preliminary Concepts

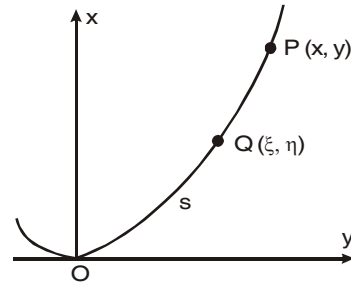
1.1 INTRODUCTION.

Many physical problems of science and technology which were solved with the help of theory of ordinary and partial differential equations can be solved by better methods of theory of integral equations. For example, while searching for the representation formula for the solution of linear differential equation in such a manner so as to include boundary conditions or initial conditions explicitly, we arrive at an integral equation. The solution of the integral equation is much easier than the original boundary value or initial value problem. The theory of integral equations is very useful tool to deal with problems in applied mathematics, theoretical mechanics, and mathematical physics. Several situations of science lead to integral equations, e.g., neutron diffusion problem and radiation transfer problem etc.

1.2. ABEL'S PROBLEM.

We propose to give an example of a situation which leads to an integral equation. Consider the following problem in mechanics.

Consider a given smooth curve in a vertical plane and suppose a material point start from rest at any point P under the influence of gravity along the curve. Let T be the time taken by the particle from P to the lowest point O . Treat O as the origin of coordinates, the x -axis vertically upward, and the y -axis horizontal. Let the coordinates of P and Q be (x, y) and (ξ, η) respectively. Let arc $OQ = s$.



Then the velocity of the particle at Q is given by

$$\frac{ds}{dt} = -\sqrt{2g(x-\xi)} \quad \text{so that} \quad t = -\int_P^Q \frac{ds}{\sqrt{2g(x-\xi)}}.$$

Hence,

$$T = \int_Q^P \frac{ds}{\sqrt{2g(x-\xi)}}. \quad \dots (1)$$

If the shape of the curve is given, then s can be expressed in terms of ξ and hence ds can be expressed in terms of ξ . So, let

$$ds = u(\xi) d\xi.$$

$\therefore$ from (1),

$$T = \int_0^x \frac{u(\xi) d\xi}{\sqrt{2g(x-\xi)}}. \quad \dots (2)$$

Able treated the above problem in modified form by finding that curve for which the time T of descent is a given function of x , say $f(x)$. Thus, we are led to the problem of finding the unknown function u from the equation

$$f(x) = \int_0^x \frac{1}{\sqrt{2g(x-\xi)}} u(\xi) d\xi. \quad \dots (3)$$

Equation (3) is called *Able integral equation*.

1.3. INTEGRAL EQUATION. DEFINITION.

[Meerut 2005, 08, 12]

An integral equation is an equation in which an unknown function appears under one or more integral signs.

For example, for $a \leq x \leq b$, $a \leq t \leq b$, the equations

$$\int_a^b K(x, t) y(t) dt = f(x) \quad \dots (1)$$

$$y(x) - \lambda \int_a^b K(x, t) y(t) dt = f(x) \quad \dots (2)$$

and

$$y(x) = \int_a^b K(x, t) [y(t)]^2 dt, \quad \dots (3)$$

where the function $y(x)$, is the unknown function while the functions $f(x)$ and $K(x, t)$ are known functions and λ , a and b are constants, are all integral equations. The above mentioned functions may be complex-valued functions of the real variables x and t .

1.4. LINEAR AND NON-LINEAR INTEGRAL EQUATIONS. DEFINITIONS.

An integral equation is called *linear* if only linear operations are performed in it upon the unknown function. An integral equation which is not linear is known as a *non-linear integral equation*. By writing either

$$L(y) = \int_a^b K(x, t) y(t) dt \quad \text{or} \quad L(y) = y(x) - \lambda \int_a^b K(x, t) y(t) dt,$$

we can easily verify that L is a linear integral operator. In fact, for any constants c_1 and c_2 , we have

$$L\{c_1 y_1(x) + c_2 y_2(x)\} = c_1 L\{y_1(x)\} + c_2 L\{y_2(x)\},$$

which is well known general criterion for a *linear operator*. In this book, we shall study only linear integral equations.

For example, the integral equations (1) and (2) of Art. 1.3 are linear integral equations while the integral equation (3) is non-linear integral equation.

The most general type of linear integral equation is of the form

$$g(x)y(x) = f(x) + \lambda \int_a^b K(x, t)y(t) dt, \quad \dots (1)$$

where the upper limit may be either variable x or fixed. The functions f , g and K are known functions while y is to be determined; λ is a non-zero real or complex, *parameter*. The function $K(x, t)$ is known as the *kernel* of the integral equation.

Remark 1. The constant λ can be incorporated into the kernel $K(x, t)$ in (1). However, in many applications λ represents a significant parameter which may take on various values in a discussion being considered. For theoretical discussion of integral equations, λ plays an important role.

Remark 2. If $g(x) \neq 0$, (1) is known as *linear integral equation of the third kind*. When $g(x) = 0$, (1) reduces to

$$f(x) + \lambda \int_a^b K(x, t)y(t) dt = 0, \quad \dots (2)$$

which is known as *linear integral equation of the first kind*. Again, when $g(x) = 1$, (1) reduces to

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt, \quad \dots (3)$$

which is known as *linear integral equation of the second kind*.

In the present book, we shall study in details equations of the form (2) and (3) only. In next two articles, we discuss special cases of (2) and (3).

1.5. FREDHOLM INTEGRAL EQUATION. DEFINITION.

(Kanpur 2010, 2011)

A linear integral equation of the form

$$g(x) y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt, \quad \dots (1)$$

where a, b are both constants, $f(x)$, $g(x)$ and $K(x, t)$ are known functions while $y(x)$ is unknown function and λ is a non-zero real or complex parameter, is called *Fredholm integral equation of third kind*. The function $K(x, t)$ is known as the kernel of the integral equation.

The following special cases of (1) are of our main interest.

(i) Fredholm integral equation of the first kind.

A linear integral equation of the form (by setting $g(x) = 0$ in (1))

$$f(x) + \lambda \int_a^b K(x, t) y(t) dt = 0, \quad \dots (2)$$

is known as *Fredholm integral equation of the first kind*.

(ii) Fredholm integral equation of the second kind.

A linear integral equation of the form (by setting $g(x) = 1$ in (1))

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt, \quad \dots (3)$$

is known as *Fredholm integral equation of the second kind*.

(iii) Homogeneous Fredholm integral equation of the second kind.

A linear integral equation of the form (by setting $f(x) = 0$ in (3)).

$$y(x) = \lambda \int_a^b K(x, t) y(t) dt, \quad \dots (4)$$

is known as the *homogeneous Fredholm integral equation of the second kind*.

1.6. VOLTERRA INTEGRAL EQUATION. DEFINITION.

A linear integral equation of the form

$$g(x) y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt, \quad \dots (1)$$

where a, b are both constants, $f(x)$, $g(x)$ and $K(x, t)$ are known functions while $y(x)$ is unknown function; λ is a non-zero real or complex parameter is called *Volterra integral equation of third kind*. The function $K(x, t)$ is known as the kernel of the integral equation.

The following special cases of (1) are of our main interest.

(i) Volterra integral equation of the first kind.

A linear integral equation of the form (by setting $g(x) = 0$ in (1))

$$f(x) + \lambda \int_a^x K(x, t) y(t) dt = 0, \quad \dots (2)$$

is known as *Volterra integral equation of the first kind*.

(ii) Volterra integral equation of the second kind.

A linear integral equation of the form (by setting $g(x) = 1$)

$$y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt, \quad \dots (3)$$

is known as *Volterra integral equation of the second kind*.

(iii) Homogeneous Volterra integral equation of the second kind.

A linear integral equation of the form (by setting $f(x) = 0$ in (3))

$$y(x) = \lambda \int_a^x K(x, t) y(t) dt, \quad \dots (4)$$

is known as the *homogeneous Volterra integral equation of the second kind*.

1.7. SINGULAR INTEGRAL EQUATION. DEFINITION.

[Meerut 2008]

When one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is known as *singular integral equation*. For example, the integral equations

$$y(x) = f(x) + \lambda \int_{-\infty}^{\infty} e^{-|x-t|} y(t) dt$$

and

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} y(t) dt, 0 < \alpha < 1$$

are singular integral equations.

1.8. SPECIAL KINDS OF KERNELS.

The following special cases of the kernel of an integral equation are of main interest and we shall frequently come across with such kernels throughout the discussion of this book.

(i) Symmetric kernel. Definition.

A kernel $K(x, t)$ is symmetric (or complex symmetric or Hermitian) if

$$K(x, t) = \bar{K}(t, x)$$

where the bar denotes the complex conjugate. A real kernel $K(x, t)$ is symmetric if

$$K(x, t) = K(t, x).$$

For example, $\sin(x+t)$, $\log(x+t)$, $x^2t^2 + xt + 1$ etc. are all symmetric kernels. Again, $\sin(2x+3t)$ and $x^2t^3 + 1$ are not symmetric kernels.

Again $i(x-t)$ is a symmetric kernel, since in this case, if $K(x, t) = i(x-t)$, then $k(t, x) = i(t-x)$ and so $\bar{K}(t, x) = -i(t-x) = i(x-t) = K(x, t)$. On the other hand, $i(x+t)$ is not a symmetric kernel, since in this case, if $K(x, t) = i(x+t)$, then $\bar{K}(t, x) = \overline{i(t+x)} = -i(t+x) = -K(x, t)$ and so $K(x, t) \neq \bar{K}(x, t)$

(ii) Separable or degenerate kernel. Definition.

[Meerut 2000]

A kernel $K(x, t)$ is called separable if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x only and a function of t only, i.e.,

$$K(x, t) = \sum_{i=1}^n g_i(x) h_i(t). \quad \dots (1)$$

Remark. The functions $g_i(x)$ can be regarded as linearly independent, otherwise the number of terms in relation (1) can be further reduced. Recall that the set of functions $g_i(x)$ is said to be linearly independent, if $c_1 g_1(x) + c_2 g_2(x) + \dots + c_n g_n(x) = 0$, where $c_1, c_2, \dots, c_n$ are arbitrary constants, then $c_1 = c_2 = \dots = c_n = 0$.

1.9. INTEGRAL EQUATIONS OF THE CONVOLUTION TYPE. DEFINITION.

Consider an integral equation in which the kernel $K(x, t)$ is dependent solely on the difference $x - t$, i.e.,

$$K(x, t) = K(x - t), \quad \dots (1)$$

where K is a certain function of one variable. Then integral equations

$$y(x) = f(x) + \lambda \int_a^x K(x-t)y(t) dt, \quad \dots (2)$$

and

$$y(x) = f(x) + \lambda \int_a^b K(x-t)y(t) dt \quad \dots (3)$$

are called *integral equations of the convolution type*. $K(x - t)$ is called *difference kernel*.

Let $y_1(x)$ and $y_2(x)$ be two continuous functions defined for $x \geq 0$. Then the *convolution or Faltung* of y_1 and y_2 is denoted and defined by

$$y_1 * y_2 = \int_0^x y_1(x-t) y_2(t) dt = \int_0^x y_1(t) y_2(x-t) dt. \quad \dots (4)$$

The integrals occurring in (4) are called the *convolution integrals*.

Note that the convolution defined by relation (4) is a particular case of the standard convolution.

$$y_1 * y_2 = \int_{-\infty}^{\infty} y_1(x-t) y_2(t) dt = \int_{-\infty}^{\infty} y_1(t) y_2(x-t) dt. \quad \dots (5)$$

By setting $y_1(t) = y_2(t) = 0$, for $t < 0$ and $t > x$, the integrals in (4) can be obtained from those in (5).

1.10. ITERATED KERNELS OR FUNCTIONS. DEFINITION.

(i) Consider Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt \quad \dots (1)$$

Then, the iterated kernels $K_n(x, t)$, $n = 1, 2, 3, \dots$ are defined as follows :

$$\left. \begin{aligned} K_1(x, t) &= K(x, t) \\ K_n(x, t) &= \int_a^b K(x, z) K_{n-1}(z, t) dz, n = 2, 3, \dots \end{aligned} \right\} \quad \dots (2)$$

(ii) Consider Volterra integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt. \quad \dots (3)$$

Then, the iterated kernels $K_n(x, t)$, $n = 1, 2, 3, \dots$ are defined as follows :

$$\left. \begin{aligned} K_1(x, t) &= K(x, t) \\ K_n(x, t) &= \int_t^x K(x, z) K_{n-1}(z, t) dz, n = 2, 3, \dots \end{aligned} \right\} \quad \dots (4)$$

1.11. RESOLVENT KERNEL OR RECIPROCAL KERNEL. DEFINITION.

Suppose solution of integral equations

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt \quad \dots (1)$$

and
$$y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt \quad \dots (2)$$

be respectively

$$y(x) = f(x) + \lambda \int_a^b R(x, t; \lambda) f(t) dt, \quad \dots (3)$$

and
$$y(x) = f(x) + \lambda \int_a^x \Gamma(x, t; \lambda) f(t) dt, \quad \dots (4)$$

then $R(x, t; \lambda)$ or $\Gamma(x, t; \lambda)$ is called the *resolvent kernel* or *reciprocal kernel* of the given integral equation.

1.12 EIGENVALUES (OR CHARACTERISTIC VALUES OR CHARACTERISTIC NUMBERS). EIGENFUNCTIONS (OR CHARACTERISTIC FUNCTIONS OR FUNDAMENTAL FUNCTIONS). DEFINITIONS.

Consider the homogeneous Fredholm integral equation

$$y(x) = \lambda \int_a^b K(x, t) y(t) dt. \quad \dots (1)$$

Then (1) has the obvious solution $y(x) = 0$, which is called the *zero or trivial solution* of (1). The values of the parameter λ for which (1) has a non-zero solution $y(x) \neq 0$ are called *eigenvalues* of (1) or of the kernel (x, t) , and every *non-zero solution* of (1) is called an *eigenfunction* corresponding to the eigenvalue λ .

Remark 1. The number $\lambda = 0$ is not an eigenvalue since for $\lambda = 0$ it follows from (1) that $y(x) = 0$.

Remark 2. If $y(x)$ is an eigenfunction of (1), then $c y(x)$, where c is an arbitrary constant, is also an eigenfunction of (1), which corresponds to the same eigenvalue λ .

Remark 3. A homogeneous Fredholm integral equation of the second kind may, generally, have no eigenvalue and eigenfunction, or it may not have any real eigenvalue or eigenfunction.

1.13. LEIBNITZ'S RULE OF DIFFERENTIATION UNDER INTEGRAL SIGN

Let $F(x, t)$ and $\partial F / \partial x$ be continuous functions of both x and t and let the first derivatives of $G(x)$ and $H(x)$ be continuous. Then

$$\frac{d}{dx} \int_{G(x)}^{H(x)} F(x, t) dt = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x} dt + F[x, H(x)] \frac{dH}{dx} - F[x, G(x)] \frac{dG}{dx} \quad \dots (1)$$

Particular Case : If G and H are absolute constants, then (1) reduces to

$$\frac{d}{dx} \int_G^H F(x, t) dt = \int_G^H \frac{\partial F}{\partial x} dt \quad \dots (2)$$

1.14. AN IMPORTANT FORMULA FOR CONVERTING A MULTIPLE INTEGRAL INTO A SINGLE ORDINARY INTEGRAL.

$$\int_a^x y(t) dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} y(t) dt.$$

Note that the integral on the L.H.S. is a multiple integral of order n while the integral on the R.H.S is ordinary integral of order one.

Proof. Let
$$I_n(x) = \int_a^x (x-t)^{n-1} y(t) dt, \quad \dots (1)$$

where n is a positive integer and a is constant.

Differentiating (1) with respect to x and using Leibnitz's rule, we have

$$\frac{dI_n}{dx} = (n-1) \int_a^x (x-t)^{n-2} y(t) dt + (x-x)^{n-1} y(x) \cdot \frac{dx}{dx} - (x-0)^{n-1} y(0) \frac{d0}{dx}$$

i.e.,
$$dI_n/dx = (n-1) I_{n-1}, \quad n > 1 \quad \dots (2)$$

From (1), $I_1 = \int_a^x y(t) dt$ so that
$$\frac{dI_1}{dx} = y(x) \quad \dots (3)$$

Now, differentiating (2) with respect to x successively k times, we have

$$d^k I_n / dx^k = (n-1)(n-2) \dots (n-k) I_{n-k}, \quad n > k \quad \dots (4)$$

Using (4) for $k = n-1$, we have

$$d^{n-1} I_n / dx^{n-1} = (n-1)! I_1 \quad \dots (5)$$

Differentiating (5) w.r.t. 'x' and using (3), we obtain

$$d^n I_n / dx^n = (n-1)! y(x) \quad \dots (6)$$

From (1), (4) and (5), it follows that $I_n(x)$ and its first $n-1$ derivatives all vanish when $x = a$. Hence using (3) and (6), we obtain

$$I_1(x) = \int_a^x y(t_1) dt_1$$

$$I_2(x) = \int_a^x I_1(t_2) dt_2 = \int_a^x \int_a^{t_2} y(t_1) dt_1 dt_2$$

Proceeding likewise, we obtain

$$I_n(x) = (n-1)! \int_a^x \int_a^{t_n} \dots \int_a^{t_3} \int_a^{t_2} y(t_1) dt_1 dt_2 \dots dt_{n-1} dt_n \quad \dots (7)$$

Combining (1) and (7), we obtain

$$\int_a^x \int_a^{t_n} \dots \int_a^{t_3} \int_a^{t_2} y(t_1) dt_1 dt_2 \dots dt_{n-1} dt_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} y(t) dt \quad \dots (8)$$

From (8), we obtain

$$\int_a^x y(t) dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} y(t) dt$$

1.15. Regularity conditions.

In this book we shall deal with functions which are either continuous, or integrable or square-integrable. We know that if an integral sign is used, the Lebesgue integral is understood. Furthermore, if a function is Riemann-integrable, it is also Lebesgue integrable. However there exist functions that are Lebesgue-integrable but not Riemann-integrable. Fortunately, we shall not come across with such functions in this book.

Square-integrable function or $\mathcal{L}_2$ -function. Definition.

A given function $y(x)$ is said to be square-integrable if

$$\int_a^b |y(x)|^2 dx < \infty \quad \dots (i)$$

The regularity conditions on the kernel $K(x, t)$ as a function of two variables are similar. Thus, $K(x, t)$ is an $\mathcal{L}_2$ -function if

(i) for each set of values of x, t in the square $a \leq x \leq b, a \leq t \leq b$,

$$\int_a^b \int_a^b |K(x, t)|^2 dx dt < \infty \quad \dots (ii)$$

(ii) for each value of x in $a \leq x \leq b$,

$$\int_a^b |K(x, t)|^2 dt < \infty \quad \dots (iii)$$

(iii) for each value of t in $a \leq t \leq b$,

$$\int_a^b |K(x, t)|^2 dx < \infty \quad \dots (iv)$$

1.16. THE INNER OR SCALAR PRODUCT OF TWO FUNCTIONS.

The *inner or scalar product* (f, g) of two complex $\mathcal{L}_2$ -functions f and g of a real variable x , $a \leq x \leq b$, is defined as

$$(f, g) = \int_a^b f(x) \overline{g(x)} dx, \quad \dots (i)$$

where the bar denotes the complex conjugate.

The given functions f and g are called *orthogonal* if their inner product is zero, i.e., if

$$(f, g) = 0, \quad \text{i.e.,} \quad \int_a^b f(x) \overline{g(x)} dx = 0$$

The *norm* of a function $f(x)$ is denoted by $\|f(x)\|$ and is defined as

$$\|f(x)\| = \left[\int_a^b f(x) \overline{f(x)} dx \right]^{1/2} = \left[\int_a^b |f(x)|^2 dx \right]^{1/2} \quad \dots (ii)$$

A function $f(x)$ is called *normalized* if $\|f(x)\| = 1$. From this definition, it follows that a non null function (whose norm is not zero) can be normalized by dividing it by its norm.

In our subsequent analysis, we shall require the following two inequalities :

Schwarz inequality $| (f, g) | \leq \|f\| \|g\|$

Minkowski inequality $\|f + g\| \leq \|f\| + \|g\|$

1.17. SOLUTION OF AN INTEGRAL EQUATION. DEFINITION.

Consider the linear integral equations :

$$g(x) y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt \quad \dots (1)$$

and

$$g(x) y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt \quad \dots (2)$$

A solution of the integral equation (1) or (2) is a function $y(x)$, which, when substituted into the equation, reduces it to an identity (with respect to x).

1.18. SOLVED EXAMPLES BASED ON ART 1.17

Ex. 1. Show that the function $y(x) = (1 + x^2)^{-3/2}$ is a solution of the Volterra integral equation

$$y(x) = \frac{1}{1 + x^2} - \int_0^x \frac{t}{1 + x^2} y(t) dt \quad [Kanpur 2009; Meerut 2003]$$

Sol. Given integral equation is $y(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} y(t) dt$... (1)

Also, given $y(x) = (1+x^2)^{-3/2}$... (2)

From (2), $y(t) = (1+t^2)^{-3/2}$... (3)

Then, R.H.S. of (1) $= \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} (1+t^2)^{-3/2} dt$, using (3)

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} \int_0^{x^2} (1+u)^{-3/2} \cdot \frac{1}{2} du \quad (\text{on putting } t^2 = u \text{ and } 2tdt = du)$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} \cdot \frac{1}{2} \cdot \left[\frac{(1+u)^{-1/2}}{(-1/2)} \right]_0^{x^2}$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2} \left[\frac{1}{(1+u)^{1/2}} \right]_0^{x^2} = \frac{1}{1+x^2} + \frac{1}{1+x^2} \left[\frac{1}{(1+x^2)^{1/2}} - 1 \right]$$

$$= (1+x^2)^{-3/2} = y(x), \text{ by (2)}$$

= L.H.S. of (1)

Hence (2) is a solution of given integral equation (1).

Ex. 2. Show that the function $y(x) = xe^x$ is a solution of the Volterra integral equation.

$$y(x) = \sin x + 2 \int_0^x \cos(x-t) y(t) dt \quad [\text{Meerut 2009, 10, 11; Kanpur 2005, 10}]$$

Sol. Given integral equation is $y(x) = \sin x + 2 \int_0^x \cos(x-t) y(t) dt$ (1)

Also, given $y(x) = xe^x$ (2)

From (1) $y(t) = te^t$ (3)

Again, we know the following standard results :

$$\int e^{ax} \sin(bx+c) dx = \frac{e^{ax}}{a^2+b^2} [a \sin(bx+c) - b \cos(bx+c)] \quad \dots (4)$$

and $\int e^{ax} \cos(bx+c) dx = \frac{e^{ax}}{a^2+b^2} [a \cos(bx+c) + b \sin(bx+c)]. \quad \dots (5)$

Then R.H.S. of (1)

$$= \sin x + 2 \int_0^x \{\cos(x-t) \times te^t\} dt = \sin x + 2 \int_0^x t \{e^t \cos(t-x)\} dt$$

$$= \sin x + 2 \left\{ \left[t \frac{e^t}{2} \{\cos(t-x) + \sin(t-x)\} \right]_0^x - \int_0^x 1 \cdot \frac{e^t}{2} \{\cos(t-x) + \sin(t-x)\} dt \right\},$$

[Integrating by parts and using formula (5)]

$$= \sin x + xe^x - \int_0^x e^t \cos(t-x) dt - \int_0^x e^t \sin(t-x) dt$$

$$\begin{aligned}
&= \sin x + xe^x - \left[\frac{e^t}{2} \{ \cos(t-x) + \sin(t-x) \} \right]_0^x - \left[\frac{e^t}{2} \{ \sin(t-x) - \cos(t-x) \} \right]_0^x \\
&\quad \text{[using formulas (4) and (5)]} \\
&= \sin x + xe^x - \left[\frac{e^x}{2} - \frac{1}{2}(\cos x - \sin x) \right] - \left[-\frac{e^x}{2} - \frac{1}{2}(-\sin x - \cos x) \right] = xe^x = y(x), \text{ by (2)} \\
&= \text{L.H.S. of (1).}
\end{aligned}$$

Hence (2) is a solution of (1).

Ex. 3. Show that $y(x) = \cos 2x$ is a solution of the integral equation

$$y(x) = \cos x + 3 \int_0^\pi K(x, t) y(t) dt \quad \text{where} \quad K(x, t) = \begin{cases} \sin x \cos t, & 0 \leq x \leq t \\ \cos x \sin t, & t \leq x \leq \pi. \end{cases}$$

[Garhwal 1998, Kanpur 2005, 08, 09; Meerut 2004, 2008, 2012]

Sol. Given integral equation is $y(x) = \cos x + 3 \int_0^\pi K(x, t) y(t) dt, \quad \dots (1)$

where $K(x, t) = \begin{cases} \sin x \cos t, & 0 \leq x \leq t \\ \cos x \sin t, & t \leq x \leq \pi. \end{cases} \quad \dots (2)$

Also given, $y(x) = \cos 2x \quad \dots (3)$

From (3), $y(t) = \cos 2t \quad \dots (4)$

Then, R.H.S. of (1)

$$\begin{aligned}
&= \cos x + 3 \left[\int_0^x K(x, t) y(t) dt + \int_x^\pi K(x, t) y(t) dt \right] \\
&= \cos x + 3 \left[\int_0^x \cos x \sin t \cos 2t dt + \int_x^\pi \sin x \cos t \cos 2t dt \right], \text{ by (2) and (4)} \\
&= \cos x + 3 \cos x \int_0^x \cos 2t \sin t dt + 3 \sin x \int_x^\pi \cos 2t \cos t dt \\
&= \cos x + \frac{3}{2} \cos x \int_0^x (\sin 3t - \sin t) dt + \frac{3}{2} \sin x \int_x^\pi (\cos 3t + \cos t) dt \\
&= \cos x + \frac{3}{2} \cos x \left[-\frac{1}{3} \cos 3t + \cos t \right]_0^x + \frac{3}{2} \sin x \left[\frac{1}{3} \sin 3t + \sin t \right]_x^\pi \\
&= \cos x + \frac{3}{2} \cos x \left[-\frac{1}{3} \cos 3x + \cos x + \frac{1}{3} - 1 \right] + \frac{3}{2} \sin x \left[-\frac{1}{3} \sin 3x - \sin x \right] \\
&= \cos x - \frac{1}{2} (\cos 3x \cos x + \sin 3x \sin x) + \frac{3}{2} (\cos^2 x - \sin^2 x) - \cos x \\
&= -\frac{1}{2} \cos (3x - x) + \frac{3}{2} \cos 2x = -\frac{1}{2} \cos 2x + \frac{3}{2} \cos 2x \\
&= \cos 2x = y(x), \text{ by (3) = L.H.S. of (1).}
\end{aligned}$$

Hence (3) is a solution of (1).

Ex. 7. Show that the function $y(x) = \sin(\pi x / 2)$ is a solution of the Fredholm integral

equation $y(x) - \frac{\pi^2}{4} \int_0^1 K(x, t) y(t) dt = \frac{x}{2}$, where the kernel $K(x, t)$ is of the form

$$K(x, t) = \begin{cases} (1/2) \times x(2-t), & 0 \leq x \leq t \\ (1/2) \times t(2-x), & t \leq x \leq 1. \end{cases} \quad \text{[Kanpur 2011; Meerut 2005]}$$

Sol. Given integral equation is $y(x) - \frac{\pi^2}{4} \int_0^1 K(x, t) y(t) dt = \frac{x}{2}, \quad \dots (1)$

where $K(x, t) = \begin{cases} (1/2) \times x(2-t), & 0 \leq x \leq t \\ (1/2) \times t(2-x), & t \leq x \leq 1. \end{cases} \quad \dots (2)$

Given $y(x) = \sin(\pi x/2). \quad \dots (3)$

From (3), $y(t) = \sin(\pi t/2). \quad \dots (4)$

Then, L.H.S. of (1)

$$\begin{aligned}
 &= \sin \frac{\pi x}{2} - \frac{\pi^2}{4} \left[\int_0^x K(x, t) y(t) dt + \int_x^1 K(x, t) y(t) dt \right], \text{ using (3)} \\
 &= \sin \frac{\pi x}{2} - \frac{\pi^2}{4} \left[\int_0^x \left\{ \frac{1}{2} t(2-x) \right\} \sin \frac{\pi t}{2} dt + \int_x^1 \left\{ \frac{1}{2} x(2-t) \right\} \sin \frac{\pi t}{2} dt \right], \text{ by (2) and (4)} \\
 &= \sin \frac{\pi x}{2} - \frac{\pi^2}{8} (2-x) \int_0^x t \sin \frac{\pi t}{2} dt - \frac{\pi^2 x}{8} \int_x^1 (2-t) \sin \frac{\pi t}{2} dt \\
 &= \sin \frac{\pi x}{2} - \frac{\pi^2 (2-x)}{8} \left[\left[t \left\{ -\frac{\cos(\pi t/2)}{\pi/2} \right\} \right]_0^x - \int_0^x 1 \left\{ -\frac{\cos(\pi t/2)}{\pi/2} \right\} dt \right] \\
 &\quad - \frac{\pi^2 x}{8} \left[\left[(2-t) \left\{ -\frac{\cos(\pi t/2)}{\pi/2} \right\} \right]_x^1 - \int_x^1 (-1) \left\{ -\frac{\cos(\pi t/2)}{\pi/2} \right\} dt \right] \\
 &= \sin \frac{\pi x}{2} - \frac{\pi^2 (2-x)}{8} \left[-\frac{2x}{\pi} \cos \frac{\pi x}{2} + \left[\frac{\sin(\pi t/2)}{(\pi/2)^2} \right]_0^x \right] \\
 &\quad - \frac{\pi^2 x}{8} \left[\frac{2(2-x)}{\pi} \cos \frac{\pi x}{2} - \left[\frac{\sin(\pi x/2)}{(\pi/2)^2} \right]_x^1 \right] \\
 &= \sin \frac{\pi x}{2} - \frac{\pi^2 (2-x)}{8} \left[-\frac{2x}{\pi} \cos \frac{\pi x}{2} + \frac{4}{\pi^2} \sin \frac{\pi x}{2} \right] \\
 &\quad - \frac{\pi^2 x}{8} \left[\frac{2(2-x)}{\pi} \cos \frac{\pi x}{2} - \frac{4}{\pi^2} + \frac{4}{\pi^2} \sin \frac{\pi x}{2} \right] \\
 &= \sin \frac{\pi x}{2} \left\{ 1 - \frac{1}{2}(2-x) - \frac{x}{2} \right\} + \frac{x}{2} = \frac{x}{2} = \text{R.H.S. of (1).}
 \end{aligned}$$

Hence (3) is a solution of (1).

EXERCISE

Verify that the given functions are solutions of the corresponding integral equations.

1. $y(x) = 1 - x; \int_0^x e^{x-t} y(t) dt = x$ (Kanpur 2007)
2. $y(x) = \frac{1}{2}; \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = \sqrt{x}$
3. $y(x) = 3; x^3 = \int_0^x (x-t)^2 y(t) dt.$ (Kanpur 2011)

$$4. \quad y(x) = x - \frac{x^3}{6}; \quad y(x) = x - \int_0^x \sinh(x-t)y(t) dt.$$

$$5. \quad y(x) = xe^x; \quad y(x) = e^x \sin x + 2 \int_0^x \cos(x-t)y(t) dt$$

$$6. \quad y(x) = x/(1+x^2)^{5/2}; \quad y(x) = \frac{3x+2x^3}{3(1+x^2)^2} - \int_0^x \frac{3x+2x^3-t}{(1+x^2)^2} y(t) dt.$$

$$7. \quad y(x) = e^x (\cos e^x - e^x \sin e^x); \quad y(x) = (1 - xe^{2x}) \cos 1 - e^{2x} \sin 1 + \int_0^x \{1 - (x-t)e^{2x}\} y(t) dt$$

$$8. \quad y(x) = e^x; \quad y(x) + \lambda \int_0^1 \sin xt y(t) dt = 1.$$

$$9. \quad y(x) = \cos x; \quad y(x) - \int_0^x (x^2 + t) \cos t y(t) dt = \sin x.$$

$$10. \quad y(x) = xe^{-x}; \quad y(x) - 4 \int_0^\infty e^{-(x+t)} y(t) dt = (x-1)e^{-x}$$

$$11. \quad y(x) = 1 - \frac{2 \sin x}{(1-\pi/2)}; \quad y(x) - \int_0^\pi \cos(x+t) y(t) dt = 1.$$

$$12. \quad y(x) = \frac{1}{\pi\sqrt{x}}; \quad \int_0^1 \frac{y(t)}{(x-t)^{1/2}} dt = 1 \quad [\text{Kanpur 2006}]$$

$$13. \quad y(x) = \frac{4c}{\pi} \sin x, \text{ (c being an arbitrary constant)}; \quad y(x) - \frac{4}{\pi} \int_0^\infty \sin x \frac{\sin^2 t}{t} y(t) dt = 0.$$

$$14. \quad y(x) = \sqrt{x}; \quad y(x) - \int_0^1 K(x,t)y(t) dt = \sqrt{x} + \frac{x}{15}(4x^{3/2} - 7), \text{ where}$$

$$K(x,t) = \begin{cases} (1/2) \times x(2-t), & 0 \leq x \leq t \\ (1/2) \times t(2-x), & t \leq x \leq 1. \end{cases}$$

$$15. \quad y(x) = e^x(2x-2/3); \quad y(x) + 2 \int_0^1 e^{x-t} y(t) dt = 2xe^x \quad [\text{Kanpur 2006, 10}]$$

$$16. \quad y(x) = 1; \quad y(x) + \int_0^1 x(e^{xt} - 1) y(t) dt = e^x - x$$

17. For what value of λ , the function $y(x) = 1 + \lambda x$ is a solution of the integral equation

$$x = \int_0^x e^{x-t} y(t) dt \quad ?$$

Hint : Proceed as in solved Ex. 1 on page 1.8

Ans. $\lambda = -1$.

CHAPTER 2

Conversion of Ordinary differential equations into integral equations

2.1. INTRODUCTION

While searching for the representation formula for the solution of an ordinary differential equation in such a manner so as to include the boundary conditions or initial conditions explicitly, we always arrive at integral equations. Thus, a boundary value or an initial value problem is converted to an integral equation. Later on in this chapter, the reader will notice that an initial value problem is always converted into a Volterra integral equation and a boundary value problem is always converted into a Fredholm integral equation. After converting an initial value or a boundary value problem into an integral equation, it can be solved by shorter methods of solving integral equations.

2.2. INITIAL VALUE PROBLEM. DEFINITION.

When an ordinary differential equation is to be solved under conditions involving dependent variable and its derivative at the same value of the independent variable, then the problem under consideration is said to be an *initial value problem*.

For example, $\frac{d^2y}{dx^2} + y = x$, $y(0) = 2$, $y(0) = 3$... (1)

and $\frac{d^2y}{dx^2} + y = x$, $y(1) = 2$, $y(1) = 2$... (2)

are both initial value problems. Note that in (1), the same value $x = 0$ of the independent variable is involved whereas in (2), the same value $x = 1$ of the independent variable is involved.

2.3. METHOD OF CONVERTING AN INITIAL VALUE PROBLEM INTO A VOLTERRA INTEGRAL EQUATION.

This method is illustrated with the help of the following solved examples.

Ex. 1. Convert the following differential equation into integral equation :

$$y'' + y = 0 \quad \text{when } y(0) = y'(0) = 0.$$

Sol. Given $y''(x) + y(x) = 0$, ... (1)

with initial conditions $y(0) = 0$... 2(a)

and $y'(0) = 0$... 2(b)

From (1), $y''(x) = -y(x)$... (3)

Integrating both sides of (3) w.r.t. 'x' from 0 to x, we have

$$\int_0^x y''(x) dx = -\int_0^x y(x) dx \quad \text{or} \quad [y'(x)]_0^x = -\int_0^x y(x) dx$$

$$\text{or} \quad y'(x) - y'(0) = -\int_0^x y(x) dx \quad \text{or} \quad y'(x) = -\int_0^x y(x) dx, \text{ using 2(b)} \quad \dots(4)$$

2.2

Conversion of Ordinary differential equation into integral equation

Integrating both sides of (4) w.r.t. 'x' from 0 to x, we have

$$\int_0^x y'(x) dx = -\int_0^x y(x) dx^2 \quad \text{or} \quad [y(x)]_0^x = -\int_0^x y(x) dx^2$$

$$\text{or} \quad y(x) - y(0) = -\int_0^x y(x) dx^2 \quad \text{or} \quad y(x) = -\int_0^x y(t) dt^2, \text{ using 2 (a)}$$

$$\text{or} \quad y(x) = -\int_0^x (x-t) y(t) dt, \text{ using result of Art. 1.14.}$$

which is the desired integral equation.

Ex. 2. Convert the following differential equation into an integral equation :

$$y'' + \lambda x y = f(x), \quad y(0) = 1, \quad y'(0) = 0$$

$$\text{Sol. Given} \quad y''(x) + \lambda x y(x) = f(x) \quad \dots (1)$$

$$\text{with initial conditions} \quad y(0) = 1 \quad \dots 2(a)$$

$$\text{and} \quad y'(0) = 0. \quad \dots 2(b)$$

$$\text{From (1),} \quad y''(x) = f(x) - \lambda x y(x) \quad \dots (3)$$

Integrating both sides of (3) w.r.t. 'x' from 0 to x, we have

$$\int_0^x y''(x) dx = \int_0^x [f(x) - \lambda x y(x)] dx \quad \text{or} \quad [y'(x)]_0^x = \int_0^x [f(x) - \lambda x y(x)] dx$$

$$\text{or} \quad y'(x) - y'(0) = \int_0^x [f(x) - \lambda x y(x)] dx \quad \text{or} \quad y'(x) = \int_0^x [f(x) - \lambda x y(x)] dx, \text{ using (2b)} \dots (4)$$

Integrating both sides of (4) w.r.t. 'x' from 0 to x, we have

$$\int_0^x y'(x) dx = \int_0^x [f(x) - \lambda x y(x)] dx^2 \quad \text{or} \quad [y(x)]_0^x = \int_0^x [f(x) - \lambda x y(x)] dx^2$$

$$\text{or} \quad y(x) - y(0) = \int_0^x [f(x) - \lambda x y(x)] dx^2 \quad \text{or} \quad y(x) = 1 + \int_0^x [f(t) - \lambda t y(t)] dt^2, \text{ using 2(a)}$$

$$\text{or} \quad y(x) = 1 + \int_0^x (x-t)[f(t) - \lambda t y(t)] dt, \text{ using result of Art. 1.14}$$

which is the required integral equation.

Ex. 3. Convert the following initial value problem into an integral equation :

$$\frac{d^2 y}{dx^2} + A(x) \frac{dy}{dx} + B(x)y = f(x), \quad y(a) = y_0, \quad y'(a) = y'_0.$$

$$\text{Sol. Given} \quad y''(x) + A(x)y'(x) + B(x)y(x) = f(x) \quad \dots (1)$$

$$\text{with initial conditions :} \quad y(a) = y_0 \quad \dots 2(a)$$

$$\text{and} \quad y'(a) = y'_0 \quad \dots 2(b)$$

$$\text{From (1),} \quad y''(x) = f(x) - B(x)y(x) - A(x)y'(x). \quad \dots (3)$$

Integrating both sides of (3) w.r.t. 'x' from a to x, we have

$$[y'(x)]_a^x = \int_a^x [f(x) - B(x)y(x)] dx - \int_a^x A(x)y'(x) dx$$

$$\text{or} \quad y'(x) - y'(a) = \int_a^x [f(x) - B(x)y(x)] dx - \left\{ [A(x)y(x)]_a^x - \int_a^x A'(x)y(x) dx \right\}$$

[on integrating by parts the second terms on R.H.S.]

$$\text{or} \quad y'(x) - y'_0 = \int_a^x [f(x) - B(x)y(x)] dx - \left\{ A(x)y(x) - A(a)y(a) - \int_a^x A'(x)y(x) dx \right\}, \text{ by 2 (b)}$$

$$\text{or} \quad y'(x) = y'_0 - A(x)y(x) + A(a)y_0 + \int_a^x \{f(x) - B(x)y(x) + A'(x)y(x)\} dx, \text{ by 2(a)} \quad \dots (4)$$

Integrating both sides of (4) w.r.t. 'x' from a to x , we have

$$\int_a^x y'(x) dx = \int_a^x [y'_0 + y_0 A(a)] dx - \int_a^x A(x)y(x) dx + \int_a^x \{f(x) - B(x)y(x) + A'(x)y(x)\} dx^2$$

$$\text{or} \quad [y(x)]_a^x = [y'_0 + y_0 A(a)](x-a) - \int_a^x A(x)y(x) dx + \int_a^x \{f(t) - B(t)y(t) + A'(t)y(t)\} dt^2$$

$$\text{or} \quad y(x) - y(a) = [y'_0 + y_0 A(a)](x-a) - \int_a^x A(t)y(t) dt + \int_a^x (x-t)\{f(t) - B(t)y(t) + A'(t)y(t)\} dt$$

[using result of Art. 1.14]

$$\text{or} \quad y(x) = y_0 + [y'_0 + y_0 A(a)](x-a) + \int_a^x (x-t)f(t) dt - \int_a^x [A(t) + (x-t)\{B(t) - A'(t)\}]y(t) dt,$$

which is Volterra integral equation of the second kind.

Ex. 4. Convert $y'' - \sin x y' + e^x y = x$ with initial conditions $y(0) = 1, y'(0) = -1$ to a Volterra integral equation of the second kind. Conversely, derive the original differential equation with the initial conditions from the integral equation obtained. [Meerut 2002, 04, 07, 11]

$$\text{Sol. Given} \quad y''(x) - \sin x y'(x) + e^x y(x) = x \quad \dots (1)$$

$$\text{with initial conditions :} \quad y(0) = 1 \quad \dots 2(a)$$

$$\text{and} \quad y'(0) = -1 \quad \dots 2(b)$$

$$\text{From (1),} \quad y''(x) = x - e^x y(x) + \sin x y'(x). \quad \dots (3)$$

Integrating both sides of (3) w.r.t. 'x' from 0 to x , we have

$$\int_0^x y''(x) dx = \int_0^x x dx - \int_0^x e^x y(x) dx + \int_0^x \sin x y'(x) dx$$

$$\text{or} \quad [y'(x)]_0^x = \frac{x^2}{2} - \int_0^x e^x y(x) dx + [\sin x y(x)]_0^x - \int_0^x \cos x y(x) dx$$

[Integrating by parts the third term on R.H.S.]

$$y'(x) - y'(0) = \frac{x^2}{2} - \int_0^x e^x y(x) dx + \sin x y(x) - \int_0^x \cos x y(x) dx$$

$$\text{or} \quad y'(x) + 1 = \frac{x^2}{2} + \sin x y(x) - \int_0^x (e^x + \cos x)y(x) dx, \text{ using 2 (b)}$$

or
$$y'(x) = \frac{x^2}{2} - 1 + \sin x \cdot y(x) - \int_0^x (e^t + \cos t) y(t) dt \quad \dots (4)$$

Integrating both sides of (4) w.r.t. 'x' from 0 to x, we have

$$\int_0^x y'(x) dx = \int_0^x \left(\frac{x^2}{2} - 1 \right) dx + \int_0^x \sin x \cdot y(x) dx - \int_0^x (e^x + \cos x) y(x) dx$$

or
$$[y(x)]_0^x = \left[\frac{x^3}{6} - x \right]_0^x + \int_0^x \sin t \cdot y(t) dt - \int_0^x (e^t + \cos t) y(t) dt$$

or
$$y(x) - y(0) = \frac{x^3}{6} - x + \int_0^x \sin t \cdot y(t) dt - \int_0^x (x-t)(e^t + \cos t) y(t) dt, \text{ using result of Art. 1.14}$$

or
$$y(x) - 1 = \frac{x^3}{6} - x + \int_0^x \{ \sin t - (x-t)(e^t + \cos t) \} y(t) dt, \text{ by 2(a)}$$

or
$$y(x) = \frac{x^3}{6} - x + 1 + \int_0^x [\sin t - (x-t)(e^t + \cos t)] y(t) dt, \quad \dots (5)$$

which is the required Volterra integral equation of the second kind.

Second part : Derivation of the given differential equation together with given initial conditions from integral equation (5) :

Differentiating both sides of (5) w.r.t. 'x', we get

$$y'(x) = \frac{x^2}{2} - 1 + \frac{d}{dx} \int_0^x [\sin t - (x-t)(e^t + \cos t)] y(t) dt$$

or
$$y'(x) = \frac{x^2}{2} - 1 + \int_0^x \frac{\partial}{\partial x} [\{ \sin t - (x-t)(e^t + \cos t) \} y(t)] dt$$

$$+ [\sin x - (x-x)(e^x + \cos x)] y(x) \frac{dx}{dx} - [\sin 0 - (x-0)(e^0 + \cos 0)] y(0) \frac{d0}{dx}$$

[using Leibnitz's rule of differentiation under integral sign (refer Art. 1.13)]

or
$$y'(x) = \frac{x^2}{2} - 1 + \int_0^x (e^t + \cos t) y(t) dt + \sin x \cdot y(x). \quad \dots (6)$$

Differentiating both sides of (6) with respect to 'x' we get

$$y''(x) = x + \cos x \cdot y(x) + \sin x \cdot y'(x) - \frac{d}{dx} \int_0^x (e^t + \cos t) y(t) dt$$

or
$$y''(x) = x + \cos x \cdot y(x) + \sin x \cdot y'(x) - \left[\int_0^x \frac{\partial}{\partial x} \{ (e^t + \cos t) y(t) \} dt \right.$$

$$\left. + (e^x + \cos x) y(x) \frac{dx}{dx} - (e^0 + \cos 0) y(0) \frac{d0}{dx} \right], \text{ using Leibnitz's rule}$$

or
$$y''(x) = x + \cos x \cdot y(x) + \sin x \cdot y'(x) - [0 + (e^x + \cos x) y(x) + 0]$$

or
$$y''(x) - \sin x \cdot y'(x) + e^x \cdot y(x) = x, \quad \dots (7)$$

which is the same as given differential equation (1).

Putting $x = 0$ on both sides of (5) and (6), we easily obtain

$$y(0) = 1 \quad \text{and} \quad y'(0) = -1. \quad \dots (8)$$

(7) and (8) together give us the given differential equation and initial conditions.

Ex. 5. Convert $y''(x) - 3y'(x) + 2y(x) = 4 \sin x$ with initial conditions $y(0) = 1$, $y'(0) = -2$ into a Volterra integral equation of the second kind. Conversely, derive the original differential equation with initial conditions from the integral equation obtained.

[Meerut 2003, 06; Kanpur 2009]

Sol. Given $y''(x) - 3y'(x) + 2y(x) = 4 \sin x \quad \dots (1)$

with initial conditions : $y(0) = 1 \quad \dots 2(a)$

and $y'(0) = -2 \quad \dots 2(b)$

From (1), $y''(x) = 4 \sin x - 2y(x) + 3y'(x) \quad \dots (3)$

Integrating both sides of (3) w.r.t. 'x' from 0 to x, we have

$$\int_0^x y''(x) dx = 4 \int_0^x \sin x dx - 2 \int_0^x y(x) dx + 3 \int_0^x y'(x) dx$$

or $[y'(x)]_0^x = 4[-\cos x]_0^x - 2 \int_0^x y(x) dx + 3[y(x)]_0^x$

or $y'(x) - y'(0) = 4(-\cos x + 1) - 2 \int_0^x y(x) dx + 3[y(x) - y(0)]$

or $y'(x) + 2 = 4 - 4 \cos x - 2 \int_0^x y(x) dx + 3y(x) - 3$, using 2 (a) and 2 (b)

or $y'(x) = -1 - 4 \cos x - 3y(x) - 2 \int_0^x y(x) dx \quad \dots (4)$

Integrating both sides of (4) w.r.t. 'x' from 0 to x, we have

$$\int_0^x y'(x) dx = -\int_0^x dx - 4 \int_0^x \cos x dx + 3 \int_0^x y(x) dx - 2 \int_0^x y(x) dx^2$$

or $[y(x)]_0^x = -x - 4[\sin x]_0^x + 3 \int_0^x y(x) dx - 2 \int_0^x y(t) dt^2$

or $y(x) - y(0) = -x - 4 \sin x + 3 \int_0^x y(t) dt - 2 \int_0^x (x-t)y(t) dt$, by result of Art. 1.14

or $y(x) = 1 - x - 4 \sin x + \int_0^x [3 - 2(x-t)]y(t) dt$, using 2 (a) $\dots (5)$

which is the required Volterra integral equation of the second kind.

Second Part : Derivation of the given differential equation together with given initial conditions from integral equation (5).

Differentiating both sides of (5) w.r.t. 'x', we get

$$y'(x) = -1 - 4 \cos x + \frac{d}{dx} \int_0^x [3 - 2(x-t)]y(t) dt$$

or $y'(x) = -1 - 4 \cos x + \int_0^x \frac{\partial}{\partial x} [3 - 2(x-t)]y(t) dt + [3 - 2(x-x)]y(x) \frac{dx}{dx} - [3 - 2(x-0)]y(0) \frac{d0}{dx}$

[using Leibnitz's rule of differentiation under integral sign (refer Art. 1.13)]

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Conversion of Ordinary differential equation into integral equation

or
$$y'(x) = -1 - 4 \cos x + \int_0^x (-2)y(t) dt + 3y(x)$$

or
$$y'(x) = -1 - 4 \cos x + 3y(x) - 2 \int_0^x y(t) dt. \quad \dots (6)$$

Differentiating both sides of (6) w.r.t. 'x', we get

$$y''(x) = 4 \sin x + 3y'(x) - 2 \frac{d}{dx} \int_0^x y(t) dt$$

or
$$y''(x) = 4 \sin x + 3y'(x) - 2 \left[\int_0^x \frac{\partial}{\partial x} y(t) dt + y(x) \frac{dx}{dx} - y(0) \frac{d0}{dx} \right] \text{ by Leibnitz's rule}$$

$$y''(x) = 4 \sin x + 3y'(x) - 2[0 + y(x) - 0]$$

or
$$y''(x) - 3y'(x) + 2y(x) = 4 \sin x, \quad \dots (7)$$

which is the same as the given differential equation.

Putting $x = 0$ in (5), we get $y(0) = 1$. Further putting $x = 0$ in (6), we get

$$y'(0) = -1 - 4 + 3y(0) \quad \text{or} \quad y'(0) = -1 - 4 + 3 = -2, \text{ using } 2(a)$$

Thus, $y(0) = 1$ and $y'(0) = -2. \quad \dots (8)$

(7) and (8) together give us the given differential equation and initial conditions.

Ex. 6. The initial value problem corresponding to the integral equation $y(x) = 1 + \int_0^x y(t) dt$ is

(a) $y' - y = 0, \quad y(0) = 1$ (b) $y' + y = 0, \quad y(0) = 0$

(c) $y' - y = 0, \quad y(0) = 0$ (d) $y' + y = 0, \quad y(0) = 1$ [GATE 2001]

Sol. Ans (a) Given
$$y(x) = 1 + \int_0^x y(t) dt. \quad \dots (1)$$

Differentiating both sides of (1) with respect to x and using the Leibnitz's rule of differentiation under the sign of integral (refer Art. 1.13), we obtain

$$y'(x) = 0 + \int_0^x \frac{\partial y(t)}{\partial x} dt + y(x) \frac{dx}{dx} - y(0) \frac{d0}{dx}$$

or
$$y'(x) = y(x), \quad \text{i.e.,} \quad y' - y = 0 \quad \dots (2)$$

From (1),
$$y(0) = 1 + \int_0^0 y(t) dt = 1, \quad \text{i.e.,} \quad y(0) = 1 \quad \dots (3)$$

(2) and (3) show that result (a) is true.

EXERCISE-2A

1. (a) Show that, if $y(x)$ satisfies the differential equation $(d^2y/dx^2) + xy = 1$ and the conditions

$y(0) = y'(0) = 0$, then y also satisfies the Volterra equation
$$y(x) = \frac{1}{2}x^2 + \int_0^x t(t-x)y(t) dt.$$

(b) Prove that the converse of the preceding statement is also true. (Meerut 2011)

2. (a) If $y''(x) = F(x)$, and y satisfies the initial conditions $y(0) = y_0$ and $y'(0) = y'_0$, show

that
$$y(x) = y_0 + xy'_0 + \int_0^x (x-t)F(t) dt.$$

(b) Verify that this expression satisfies the prescribed differential equation and initial conditions.

3. Convert $y''(x) - 2xy'(x) - 3y(x) = 0$ with initial conditions $y(0) = 1$, $y'(0) = 0$ to a Volterra integral equation of the second kind. Conversely, derive the original differential equation with initial conditions from the integral equation obtained. **Ans.** $y(x) = 1 + \int_0^x (x+t)y(t) dt$

4. Reduce the following initial value problem into an integral equation

$$\frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0, \quad y(0) = 1, \quad y'(0) = 1. \quad \text{Ans. } y(x) = 1 + x - \int_0^x t y(t) dt$$

[Kanpur 2007, 11; Meerut 2009]

5. Show that the solution of the Volterra equation $y(x) = 1 + \int_0^x (t-x)y(t) dt$ satisfies the differential equation $y''(x) + y(x) = 0$ and the boundary conditions $y(0) = 1$, $y'(0) = 1$

2.4. ALTERNATIVE METHOD OF CONVERTING AN INITIAL VALUE PROBLEM INTO A VOLTERRA INTEGRAL EQUATION.

This method is somewhat simpler than the method outlined in Art. 2.3. However, the method explained in Art. 2.3 is very useful in problem where we are required to derive the original differential equation together with initial conditions from the integral equation obtained.

Consider the ordinary linear differential equation of order n :

$$\frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + a_2(x) \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n(x) y = \phi(x) \quad \dots (1)$$

with the initial conditions

$$y(a) = q_0, \quad y'(a) = q_1, \quad y''(a) = q_2, \quad \dots, \dots \quad y^{(n-1)}(a) = q_{n-1}, \quad \dots (2)$$

where the functions $a_1(x)$, ..., $a_n(x)$ and $\phi(x)$ are defined and continuous in $a \leq x \leq b$.

In order to reduce the initial value problem (1)–(2) to the Volterra integral equation, we introduce an unknown function $u(x)$. Thus, we take

$$d^n y / dx^n = u(x) \quad \dots (A_n)$$

Integrating both sides of equation (A_n) w.r.t. 'x' from a to x , we have

$$\left[\frac{d^{n-1} y}{dx^{n-1}} \right]_a^x = \int_a^x u(x) dx \quad \text{or} \quad \frac{d^{n-1} y}{dx^{n-1}} - y^{(n-1)}(a) = \int_a^x u(x) dx$$

$$\text{or} \quad \frac{d^{n-1} y}{dx^{n-1}} = \int_a^x u(x) dx + q_{n-1}, \quad \text{using (2)} \quad \dots (A_{n-1})'$$

$$\text{or} \quad \frac{d^{n-1} y}{dx^{n-1}} = \int_a^x u(t) dt + q_{n-1} \quad \dots (A_{n-1})$$

Integrating both sides of equation $(A_{n-1})'$ w.r.t. 'x' from a to x , we have

$$\left[\frac{d^{n-2} y}{dx^{n-2}} \right]_a^x = \int_a^x u(x) dx^2 + q_{n-1} \int_a^x dx$$

$$\begin{aligned}
&\text{or} \quad \frac{d^{n-2}y}{dx^{n-2}} - y^{(n-2)}(a) = \int_a^x u(x) dx^2 + q_{n-1} [x]_a^x \\
&\text{or} \quad \frac{d^{n-2}y}{dx^{n-2}} = \int_a^x u(x) dx^2 + (x-a)q_{n-1} + q_{n-2}, \text{ using (2)} \quad \dots (A_{n-2})' \\
&\text{or} \quad \frac{d^{n-2}y}{dx^{n-2}} = \int_a^x u(t) dt^2 + (x-a)q_{n-1} + q_{n-2} \\
&\text{or} \quad \frac{d^{n-2}y}{dx^{n-2}} = \int_a^x (x-t)u(t) dt + (x-a)q_{n-1} + q_{n-2} \quad \dots (A_{n-2}) \\
&\quad \quad \quad \text{[using result of Art. 1.14]}
\end{aligned}$$

Integrating both sides of equation $(A_{n-2})'$ w.r.t. 'x' from a to x , we have

$$\begin{aligned}
&\left[\frac{d^{n-3}y}{dx^{n-3}} \right]_a^x = \int_a^x u(x) dx^3 + q_{n-1} \int_a^x (x-a) dx + q_{n-2} \int_a^x dx \\
&\text{or} \quad \frac{d^{n-3}y}{dx^{n-3}} - y^{(n-3)}(a) = \int_a^x u(x) dx^3 + q_{n-1} \left[\frac{(x-a)^2}{2} \right]_a^x + q_{n-2} [x]_a^x \\
&\text{or} \quad \frac{d^{n-3}y}{dx^{n-3}} = \int_a^x u(x) dx^3 + q_{n-1} \frac{(x-a)^2}{2!} + q_{n-2} \frac{(x-a)}{1!} + q_{n-3}, \text{ using (2)} \quad \dots (A_{n-3})' \\
&\text{or} \quad \frac{d^{n-3}y}{dx^{n-3}} = \int_a^x u(t) dt^3 + q_{n-1} \frac{(x-a)^2}{2!} + q_{n-2} \frac{(x-a)}{1!} + q_{n-3} \\
&\text{or} \quad \frac{d^{n-3}y}{dx^{n-3}} = \int_a^x \frac{(x-t)^2}{2!} u(t) dt + q_{n-1} \frac{(x-a)^2}{2!} + q_{n-2} \frac{(x-a)}{1!} + q_{n-3} \quad \dots (A_{n-3}) \\
&\quad \quad \quad \text{[using result of Art. 1.14 again]}
\end{aligned}$$

Ans so on. Finally, we arrive at :

$$\begin{aligned}
&\frac{dy}{dx} = \int_a^x \frac{(x-t)^{n-2}}{(n-2)!} u(t) dt + q_{n-1} \frac{(x-a)^{n-2}}{(n-2)!} + q_{n-2} \frac{(x-a)^{n-3}}{(n-3)!} + \dots + q_2 (x-a) + q_1 \dots (A_1) \\
&\text{and} \quad y = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} u(t) dt + q_{n-1} \frac{(x-a)^{n-1}}{(n-1)!} + q_{n-2} \frac{(x-a)^{n-2}}{(n-2)!} + \dots + q_1 (x-a) + q_0 \dots (A_0)
\end{aligned}$$

Multiplying (A_n) , (A_{n-1}) , ..., (A_1) and (A_0) by 1, $a_1(x)$, ..., $a_{n-1}(x)$ and $a_n(x)$ respectively and adding, we get

$$\begin{aligned}
&\frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n(x) y = u(x) + \left[q_{n-1} a_1(x) + \{q_{n-2} + (x-a)q_{n-1}\} a_2(x) + \dots \right. \\
&\quad \left. + \left\{ q_0 + q_1(x-a) + \dots + q_{n-1} \frac{(x-a)^{n-1}}{(n-1)!} \right\} a_n(x) \right] + \int_a^x \left[a_1(x) + (x-t)a_2(x) + \frac{(x-t)^2}{2!} a_3(x) \right. \\
&\quad \left. + \dots + \frac{(x-t)^{n-1}}{(n-1)!} a_n(x) \right] u(t) dt
\end{aligned}$$

or
$$\phi(x) = u(x) + \psi(x) - \int_a^x K(x, t) u(t) dt, \quad \dots (3)$$

where we have used (1) and assumed the following :

$$\psi(x) = q_{n-1}a_1(x) + \{q_{n-2} + (x-a)q_{n-1}\}a_2(x) + \dots + \left\{q_0 + q_1(x-a) + \dots + q_{n-1}\frac{(x-a)^{n-1}}{(n-1)!}\right\}a_n(x) \quad \dots (4)$$

and
$$K(x, t) = - \left[a_1(x) + (x-t)a_2(x) + \dots + \frac{(x-t)^{n-1}}{(n-1)!} a_n(x) \right] \quad \dots (5)$$

Again, let
$$\phi(x) - \psi(x) = f(x). \quad \dots (6)$$

Using (6), (3) reduces to
$$u(x) = f(x) + \int_a^x K(x, t) u(t) dt, \quad \dots (7)$$

which is the required Volterra integral equation of the second kind. Thus, the initial value problem (1) – (2) has been converted into Volterra integral equation of the second kind (7).

SOLVED EXAMPLES BASED ON ART. 2.4

Ex. 1. Form an integral equation corresponding to the differential equation $y'' + xy' + y = 0$, with the initial conditions : $y(0) = 1$, $y'(0) = 0$.

Sol. Given differential equation is
$$d^2y/dx^2 + x(dy/dx) + y = 0, \quad \dots (1)$$
 subject to the initial conditions :

$$y(0) = 1 \quad \dots 2(a)$$

and
$$y'(0) = 0. \quad \dots 2(b)$$

Suppose that
$$d^2y/dx^2 = u(x) \quad \dots (A_2)$$

Integrating both sides of (A_2) w.r.t. 'x' from 0 to x, we have

$$\left[\frac{dy}{dx} \right]_0^x = \int_0^x u(x) dx \quad \text{or} \quad \frac{dy}{dx} - y'(0) = \int_0^x u(x) dx$$

or
$$\frac{dy}{dx} = \int_0^x u(x) dx, \text{ using } 2(b) \quad \dots (A_1)'$$

or
$$\frac{dy}{dx} = \int_0^x u(t) dt \quad \dots (A_1)$$

Integrating both sides of $(A_1)'$ w.r.t. 'x' from 0 to x, we have

$$y(x) - y(0) = \int_0^x u(x) dx^2 \quad \text{or} \quad y(x) - 1 = \int_0^x u(t) dt^2, \text{ using } 2(a)$$

or
$$y(x) = 1 + \int_0^x (x-t)u(t) dt, \text{ using result of Art. 1.14} \quad \dots (A_0)$$

Putting values of d^2y/dx^2 , dy/dx and y given by (A_2) , (A_1) and (A_0) respectively in (1), we get

$$u(x) + x \int_0^x u(t) dt + 1 + \int_0^x (x-t)u(t) dt = 0 \quad \text{or} \quad u(x) + 1 + \int_0^x xu(t) dt + \int_0^x (x-t)u(t) dt = 0$$

or
$$u(x) + 1 + \int_0^x \{x + (x-t)\}u(t) dt = 0 \quad \text{or} \quad u(x) = -1 - \int_0^x (2x-t)u(t) dt, \dots (3)$$

which is the required Volterra integral equation of the second kind.

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Conversion of Ordinary differential equation into integral equation

Ex. 2. Form an integral equation corresponding to the differential equation $(d^2y/dx^2) - \sin x (dy/dx) + e^x y = x$, with the initial conditions $y(0) = 1$, $y'(0) = -1$.

Sol. Given differential equation is $d^2y/dx^2 - \sin x (dy/dx) + e^x y = x$, ... (1)
subject to the initial conditions : $y(0) = 1$... 2(a)

and $y'(0) = -1$... 2(b)

Suppose that $d^2y/dx^2 = u(x)$ (A₂)

Integrating (A₂) w.r.t. 'x' from 0 to x, we get

$$\left[\frac{dy}{dx} \right]_0^x = \int_0^x u(x) dx \quad \text{or} \quad \frac{dy}{dx} - y'(0) = \int_0^x u(x) dx$$

$$\text{or} \quad \frac{dy}{dx} + 1 = \int_0^x u(x) dx, \text{ by 2 (b)} \quad \text{or} \quad \frac{dy}{dx} = -1 + \int_0^x u(x) dx \quad \dots (A_1)'$$

$$\text{or} \quad \frac{dy}{dx} = -1 + \int_0^x u(t) dt. \quad \dots (A_1)$$

Integrating (A₁)' w.r.t. 'x', we get

$$[y(x)]_0^x = -[x]_0^x + \int_0^x u(x) dx^2 \quad \text{or} \quad y(x) - y(0) = -x + \int_0^x u(t) dt^2$$

$$\text{or} \quad y(x) - 1 = -x + \int_0^x u(t) dt^2, \text{ using 2 (a)}$$

$$\text{or} \quad y(x) = 1 - x + \int_0^x (x-t)u(t) dt, \text{ using result (1) of Art. 1.14} \quad \dots (A_0)$$

Putting values of d^2y/dx^2 , dy/dx and y given by (A₂), (A₁) and (A₀) respectively in (1), we get

$$u(x) - \sin x \left[-1 + \int_0^x u(t) dt \right] + e^x \left[1 - x + \int_0^x (x-t)u(t) dt \right] = x$$

$$\text{or} \quad u(x) = x - \sin x - e^x(1-x) + \int_0^x \sin x u(t) dt - \int_0^x e^x(x-t)u(t) dt$$

$$\text{or} \quad u(x) = x - \sin x - e^x(1-x) + \int_0^x [\sin x - e^x(x-t)] u(t) dt, \quad \dots (3)$$

which is the required Volterra integral equation of the second kind.

Ex. 3. Form the integral equation corresponding to the following differential equation with the given initial conditions : $y''' - 2xy = 0$; $y(0) = 1/2$, $y'(0) = y''(0) = 1$.

Sol. Given differential equation is $d^3y/dx^3 - 2xy = 0$, ... (1)
subject to the initial conditions : $y(0) = 1/2$, ... 2(a)

$$y'(0) = 1 \quad \dots 2(b)$$

$$\text{and} \quad y''(0) = 1 \quad \dots 2(c)$$

$$\text{Suppose that} \quad d^3y/dx^3 = u(x) \quad \dots (A_3)$$

Integrating (A₃) w.r.t. 'x' from 0 to x, we get

$$\left[\frac{d^2y}{dx^2} \right]_0^x = \int_0^x u(x) dx \quad \text{or} \quad \frac{d^2y}{dx^2} - y''(0) = \int_0^x u(x) dx$$

$$\text{or} \quad \frac{d^2 y}{dx^2} = 1 + \int_0^x u(x) dx, \text{ by 2 (c)} \quad \dots (A_2)'$$

$$\text{or} \quad \frac{d^2 y}{dx^2} = 1 + \int_0^x u(t) dt. \quad \dots (A_2)$$

Integrating $(A_2)'$ w.r.t. 'x' from 0 to x, we get

$$\left[\frac{dy}{dx} \right]_0^x = \int_0^x dx + \int_0^x u(x) dx^2 \quad \text{or} \quad \frac{dy}{dx} - y'(0) = x + \int_0^x u(x) dx^2$$

$$\text{or} \quad \frac{dy}{dx} = 1 + x + \int_0^x u(x) dx^2, \text{ by 2 (b)} \quad \dots (A_1)'$$

$$\text{or} \quad \frac{dy}{dx} = 1 + x + \int_0^x u(t) dt^2$$

$$\text{or} \quad \frac{dy}{dx} = 1 + x + \int_0^x (x-t) u(t) dt, \text{ using result (1) of Art. 1.14} \quad \dots (A_1)$$

Integrating $(A_1)'$ w.r.t. 'x', from 0 to x, we get

$$y(x) - y(0) = \int_0^x (1+x) dx + \int_0^x u(x) dx^3 \quad \text{or} \quad y(x) = \frac{1}{2} + \left[x + \frac{1}{2} x^2 \right]_0^x + \int_0^x u(t) dt^3, \text{ by 2 (a)}$$

$$\text{or} \quad y(x) = \frac{1}{2} + x + \frac{1}{2} x^2 + \int_0^x \frac{(x-t)^2}{2!} u(t) dt, \text{ using result of Art. 1.14} \quad \dots (A_0)$$

Putting values of $d^3 y/dx^3$ and y given by (A_3) and (A_0) respectively in (1), we have

$$u(x) - 2x \left[\frac{1}{2} + x + \frac{1}{2} x^2 + \frac{1}{2} \int_0^x (x-t)^2 u(t) dt \right] = 0 \quad \text{or} \quad u(x) = x(1 + 2x + x^2) + x \int_0^x (x-t)^2 u(t) dt$$

$$\text{or} \quad u(x) = x(x+1)^2 + \int_0^x x(x-t)^2 u(t) dt,$$

which is the required integral equation.

Ex. 4. Form an integral equation corresponding to the differential equation $y''' + x y'' + (x^2 - x)y = x e^x + 1$ with initial conditions : $y(0) = 1 = y'(0)$, $y''(0) = 0$. [Meerut 2009]

Sol. Given differential equation is $d^3 y/dx^3 + x (d^2 y/dx^2) + (x^2 - x)y = x e^x + 1 \quad \dots (1)$
subject to the initial conditions : $y(0) = 1, \quad \dots 2(a)$

$$y'(0) = 1 \quad \dots 2(b)$$

$$\text{and} \quad y''(0) = 0 \quad \dots 2(c)$$

$$\text{Suppose that} \quad d^3 y/dx^3 = u(x). \quad \dots (A_3)$$

Integrating (A_3) w.r.t. 'x' from 0 to x, we get

$$\left[\frac{d^2 y}{dx^2} \right]_0^x = \int_0^x u(x) dx \quad \text{or} \quad \frac{d^2 y}{dx^2} - y''(0) = \int_0^x u(x) dx$$

$$\text{or} \quad \frac{d^2 y}{dx^2} = \int_0^x u(x) dx, \text{ by 2 (c)} \quad \dots (A_2)'$$

or
$$\frac{d^2 y}{dx^2} = \int_0^x u(t) dt \quad \dots (A_2)$$

Integrating $(A_2)'$ w.r.t. 'x' from 0 to x, we get

$$\left[\frac{dy}{dx} \right]_0^x = \int_0^x u(x) dx^2 \quad \text{or} \quad \frac{dy}{dx} - y'(0) = \int_0^x u(x) dx^2$$

or
$$\frac{dy}{dx} = 1 + \int_0^x u(x) dx^2, \text{ by 2 (b)} \quad \dots (A_1)'$$

or
$$\frac{dy}{dx} = 1 + \int_0^x u(t) dt^2 \quad \text{or} \quad \frac{dy}{dx} = 1 + \int_0^x (x-t) u(t) dt, \text{ using result of Art. 1.14} \quad \dots (A_1)$$

Integrating $(A_1)'$ w.r.t. 'x' from 0 to x, we get

$$y(x) - y(0) = \int_0^x dx + \int_0^x u(x) dx^3 \quad \text{or} \quad y(x) = 1 + x + \int_0^x u(t) dt^3, \text{ by 2(a)}$$

or
$$y(x) = 1 + x + \int_0^x \frac{(x-t)^2}{2!} u(t) dt, \text{ using result of Art 1.14} \quad \dots (A_0)$$

Putting values of $d^3 y/dx^3$, $d^2 y/dx^2$ and $y(x)$ given by (A_3) , (A_2) and (A_0) respectively in (1),

we get
$$u(x) + x \int_0^x u(t) dt + (x^2 - x) \left[1 + x + \frac{1}{2} \int_0^x (x-t)^2 u(t) dt \right] = x e^x + 1$$

or
$$u(x) = x e^x + 1 - (x^2 - x)(x+1) - \int_0^x x u(t) dt - \int_0^x \frac{1}{2} (x^2 - x)(x-t)^2 u(t) dt$$

or
$$u(x) = x e^x + 1 - x(x^2 - 1) - \int_0^x \left[x + \frac{1}{2} (x^2 - x)(x-t)^2 \right] u(t) dt,$$

which is the required integral equation.

Ex. 5. Prove that the linear differential equation of second order

$$\frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = F(x),$$

with initial conditions $y(0) = C_0$ and $y'(0) = C_1$ can be transformed into non-homogeneous Volterra's integral equation of second kind. [GATE 2006]

Sol. Given differential equation is
$$\frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = F(x), \quad \dots (1)$$

subject to the boundary conditions :
$$y(0) = C_0 \quad \dots 2(a)$$

and
$$y'(0) = C_1 \quad \dots 2(a)$$

Suppose that
$$d^2 y / dx^2 = u(x). \quad \dots (A_2)$$

Integrating (A_2) w.r.t. 'x' from 0 to x, we get

$$\left[\frac{dy}{dx} \right]_0^x - \int_0^x u(x) dx \quad \text{or} \quad \frac{dy}{dx} - y'(0) = \int_0^x u(x) dx$$

or
$$\frac{dy}{dx} = C_1 + \int_0^x u(x) dx, \text{ by 2 (b)} \quad \dots (A_1)'$$

or
$$\frac{dy}{dx} = C_1 + \int_0^x u(t) dt. \quad \dots (A_1)$$

Integrating $(A_1)'$ w.r.t. 'x' from 0 to x, we get

$$y(x) - y(0) = C_1 \int_0^x dx + \int_0^x u(x) dx^2 \quad \text{or} \quad y(x) = C_0 + C_1 x + \int_0^x u(t) dt^2, \text{ by 2 (a)}$$

or
$$y(x) = C_0 + C_1 x + \int_0^x (x-t) u dt, \text{ using result of Art. 1.14} \quad \dots (A_0)$$

Putting values of d^2/dx^2 , dy/dx and y given by (A_2) , (A_1) and (A_0) respectively in (1), we get

$$u(x) + a_1(x) [C_1 + \int_0^x u(t) dt] + a_2(x) [C_0 + C_1 x + \int_0^x (x-t) u(t) dt] = F(x)$$

or
$$u(x) = F(x) - C_1 a_1(x) + (C_0 + C_1 x) a_2(x) - \int_0^x a_1(x) u(t) dt - \int_0^x a_2(x) (x-t) u(t) dt$$

or
$$u(x) = F(x) - C_1 a_1(x) + (C_0 + C_1 x) a_2(x) - \int_0^x [a_1(x) + a_2(x) (x-t)] u(t) dt,$$

which is the required non-homogeneous Volterra's integral equation of second kind.

EXERCISE-2 (b)

Reduce the following initial value problems into Volterra integral equations of the second kind :

1. $y'' + y = 0$; $y(0) = 0$, $y'(0) = 1$. **Ans.** $u(x) = -x - \int_0^x (x-t) u(t) dt$, where $u(x) = y''$

2. $y' - y = 0$; $y(0) = 1$ **Ans.** $u(x) = 1 + \int_0^x u(t) dt$, where $u(x) = y'$

3. $y'' + y = \cos x$; $y(0) = 0$, $y'(0) = 1$

Ans. $u(x) = \cos x - x - \int_0^x (x-t) u(t) dt$, where $u(x) = y''$

4. $y'' - 5y' + 6y = 0$; $y(0) = 0$, $y'(0) = -1$. (Kanpur 2008)

Ans. $u(x) = 6x - 5 + \int_0^x (5 - 6x + 6t) u(t) dt$, where $u(x) = y''$

5. $y''(x) - 3y'(x) + 2y(x) = 4 \sin x$, $y(0) = 1$, $y'(0) = -2$.

Ans. $u(x) = 4(x + \sin x - 2) + \int_0^x [3 - 2(x-t)] u(t) dt$, where $u(x) = y''$

6. $y'' + y = \cos x$; $y(0) = 0$, $y'(0) = 0$ **Ans.** $u(x) = \cos x - \int_0^x (x-t) u(t) dt$, where $u(x) = y''$

7. $y'' - 2xy' - 3y = 0$, $y(0) = 1$, $y'(0) = 0$. **Ans.** $u(x) = 3 + \int_0^x (5x - 3t) u(t) dt$, where $u(x) = y''$

8. $y'' + xy = 1$, $y(0) = y'(0) = 0$. (Kanpur 2009)

Ans. $u(x) = 1 - \int_0^x x(x-t) u(t) dt$, where $u(x) = y''$

9. $y'' + (1+x^2)y = \cos x$; $y(0) = 0$, $y'(0) = 2$

Ans. $u(x) = \cos x - 2x(1+x^2) - \int_0^x (1+x^2)(x-t) u(t) dt$, where $u(x) = y''$

10. $y''(x) + \lambda y(x) = F(x)$, $y(0) = 1$, $y'(0) = 0$. (Meerut 2012)

Ans. $u(x) = F(x) - \lambda + \lambda \int_0^x (x-t) u(t) dt$, where $u(x) = y''$

11. Show that a linear differential equation with constant coefficients reduces, under any initial conditions, to a Volterra integral of the second kind with kernel dependent solely on the difference $(x - t)$ of arguments (integral equation of the closed cycle or equation of the Faltung type, or convolution type).

12. Establish relation between linear differential equation of order n and Volterra integral equation of the second kind.

13. Reduce linear differential equation of order n into a Volterra integral equation the second kind. Hence prove that the solution of Volterra integral equation is unique.

14. Show that linear differential equation $d^n y/dx^n + a_1(x) (d^{n-1} y/dx^{n-1}) + \dots + a_n(x) y = \phi(x)$ can be converted into volterra integral equation of the type

$$u(x) + \int_0^x \left[a_1(x) + a_2(x)(x-t) + \dots + a_n(x) \frac{(x-t)^{n-1}}{(n-1)!} \right] u(t) dt = f(x).$$

2.5. BOUNDARY VALUE PROBLEM, DEFINITION.

When an ordinary differential equation is to be solved under conditions involving dependent variable and its derivatives at two different values of independent variable, then the problem under consideration is said to be a *boundary value problem*.

For example $d^2 y / dx^2 + y = 0$, $y(a) = y_1$, $y(b) = y_2$ is a boundary value problem. Note that here different values $x = a$ and $x = b$ of the independent variable x are involved.

2.6. METHOD OF CONVERTING A BOUNDARY VALUE PROBLEM INTO A FREDHOLM INTEGRAL EQUATION.

We explain the method with help of the following solved examples.

Ex. 1. (a) Reduce the following boundary value problem into an integral equation :

$$d^2 y / dx^2 + \lambda y = 0 \quad \text{with} \quad y(0) = 0, \quad y(l) = 0 \quad [\text{Kanpur 2000}]$$

(b) Also recover the B.V.P. from the integral equation. [Meerut 2000]

Sol. Given $y''(x) + \lambda y(x) = 0$... (1)
 with boundary conditions $y(0) = 0$... 2(a)
 and $y(l) = 0$ 2(b)

From (1), $y''(x) = -\lambda y(x)$... (3)

Integrating both sides of (3) w.r.t. 'x' from 0 to x, we have

$$\int_0^x y''(x) dx = -\lambda \int_0^x y(x) dx \quad \text{or} \quad [y'(x)]_0^x = -\lambda \int_0^x y(x) dx$$

or $y'(x) - y'(0) = -\lambda \int_0^x y(x) dx$ (4)

Let $y'(0) = C$, a constant ... (5)

Using (5), (4) gives $y'(x) = C - \lambda \int_0^x y(x) dx$... (6)

Integrating both sides of (6) w.r.t. 'x', from 0 to x, we get

$$\int_0^x y'(x) dx = C \int_0^x dx - \lambda \int_0^x y(x) dx^2 \quad \text{or} \quad [y(x)]_0^x = Cx - \lambda \int_0^x y(t) dt^2$$

$$\text{or} \quad y(x) - y(0) = Cx - \lambda \int_0^x (x-t) y(t) dt, \text{ using result of Art. 1.14}$$

$$\text{or} \quad y(x) - 0 = Cx - \lambda \int_0^x (x-t) y(t) dt, \text{ by 2 (a)} \quad \text{or} \quad y(x) = Cx - \lambda \int_0^x (x-t) y(t) dt. \dots (7)$$

Putting $x = l$ in (7), we get

$$y(l) = Cl - \lambda \int_0^l (l-t) y(t) dt \quad \text{or} \quad 0 = Cl - \lambda \int_0^l (l-t) y(t) dt, \text{ using 2 (b)}$$

$$\text{or} \quad C = \frac{\lambda}{l} \int_0^l (l-t) y(t) dt \quad \dots (8)$$

Using (8), (7) reduces to

$$y(x) = \frac{\lambda}{l} x \int_0^l (l-t) y(t) dt - \lambda \int_0^x (x-t) y(t) dt \quad \dots (9)$$

$$\text{or} \quad y(x) = \int_0^l \frac{\lambda x(l-t)}{l} y(t) dt - \lambda \int_0^x (x-t) y(t) dt$$

$$\text{or} \quad y(x) = \int_0^x \frac{\lambda x(l-t)}{l} y(t) dt + \int_x^l \frac{\lambda x(l-t)}{l} y(t) dt - \int_0^x \lambda(x-t) y(t) dt$$

$$\text{or} \quad y(x) = \lambda \int_0^x \left[\frac{x(l-t)}{l} - (x-t) \right] y(t) dt + \lambda \int_x^l \frac{x(l-t)}{l} y(t) dt$$

$$\text{or} \quad y(x) = \lambda \int_0^x \frac{x(l-t) - l(x-t)}{l} y(t) dt + \lambda \int_x^l \frac{x(l-t)}{l} y(t) dt$$

$$\text{or} \quad y(x) = \lambda \left[\int_0^x \frac{x(l-t) - l(x-t)}{l} y(t) dt + \int_x^l \frac{x(l-t)}{l} y(t) dt \right]$$

$$\text{or} \quad y(x) = \lambda \int_0^l K(x, t) y(t) dt, \quad \dots (10)$$

$$\text{where} \quad K(x, t) = \begin{cases} (t/l) \times (l-x), & \text{if } 0 < t < x \\ (x/l) \times (l-t), & \text{if } x < t < l \end{cases} \quad \dots (11)$$

(10) is the required Fredholm integral equation, where $K(x, t)$ is given by (11)

(b) Refer part (c) of the next Ex. 2.

Ex. 2. (a) If $y''(x) + \lambda y(x) = 0$, and y satisfies the end condition $y(0) = 0$, $y(l) = 0$, show that

$$y(x) = \frac{\lambda x}{l} \int_0^l (l-t) y(t) dt - \lambda \int_0^x (x-t) y(t) dt.$$

$$\text{(b) Show that the result of part (a) can be written as} \quad y(x) = \lambda \int_0^l K(x, t) y(t) dt,$$

$$\text{where} \quad K(x, t) = \begin{cases} (t/l) \times (l-x), & \text{when } t < x \\ (x/l) \times (l-t) & \text{when } t > x. \end{cases}$$

(c) Verify directly that the expression obtained satisfies the prescribed differential equation and end conditions.

Sol. (a) Given $y''(x) + \lambda y(x) = 0$... (1)

with the boundary conditions $y(0) = 0$... 2 (a)

and $y(l) = 0$... 2 (b)

Now proceed as in Ex. 1 upto equation (9), i.e.,

$$y(x) = \frac{\lambda x}{l} \int_0^l (l-t) y(t) dt - \lambda \int_0^x (x-t) y(t) dt. \quad \dots (i)$$

Part (b) Proceed as in Ex. 1. Equations (10) and (11) give the required results.

Part (c) We shall now proceed with integral equation (9) of solution of Ex. 1 and obtain the given differential equation (1) together with given boundary conditions 2 (a) and 2 (b) as follows :

Re-writing (9), we have

$$y(x) = \int_0^l \frac{\lambda x(l-t)}{l} y(t) dt - \int_0^x \lambda(x-t) y(t) dt \quad \dots (ii)$$

Putting $x = 0$ and $x = l$ by turn in (ii), we get

$$y(0) = 0 \quad \text{and} \quad y(l) = 0 \quad \dots (iii)$$

Differentiating both sides of (ii) w.r.t. 'x', we get

$$y'(x) = \frac{d}{dx} \int_0^l \frac{\lambda x(l-t)}{l} y(t) dt - \frac{d}{dx} \int_0^x \lambda(x-t) y(t) dt, \quad \dots (iv)$$

Using Leibnitz's rule (refer results (1) and (2) of Art 1.13), (iv) reduces to

$$y'(x) = \int_0^l \frac{\partial}{\partial x} \left\{ \frac{\lambda x(l-t)}{l} y(t) \right\} dt - \left[\int_0^x \frac{\partial}{\partial x} \{ \lambda(x-t) y(t) \} dt + \lambda(x-x) y(x) \frac{dx}{dx} - \lambda(x-0) y(0) \frac{d0}{dx} \right]$$

or
$$y'(x) = \int_0^l \frac{\lambda(l-t)}{l} y(t) dt - \int_0^x \lambda y(t) dt. \quad \dots (v)$$

Differentiating both sides of (v) w.r.t. 'x', we get

$$y''(x) = \frac{d}{dx} \int_0^l \frac{\lambda(l-t)}{l} y(t) dt - \frac{d}{dx} \int_0^x \lambda y(t) dt. \quad \dots (vi)$$

Using Leibnitz's rule again, (vi) reduces to

$$y''(x) = \int_0^l \frac{\partial}{\partial x} \left\{ \frac{\lambda(l-t)}{l} y(t) \right\} dt - \left[\int_0^x \frac{\partial}{\partial x} \{ \lambda y(t) \} dt + \lambda y(x) \frac{dx}{dx} - \lambda y(0) \frac{d0}{dx} \right]$$

[Using results (1) and (2) of Art. 1.3]

or $y''(x) = 0 - 0 - \lambda y(x) + 0$ or $y''(x) + \lambda y(x) = 0$... (vii)

Equation (vii) together with boundary conditions (iii) show that expression (i) satisfies the prescribed differential equation and end conditions.

Ex. 3. If $y(x)$ is continuous and satisfies

$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt, \quad \text{where} \quad K(x, t) = \begin{cases} (1-t)x, & 0 \leq x \leq t \\ (1-x)t, & t \leq x \leq 1. \end{cases}$$

then prove that $y(x)$ is also the solution of the boundary value problem

$$d^2y/dx^2 + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0.$$

Sol. Given
$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt, \quad \dots (1)$$

where
$$K(x, t) = \begin{cases} (1-t)x, & 0 \leq x \leq t \\ (1-x)t, & t \leq x \leq 1. \end{cases} \quad \dots (2)$$

Re-writing (1), we have

$$y(x) = \lambda \left[\int_0^x K(x, t) y(t) dt + \int_x^1 K(x, t) y(t) dt \right]$$

or
$$y(x) = \int_0^x \lambda t(1-x) y(t) dt + \int_x^1 \lambda x(1-t) y(t) dt, \text{ by (2)} \quad \dots (3)$$

Putting $x = 0$ and $x = 1$ by turn in (3), we get

$$y(0) = 0 \quad \text{and} \quad y(1) = 0. \quad \dots (4)$$

Differentiating both sides of (3), w.r.t. 'x', we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \int_0^x \lambda t(1-x) y(t) dt + \frac{d}{dx} \int_x^1 \lambda x(1-t) y(t) dt \\ \frac{dy}{dx} &= \int_0^x \frac{\partial}{\partial x} \{ \lambda t(1-x) y(t) \} dt + \lambda x(1-x) y(x) \frac{dx}{dx} - 0 \times \frac{d0}{dx} + \int_x^1 \frac{\partial}{\partial x} \{ \lambda x(1-t) y(t) \} dt \\ &\quad + 0 \times \frac{d(1)}{dx} - \lambda x(1-x) y(x) \frac{dx}{dx}, \text{ using Leibnitz's rule} \end{aligned}$$

or
$$\frac{dy}{dx} = - \int_0^x \lambda t y(t) dt + \int_x^1 \lambda (1-t) y(t) dt \quad \dots (5)$$

Differentiating both sides of (5) w.r.t. 'x', we get

$$\begin{aligned} \frac{d^2 y}{dx^2} &= - \left[\int_0^x \frac{\partial}{\partial x} \{ \lambda t y(t) \} dt + \lambda x y(x) \frac{dx}{dx} - 0 \times \frac{d0}{dx} \right] + \int_x^1 \frac{\partial}{\partial x} \{ \lambda (1-t) y(t) \} dt + 0 \times \frac{d(1)}{dx} - \lambda (1-x) y(x) \frac{dx}{dx} \\ &\quad \text{[using Leibntiz's rule again]} \end{aligned}$$

or
$$\frac{d^2 y}{dx^2} = - \lambda x y(x) - \lambda (1-x) y(x) \quad \text{or} \quad \frac{d^2 y}{dx^2} + \lambda y = 0 \quad \dots (6)$$

(4) and (6) show that if $y(x)$ satisfies (1), then $y(x)$ is also the solution of the boundary value problem

$$d^2 y/dx^2 + \lambda y = 0, \quad y(0) = y(1) = 0$$

Ex. 4. Transform $d^2 y/dx^2 + xy = 1$, $y(0) = y(1) = 0$ into an integral equation.

Also recover the boundary value problem you obtain.

[Meerut 2001, 10, 12]

Sol. Given

$$y''(x) + x y(x) = 1 \quad \dots (1)$$

with boundary conditions

$$y(0) = 0 \quad \dots 2(a)$$

and

$$y(1) = 1. \quad \dots 2(b)$$

From (1),

$$y''(x) = 1 - x y(x). \quad \dots (3)$$

Integrating both sides of (3) w.r.t. 'x', from 0 to x, we get

$$\int_0^x y''(x) dx = \int_0^x dx - \int_0^x x y(x) dx \quad \text{or} \quad [y'(x)]_0^x = x - \int_0^x x y(x) dx$$

or
$$y'(x) - y'(0) = x - \int_0^x x y(x) dx. \quad \dots (4)$$

Let

$$y'(0) = c. \quad \dots (5)$$

Using (5), (4) gives $y'(x) = c + x - \int_0^x x y(x) dx$ (6)

Integrating both sides of (6) w.r.t. 'x', from 0 to x, we get

$$\int_0^x y'(x) dx = \int_0^x (c + x) dx - \int_0^x x y(x) dx^2 \quad \text{or} \quad [y(x)]_0^x = \left[cx + \frac{1}{2} x^2 \right]_0^x - \int_0^x t y(t) dt^2$$

or $y(x) - y(0) = cx + \frac{1}{2} x^2 - \int_0^x (x-t)t y(t) dt$, using result of Art. 1.14.

or $y(x) = cx + \frac{1}{2} x^2 - \int_0^x (x-t)t y(t) dt$, by 2 (a) ... (7)

Putting $x = 1$ in (7), we have

$$y(1) = c + \frac{1}{2} - \int_0^1 (1-t)t y(t) dt \quad \text{or} \quad 1 = c + \frac{1}{2} - \int_0^1 (1-t)t y(t) dt, \text{ by 2 (b)}$$

or $c = \frac{1}{2} + \int_0^1 (1-t)t y(t) dt$ (8)

Using (8), (7) reduces to

$$y(x) = x \left[\frac{1}{2} + \int_0^1 (1-t)t y(t) dt \right] + \frac{1}{2} x^2 - \int_0^x (x-t)t y(t) dt$$

or $y(x) = \frac{1}{2} x(1+x) + \int_0^1 xt(1-t)y(t) dt - \int_0^x t(x-t) y(t) dt$

or $y(x) = \frac{1}{2} x(1+x) + \int_0^x xt(1-t)y(t) dt + \int_x^1 xt(1-t) y(t) dt - \int_0^x t(x-t)y(t) dt$

or $y(x) = \frac{1}{2} x(1+x) + \int_0^x t y(t) \{x - xt - x + t\} dt + \int_x^1 xt(1-t)y(t) dt$

or $y(x) = \frac{1}{2} x(1+x) + \int_0^x t^2 (1-x)y(t) dt + \int_x^1 xt(1-t)y(t) dt$

or $y(x) = \frac{1}{2} x(1+x) + \int_0^1 K(x,t)y(t) dt$, ... (9)

where $K(x,t) = \begin{cases} t^2(1-x), & \text{when } t < x \\ xt(1-t), & \text{when } t > x \end{cases}$... (10)

To recover the B.V.P. from the integral equation given by (9) and (10) :

Re-writing (9), we have

$$y(x) = \frac{1}{2} (x + x^2) + \int_0^x K(x,t) y(t) dt + \int_x^1 K(x,t) y(t) dt$$

or $y(x) = \frac{1}{2} (x + x^2) + \int_0^x t^2 (1-x) y(t) dt + \int_x^1 xt(1-t) y(t) dt$, ... (11)

Differentiating both sides of (11) with respect to x and using Leibnitz's rule of differentiation under the integral sign (see Art. 1.13), we obtain

$$y'(x) = \frac{1}{2}(1+2x) + \int_0^x (-t^2) y(t) dt + x^2(1-x) y(x) \frac{dx}{dx} - (0) \times y(0) \frac{d0}{dx} \\ + \int_x^1 (t-t^2) y(t) dt + x(1-1) y(1) \frac{d(1)}{dx} - x^2(1-x) y(x) \frac{dx}{dx}$$

$$\text{or} \quad y'(x) = \frac{1}{2}(1+2x) - \int_0^x t^2 y(t) dt + \int_x^1 (t-t^2) y(t) dx \quad \dots (12)$$

Differentiating both sides of (12) w.r.t. 'x' and using Leibnitz rule as before, we have

$$y''(x) = 1 - \left[\int_0^x \frac{\partial \{t^2 y(t)\}}{\partial x} + x^2 y(x) \frac{dx}{dx} - (0) \times y(0) \frac{d0}{dx} \right] \\ + \int_x^1 \frac{\partial \{(t-t^2) y(t)\}}{\partial x} + (1-1) y(1) \frac{d1}{dx} - (x-x^2) y(x) \frac{dx}{dx}$$

$$\text{or} \quad y''(x) = 1 - x^2 y(x) - (x-x^2) y(x) \quad \text{or} \quad y'' + x y = 1 \quad \dots (13)$$

$$\text{From (11),} \quad y(0) = 0 + \int_0^0 t^2 (1-0) y(t) dt + \int_0^1 (0) y(t) dt = 0 \quad \dots (14)$$

$$\text{and} \quad y(1) = 1 + \int_0^1 t^2 (1-1) y(t) dt + \int_1^1 t (1-t) y(t) dt = 1 \quad \dots (15)$$

Thus, we have recovered the given boundary value problem with help of (13), (14) and (15).

Ex. 5. Obtain Fredholm integral equation of second kind corresponding to the boundary value problem $d^2\phi/dx^2 + \lambda\phi = x$, $\phi(0) = 0$, $\phi(1) = 1$. Also recover the boundary value problem from the integral equation obtained [Kanpur 2011; Meerut 2005]

$$\text{Sol. Given} \quad \phi''(x) = x - \lambda\phi(x) \quad \dots (1)$$

$$\text{with boundary conditions :} \quad \phi(0) = 0 \quad \dots (2)$$

$$\text{and} \quad \phi(1) = 1. \quad \dots (3)$$

Integrating both sides of (1) w.r.t. 'x' from 0 to x, we get

$$\int_0^x \phi''(x) dx = \int_0^x x dx - \lambda \int_0^x \phi(x) dx \quad \text{or} \quad [\phi'(x)]_0^x = \left[\frac{x^2}{2} \right]_0^x - \lambda \int_0^x \phi(x) dx$$

$$\text{or} \quad \phi'(x) - \phi'(0) = \frac{x^2}{2} - \lambda \int_0^x \phi(x) dx \quad \dots (4)$$

$$\text{Assume that} \quad \phi'(0) = C, C \text{ being a constant} \quad \dots (5)$$

$$\text{Using (5), (4) yields} \quad \phi'(x) = C + \frac{x^2}{2} - \lambda \int_0^x \phi(x) dx. \quad \dots (6)$$

Integrating both sides of (6) w.r.t. 'x', we obtain

$$\int_0^x \phi'(x) dx = C \int_0^x dx + \int_0^x \frac{x^2}{2} dx - \lambda \int_0^x \int_0^x \phi(x) dx^2$$

$$\text{or} \quad [\phi(x)]_0^x = C[x]_0^x + [x^3/6]_0^x - \lambda \int_0^x \int_0^x \phi(t) dt^2$$

or $\phi(x) - \phi(0) = Cx + (x^3/6) - \lambda \int_0^x (x-t)\phi(t) dt$, using result of Art. 1.14

or $\phi(x) = Cx + (x^3/6) - \lambda \int_0^x (x-t)\phi(t) dt$, by (2) ... (7)

Putting $x = 1$ in (7) and using (3), we obtain

$$1 = C + \frac{1}{6} - \lambda \int_0^1 (1-t)\phi(t) dt \quad \text{or} \quad C = \frac{5}{6} + \lambda \int_0^1 (1-t)\phi(t) dt$$

Substituting the above value of C in (7), we obtain

$$\phi(x) = x \left\{ \frac{5}{6} + \lambda \int_0^1 (1-t)\phi(t) dt \right\} + \frac{x^3}{6} - \lambda \int_0^x (x-t)\phi(t) dt$$

or $\phi(x) = \frac{5x}{6} + \frac{x^3}{6} + \int_0^1 \lambda x(1-t)\phi(t) dt - \int_0^x \lambda(x-t)\phi(t) dt$

or $\phi(x) = (5x + x^3)/6 + \int_0^x \lambda x(1-t)\phi(t) dt + \int_x^1 \lambda x(1-t)\phi(t) dt - \int_0^x \lambda(x-t)\phi(t) dt$

or $\phi(x) = (5x + x^3)/6 + \int_0^x \lambda \{ (x-xt) - (x-t) \} \phi(t) dt + \int_x^1 \lambda x(1-t)\phi(t) dt$

or $\phi(x) = (5x + x^3)/6 + \lambda \left\{ \int_0^x t(1-x)\phi(t) dt + \int_x^1 x(1-t)\phi(t) dt \right\}$

or $\phi(x) = (5x + x^3)/6 + \lambda \int_0^1 K(x,t)\phi(t) dt$, ... (8)

where $K(x,t) = \begin{cases} t(1-x), & \text{when } 0 \leq t < x \\ x(1-t), & \text{when } x < t \leq 1 \end{cases}$... (9)

The required integral equation is given by (8) and (9).

To recover the B.V.P from the integral equation given by (8) and (9) :

Re-writing (8), we have

$$\phi(x) = (5x + x^3)/6 + \lambda \left\{ \int_0^x K(x,t)\phi(t) dt + \int_x^1 K(x,t)\phi(t) dt \right\}$$

or $\phi(x) = (5x + x^3)/6 + \int_0^x \lambda t(1-x)\phi(t) dt + \int_x^1 \lambda x(1-t)\phi(t) dt$, using (9) ... (11)

Differentiating both sides of (11) w.r.t. 'x' and using Leibnitz's rule, we obtain

$$\begin{aligned} \phi'(x) = (5 + 3x^2)/6 + \int_0^x \frac{\partial \{ \lambda t(1-x)\phi(t) \}}{\partial x} dt + \lambda x(1-x)\phi(x) \frac{dx}{dx} - 0 \\ + \int_x^1 \frac{\partial \{ \lambda x(1-t)\phi(t) \}}{\partial x} dt + 0 - \lambda x(1-x)\phi(x) \frac{dx}{dx} \end{aligned}$$

or $\phi'(x) = \frac{1}{6}(5 + 3x^2) + \int_0^x (-\lambda t)\phi(t) dt + \int_x^1 \lambda(1-t)\phi(t) dt$... (12)

Differentiating both sides of (12) w.r.t. 'x' and using Leibnitz's rule, we obtain

$$\begin{aligned}\phi''(x) = x + \int_0^x \frac{\partial \{(-\lambda t)\phi(t)\}}{\partial x} dt + (-\lambda x) \phi(x) \frac{dx}{dx} - 0 \\ + \int_x^1 \frac{\partial \{\lambda(1-t)\phi(t)\}}{\partial x} dt + 0 - \lambda(1-x) \phi(x) \frac{dx}{dx}\end{aligned}$$

or $\phi''(x) = x - \lambda x \phi(x) - \lambda \phi(x) + \lambda x \phi(x)$ or $\phi'' + \lambda \phi = x$... (13)

From (11), $\phi(0) = 0$ and $\phi(1) = 1$... (14)

Thus, we have recovered the given boundary value problem with help of (13) and (14).

Ex. 6. The integral equation $y(x) = \int_0^x (x-t) y(t) dt - x \int_0^1 (1-t) y(t) dt$ is equivalent to :

- (a) $y'' - y = 0, y(0) = 0, y(1) = 0$ (b) $y'' - y = 0, y(0) = 0, y'(0) = 0$
 (c) $y'' + y = 0, y(0) = 0, y(1) = 0$ (d) $y'' + y = 0, y(0) = 0, y'(0) = 0$ [GATE 2003]

Sol. Ans. (a) Given : $y(x) = \int_0^x (x-t) y(t) dt - \int_0^1 x(1-t) y(t) dt$... (1)

Differentiating both sides of (1) w.r.t. 'x' and using Leibnitz's rule, we have

$$y'(x) = \int_0^x \frac{\partial \{(x-t) y(t)\}}{\partial x} dt + (x-x) y(x) \frac{dx}{dx} - (x-0) y(0) \frac{d0}{dx} - \int_0^1 \frac{\partial \{x(1-t) y(t)\}}{\partial x} dt$$

or $y'(x) = \int_0^x y(t) dt - \int_0^1 (1-t) y(t) dt$... (2)

Differentiating both sides of (2) w.r.t. 'x' and using Leibnitz's rule, we obtain

$$y''(x) = \int_0^x \frac{\partial y(t)}{\partial x} dt + y(x) \frac{dx}{dx} - y(0) \frac{d0}{dx} - \int_0^1 \frac{\partial \{(1-t) y(t)\}}{\partial x} dt$$

or $y''(x) = 0 + y(x) - 0 - 0$ or $y'' - y = 0$... (3)

From (1), $y(0) = \int_0^0 (-t) y(t) dt - \int_0^1 \{0 \times (1-t)\} y(t) dt = 0$... (4)

and $y(1) = \int_0^1 (1-t) y(t) dt - \int_0^1 (1-t) y(t) dt = 0$... (5)

Thus, the given integral equation is equivalent to the boundary value problem given by (3), (4) and (5). Hence alternative (a) is true.

EXERCISE-2C

1. Convert the boundary value problem $y'' + y = 0, y(0) = 1, y'(1) = 0$, into an integral equation.

Ans. $y(x) = 1 + \int_0^1 K(x,t) y(t) dt$, where $K(x,t) = \begin{cases} t, & t < x \\ x, & t > x \end{cases}$.

2. If $y(x)$ has continuous first and second derivatives and satisfies the boundary value problem $d^2y/dx^2 + \lambda y = 0, y(0) = 0, y(1) = 0$, then show that $y(x)$ is continuous and satisfies the homogeneous linear integral equation,

$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt, \quad \text{where} \quad K(x, t) = \begin{cases} (1-t)x, & \text{for } 0 \leq x \leq t \\ (1-x)t, & \text{for } t \leq x \leq 1 \end{cases} \quad (\text{Kanpur 2007})$$

3. (a) If $y''(x) = F(x)$, and y satisfies the end conditions $y(0) = 0$ and $y(1) = 0$, show that

$$y(x) = \int_0^x (x-t)F(t) dt - x \int_0^1 (1-t)F(t) dt.$$

- (b) Show that the result of part (a) can be written in the form

$$y(x) = \int_0^1 K(x, t) F(t) dt, \quad \text{where} \quad K(x, t) = \begin{cases} t(x-t), & \text{when } t < x \\ x(t-1), & \text{when } t > x \end{cases}.$$

- (c) Verify directly that the expression obtained satisfies the prescribed differential equation and end conditions.

4. Show that the boundary value problem $y'' + Ay' + By = 0$, $y(0) = 0$, $y(1) = 1$, where A and B are constants, leads to the integral equation

$$y(x) = \int_0^1 K(x, t) y(t) dt, \quad \text{where} \quad K(x, t) = \begin{cases} Bt(1-x) + Ax - A, & \text{when } t < x \\ Bx(1-t) + Ax, & \text{when } t > x. \end{cases}$$

5. Transform the boundary value problem $d^2y/dx^2 + y = x$, $y(0) = 0$, $y'(1) = 0$ to a Fredholm integral equation. [Meerut 2000, 2008]

$$\text{Ans. } y(x) = \frac{1}{6}(x^3 - 3x) + \int_0^1 K(x, t) y(t) dt, \quad \text{where} \quad K(x, t) = \begin{cases} x, & x < 1 \\ t, & x > 1 \end{cases}$$

6. Show that the boundary value problem $d^2y/dx^2 + \lambda y = x$, $y(0) = y(\pi) = 0$ can be connected into an integral equation

$$y(x) = \frac{1}{6}x^3(1 - \pi^2) + \lambda \int_0^\pi K(x, t) y(t) dt, \quad \text{where} \quad K(x, t) = \begin{cases} (x/\pi) \times (x-t), & \text{when } t > x \\ (x/\pi - 1)(x-t), & \text{when } t < x \end{cases}$$

7. Reduce the following boundary value problem into an integral equation.

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) + v y(1) = 0.$$

$$\text{Ans. } y(x) = \lambda \int_0^1 K(x, t) y(t) dt + \frac{x}{1+v}, \quad \text{where} \quad K(x, t) = \begin{cases} \frac{1+v(1-x)}{1+v} t, & t < x \\ \frac{x[1+v(1-t)]}{1+v}, & t > x \end{cases}$$

8. Reduce the boundary value problem $y'' + \lambda y = 0$, $y(0) = y(\pi/2) = 0$ to an integral equation.

$$\text{Ans. } y(x) = \lambda \int_0^{\pi/2} K(x, t) y(t) dt, \quad \text{where} \quad K(x, t) = \begin{cases} [1 - (2t/\pi)] x, & 0 \leq x < t \\ [1 - (2x/\pi)] t, & t < x \leq \pi/2. \end{cases}$$

9. Find the Fredholm integral equation of second kind corresponding to the boundary-value problem $(d^2y/dx^2) + \lambda y = 0$, $y(0) = 0$, $y(l) = 0$.

Also recover the boundary-value problem from the integral equation you obtain.

CHAPTER 3

Homogeneous Fredholm Integral Equations of the Second Kind with Separable or Degenerate Kernels

3.1 CHARACTERISTIC VALUES (OR CHARACTERISTIC NUMBERS OR EIGENVALUES). CHARACTERISTIC FUNCTIONS (OR EIGENFUNCTIONS).

Consider a homogeneous Fredholm integral equation of the second kind :

$$y(x) = \lambda \int_a^b K(x, t) y(t) dt. \quad \dots (1)$$

Then (1) has always the obvious solution $y(x) = 0$, which is known as *zero or trivial solution* of (1). The values of the parameter λ for which (1) has *non-zero (or non-trivial)* solution $y(x) \neq 0$ are known as the *eigenvalues* of (1) or of the kernel $K(x, t)$. Further, if $\phi(x)$ is continuous and $\phi(x) \neq 0$ on the interval (a, b) and

$$\phi(x) = \lambda_0 \int_a^b K(x, t) \phi(t) dt, \quad \dots (2)$$

then $\phi(x)$ is known as an *eigenfunction* of (1) corresponding to the *eigenvalue* λ_0 .

Remark 1. The number $\lambda = 0$ is not an eigenvalue since for $\lambda = 0$, (1) yields $y(x) = 0$, which is a zero solution.

Remark 2. If the kernel $K(x, t)$ is continuous in the rectangle $R : a \leq x \leq b, a \leq t \leq b$, and the numbers a and b are finite, then to every eigenvalue λ there exist a finite number of linearly independent eigenfunctions; the number of such functions is known as the *index of the eigenvalue*. Different eigenvalues have different indices.

Remark 3. If $\phi(x)$ is an eigenfunction of (1) corresponding to eigenvalue λ_0 then $C\phi(x)$ is also eigenfunction of (1) corresponding to the same eigenvalue. Here C is an arbitrary constant.

Remark 4. A homogeneous Fredholm integral equation may, generally, have no eigenvalues and eigenfunctions or it may not have any real eigenvalue and eigenfunction.

3.2 SOLUTION OF HOMOGENEOUS FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND WITH SEPARABLE (OR DEGENERATE) KERNEL. [Meerut 2000]

Consider a homogeneous Fredholm integral equation of the second kind :

$$y(x) = \lambda \int_a^b K(x, t) y(t) dt. \quad \dots (1)$$

Since kernel $K(x, t)$ is separable, we take

$$K(x, t) = \sum_{i=1}^n f_i(x) g_i(t). \quad \dots (2)$$

Using (2), (1) reduces to

$$y(x) = \lambda \int_a^b \left[\sum_{i=1}^n f_i(x) g_i(t) \right] y(t) dt \quad \text{or} \quad y(x) = \lambda \sum_{i=1}^n f_i(x) \int_a^b g_i(t) y(t) dt. \quad \dots (3)$$

[Interchanging the order of summation and integration]

$$\text{Let} \quad \int_a^b g_i(t) y(t) dt = C_i, \quad \text{where } i = 1, 2, \dots, n. \quad \dots (4)$$

$$\text{Using (4), (3) reduces to} \quad y(x) = \lambda \sum_{i=1}^n C_i f_i(x), \quad \dots (5)$$

where constants C_i ($i = 1, 2, \dots, n$) are to be determined in order to find solution of (1) in the form given by (5).

We now proceed to evaluate C_i 's as follows :

Multiplying both sides of (5) successively by $g_1(x)$, $g_2(x)$, ..., $g_n(x)$ and integrating over the interval (a, b) , we have

$$\int_a^b g_1(x) y(x) dx = \lambda \sum_{i=1}^n C_i \int_a^b g_1(x) f_i(x) dx, \quad \dots (A_1)$$

$$\int_a^b g_2(x) y(x) dx = \lambda \sum_{i=1}^n C_i \int_a^b g_2(x) f_i(x) dx, \quad \dots (A_2)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\text{and} \quad \int_a^b g_n(x) y(x) dx = \lambda \sum_{i=1}^n C_i \int_a^b g_n(x) f_i(x) dx. \quad \dots (A_n)$$

$$\text{Let} \quad \alpha_{ji} = \int_a^b g_j(x) f_i(x) dx, \quad \text{where } i, j = 1, 2, \dots, n. \quad \dots (6)$$

Using (4) and (6), (A_1) reduces to

$$C_1 = \lambda \sum_{i=1}^n C_i \alpha_{1i} \quad \text{or} \quad C_1 = \lambda [C_1 \alpha_{11} + C_2 \alpha_{12} + \dots + C_n \alpha_{1n}]$$

$$\text{or} \quad (1 - \lambda \alpha_{11}) C_1 - \lambda \alpha_{12} C_2 - \dots - \lambda \alpha_{1n} C_n = 0.$$

Similarly, we may simplify (A_2) , ..., (A_n) . Thus, we obtain the following system of homogeneous linear equations to determine $C_1, C_2, \dots, C_n$.

$$(1 - \lambda \alpha_{11}) C_1 - \lambda \alpha_{12} C_2 - \dots - \lambda \alpha_{1n} C_n = 0 \quad \dots (B_1)$$

$$-\lambda \alpha_{21} C_1 + (1 - \lambda \alpha_{22}) C_2 - \dots - \lambda \alpha_{2n} C_n = 0 \quad \dots (B_2)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$-\lambda \alpha_{n1} C_1 - \lambda \alpha_{n2} C_2 - \dots + (1 - \lambda \alpha_{nn}) C_n = 0. \quad \dots (B_n)$$

The determinant $D(\lambda)$ of this system is

$$D(\lambda) = \begin{vmatrix} 1 - \lambda\alpha_{11} & -\lambda\alpha_{12} & \dots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & 1 - \lambda\alpha_{22} & \dots & -\lambda\alpha_{2n} \\ \vdots & \vdots & \dots & \vdots \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \dots & 1 - \lambda\alpha_{nn} \end{vmatrix}. \quad \dots (7)$$

If $D(\lambda) \neq 0$, the system of equations $(B_1), (B_2), \dots, (B_n)$ has only trivial solution $C_1 = C_2 = \dots = C_n = 0$ and hence from (5) we notice that (1) has only zero or trivial solution $y(x) = 0$. However, if $D(\lambda) = 0$, at least one of the C_i 's can be assigned arbitrarily, and the remaining C_i 's can be determined accordingly. Hence when $D(\lambda) = 0$, infinitely many solutions of the integral equation (1) exist.

Those values of λ for which $D(\lambda) = 0$ are called the eigenvalues, and any non-trivial solution of (1) is called a corresponding eigenfunction of (1).

The eigenvalues of (1) are given by $D(\lambda) = 0$, i.e.,

$$\begin{vmatrix} 1 - \lambda\alpha_{11} & -\lambda\alpha_{12} & \dots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & 1 - \lambda\alpha_{22} & \dots & -\lambda\alpha_{2n} \\ \dots & \dots & \dots & \dots \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \dots & 1 - \lambda\alpha_{nn} \end{vmatrix} = 0. \quad \dots (8)$$

So the degree of equation (8) in λ is $m \leq n$. It follows that if integral equation (1) has separable kernel given by (2), then (1) has at the most n eigenvalues.

3.3. SOLVED EXAMPLES BASED ON ART. 3.1 AND 3.2

Ex. 1. Solve the homogeneous Fredholm equation

$$y(x) = \lambda \int_0^1 e^x e^t y(t) dt \quad [\text{Kanpur 2005, 2007, 10, 11}]$$

OR

Find the eigenvalues and eigenfunctions of the homogeneous integral equation.

$$y(x) = \lambda \int_0^1 e^x e^t y(t) dt$$

Sol. Given $y(x) = \lambda \int_0^1 e^x e^t y(t) dt$ or $y(x) = \lambda e^x \int_0^1 e^t y(t) dt. \quad \dots (1)$

Let $c = \int_0^1 e^t y(t) dt. \quad \dots (2)$

Then (1) reduces to $y(x) = \lambda c e^x. \quad \dots (3)$

From (3), $y(t) = \lambda c e^t. \quad \dots (4)$

Using (4), (2) becomes $c = \int_0^1 e^t (\lambda c e^t) dt$

$$\text{or} \quad c = \lambda c \left[\frac{e^{2t}}{2} \right]_0^1 = \frac{\lambda c}{2} (e^2 - 1). \quad \text{or} \quad c \left[1 - \frac{\lambda}{2} (e^2 - 1) \right] = 0. \quad \dots (5)$$

If $c = 0$ then (4) gives $y(x) = 0$. We, therefore, assume that for non-zero solution of (1), $c \neq 0$. Then (5) gives

$$1 - (\lambda/2)(e^2 - 1) = 0 \quad \text{or} \quad \lambda = 2/(e^2 - 1), \quad \dots (6)$$

which is an eigenvalue of (1).

Putting the value of λ given by (6) in (3), the corresponding eigenfunction is given by

$$y(x) = \left\{ 2c/(e^2 - 1) \right\} e^x$$

Hence, corresponding to eigenvalue $2/(e^2 - 1)$ there corresponds the eigenfunction e^x .

Remark. While writing eigenfunction the constant $2c/(e^2 - 1)$ is taken as unity.

Ex. 2. Show that the homogeneous integral equation $y(x) - \lambda \int_0^1 (3x - 2)t y(t) dt = 0$ has no characteristic numbers and eigenfunctions. (Kanpur 2010, Meerut 2011)

$$\text{Sol. Given} \quad y(x) = \lambda \int_0^1 (3x - 2)t y(t) dt \quad \text{or} \quad y(x) = \lambda (3x - 2) \int_0^1 t y(t) dt. \quad \dots (1)$$

$$\text{Let} \quad C = \int_0^1 t y(t) dt. \quad \dots (2)$$

$$\text{Then (1) reduces to} \quad y(x) = \lambda C (3x - 2). \quad \dots (3)$$

$$\text{From (3),} \quad y(t) = \lambda C (3t - 2). \quad \dots (4)$$

Using (4), (2) becomes

$$C = \int_0^1 \lambda C t (3t - 2) dt \quad \text{or} \quad C = \lambda C \left[t^3 - t^2 \right]_0^1 \quad \text{or} \quad C = 0.$$

$\therefore$ From (3) $y(x) \equiv 0$, which is zero solution of (1). Hence for any λ , (1) has only zero solution $y(x) \equiv 0$. Therefore, (1) does not possess any characteristic number or eigenfunction.

Remark. Note that the kernel $K(x, t) = (3x - 2)t$ of the above example is not symmetric. Thus we have shown that a kernel which is not symmetric does not necessarily have a characteristic constant. On the other hand, it will be shown in Art. 7.2 of Chapter 7 that a Fredholm homogeneous integral equation with symmetric kernel possesses at least one characteristic constant.

Ex. 3. Find the eigenvalues and the corresponding eigenfunctions of the homogeneous integral equation $y(x) = \lambda \int_0^1 \sin \pi x \cos \pi t y(t) dt$. (Kanpur 2009)

$$\text{Sol. Given} \quad y(x) = \lambda \int_0^1 \sin \pi x \cos \pi t y(t) dt \quad \text{or} \quad y(x) = \lambda \sin \pi x \int_0^1 \cos \pi t y(t) dt. \quad \dots (1)$$

$$\text{Let} \quad C = \int_0^1 \cos \pi t y(t) dt. \quad \dots (2)$$

$$\text{Then (1) reduces to} \quad y(x) = C \lambda \sin \pi x. \quad \dots (3)$$

$$\text{From (3),} \quad y(t) = C \lambda \sin \pi t. \quad \dots (4)$$

Using (4), (2) becomes

$$C = \int_0^1 \cos \pi t (\lambda c \sin \pi t) dt \quad \text{or} \quad C = \frac{\lambda C}{2} \int_0^1 \sin 2\pi t dt$$

or
$$C = \frac{\lambda C}{2} \left[-\frac{\cos 2\pi t}{2\pi} \right]_0^1 = \frac{\lambda C}{2} \left[-\frac{1}{2\pi} + \frac{1}{2\pi} \right].$$

Hence $C = 0$ and so from (3), $y(x) \equiv 0$. Thus for any λ , (1) has only zero solution $y(x) \equiv 0$. Therefore, (1) does not possess any characteristic number or eigenfunction.

Ex. 4. (a) The values of λ for which the integral equation $y(x) = \lambda \int_0^1 (6x-t)y(t) dt$ has a non-trivial solution, are given by the roots of the equations :

$$\begin{array}{ll} (a) (3\lambda - 1)(2 + \lambda) - \lambda^2 = 0 & (b) (3\lambda - 1)(2 + \lambda) + 2 = 0 \\ (c) (3\lambda - 1)(2 + \lambda) - 4\lambda^2 = 0 & (d) (3\lambda - 1)(2 + \lambda) + \lambda^3 = 0 \end{array} \quad [\text{GATE 2004}]$$

Sol. Ans (c). Given
$$y(x) = 6\lambda x \int_0^1 y(t) dt - \lambda \int_0^1 t y(t) dt \quad \dots(1)$$

Let
$$C_1 = \int_0^1 y(t) dt \quad \dots(2)$$

and
$$C_2 = \int_0^1 t y(t) dt \quad \dots(3)$$

Then (1) yields
$$y(x) = 6\lambda x C_1 - \lambda C_2 \quad \dots(4)$$

From (4),
$$y(t) = 6\lambda t C_1 - \lambda C_2 \quad \dots(5)$$

Using, (5), (2) becomes
$$C_1 = \int_0^1 (6\lambda t C_1 - \lambda C_2) dt$$

or
$$C_1 = [3\lambda C_1 t^2 - \lambda C_2 t]_0^1 \quad \text{or} \quad C_1 = 3\lambda C_1 - \lambda C_2$$

Thus,
$$(3\lambda - 1)C_1 - \lambda C_2 = 0 \quad \dots(6)$$

Using (5), (3) becomes
$$C_2 = \int_0^1 t(6\lambda t C_1 - \lambda C_2) dt$$

or
$$C_2 = [2\lambda C_1 t^3 - (1/2) \times \lambda C_2 t^2]_0^1 \quad \text{or} \quad C_2 = 2\lambda C_1 - (1/2) \times \lambda C_2$$

Thus,
$$-4\lambda C_1 + (2 + \lambda)C_2 = 0 \quad \dots(7)$$

Now, we have a system of homogeneous linear equations (6) and (7) to determine C_1 and C_2 . For non-trivial solution of the given integral equation, the system (6) – (7) must possess non-trivial solution and so we must have

$$\begin{vmatrix} 3\lambda - 1 & -\lambda \\ -4\lambda & 2 + \lambda \end{vmatrix} = 0 \quad \text{or} \quad (3\lambda - 1)(2 + \lambda) - 4\lambda^2 = 0,$$

which gives the desired values of λ for the required non-trivial solution of the given integral equation.

Ex. 4. (b) Find the eigenvalues and the corresponding eigenfunctions of the integral equation

$$y(x) = \lambda \int_0^1 (2xt - 4x^2) y(t) dt. \quad [\text{MEERUT 2007}]$$

Sol. Given
$$y(x) = \lambda \int_0^1 (2xt - 4x^2) y(t) dt$$

or
$$y(x) = 2\lambda x \int_0^1 t y(t) dt - 4\lambda x^2 \int_0^1 y(t) dt. \quad \dots (1)$$

Let
$$C_1 = \int_0^1 t y(t) dt \quad \dots (2)$$

and
$$C_2 = \int_0^1 y(t) dt. \quad \dots (3)$$

Then (1) reduces to
$$y(x) = 2\lambda C_1 x - 4\lambda C_2 x^2. \quad \dots (4)$$

From (4),
$$y(t) = 2\lambda C_1 t - 4\lambda C_2 t^2. \quad \dots (5)$$

Using (5), (2) becomes

$$C_1 = \int_0^1 t(2\lambda C_1 t - 4\lambda C_2 t^2) dt \quad \text{or} \quad C_1 [1 - 2\lambda \int_0^1 t^2 dt] + 4\lambda C_2 \int_0^1 t^3 dt = 0$$

or
$$C_1 (1 - 2\lambda/3) + \lambda C_2 = 0. \quad \dots (6)$$

Again, using (5), (3) becomes

$$C_2 = \int_0^1 (2\lambda C_1 t - 4\lambda C_2 t^2) dt \quad \text{or} \quad 2\lambda C_1 \int_0^1 dt - C_2 [1 + 4\lambda \int_0^1 t^2 dt] = 0$$

or
$$\lambda C_1 - C_2 (1 + 4\lambda/3) = 0. \quad \dots (7)$$

Thus, we have a system of homogeneous linear equations (6) and (7) for determining C_1 and C_2 . For non-zero solution of this system of equations, we must have

$$\begin{vmatrix} (1 - 2\lambda/3) & \lambda \\ \lambda & -(1 + 4\lambda/3) \end{vmatrix} = 0 \quad \text{or} \quad -\left(1 - \frac{2\lambda}{3}\right)\left(1 + \frac{4\lambda}{3}\right) - \lambda^2 = 0$$

or
$$\lambda^2 + 6\lambda + 9 = 0 \quad \text{so that} \quad \lambda = -3, -3$$

Hence the eigenvalues are $\lambda_1 = -3, \lambda_2 = -3$.

To determine eigenfunction corresponding to $\lambda = \lambda_1 = -3$

Putting $\lambda = \lambda_1 = -3$ in (6) and (7), we get

$$3C_1 - 3C_2 = 0 \quad \dots (8)$$

and
$$-3C_1 + 3C_2 = 0 \quad \dots (9)$$

(8) or (9) give $C_1 = C_2$. Hence from (4), we have

$$y(x) = 2C_1\lambda_1 (x - 2x^2) = -6C_1 (x - 2x^2).$$

Taking $-6C_1 = 1$, the eigenfunction is $(x - 2x^2)$.

Hence eigenfunction corresponding to eigenvalue $\lambda_1 = \lambda_2 = -3$ is $x - 2x^2$.

Ex. 5. (a) Solve the homogeneous Fredholm integral equation of the second kind :

$$y(x) = \lambda \int_0^{2\pi} \sin(x+t) y(t) dt$$

Sol. Given
$$y(x) = \lambda \int_0^{2\pi} \sin(x+t) y(t) dt$$

or
$$y(x) = \lambda \int_0^{2\pi} (\sin x \cos t + \cos x \sin t) y(t) dt$$

or
$$y(x) = \lambda \sin x \int_0^{2\pi} \cos t y(t) dt + \lambda \cos x \int_0^{2\pi} \sin t y(t) dt. \quad \dots (1)$$

Let
$$C_1 = \int_0^{2\pi} \cos t y(t) dt \quad \dots (2)$$

and
$$C_2 = \int_0^{2\pi} \sin t y(t) dt. \quad \dots (3)$$

Then (1) reduces to
$$y(x) = \lambda C_1 \sin x + \lambda C_2 \cos x. \quad \dots (4)$$

From (4),
$$y(t) = \lambda C_1 \sin t + \lambda C_2 \cos t. \quad \dots (5)$$

Using (5), (2) becomes
$$C_1 = \int_0^{2\pi} \cos t (\lambda C_1 \sin t + \lambda C_2 \cos t) dt$$

or
$$C_1 = \frac{\lambda C_1}{2} \int_0^{2\pi} \sin 2t dt + \frac{\lambda C_2}{2} \int_0^{2\pi} (1 + \cos 2t) dt \quad \text{or} \quad C_1 = \frac{\lambda C_1}{2} \left[-\frac{\cos 2t}{2} \right]_0^{2\pi} + \frac{\lambda C_2}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi}$$

or
$$C_1 = 0 + \lambda C_2 \pi \quad \text{or} \quad C_1 - \lambda \pi C_2 = 0. \quad \dots (6)$$

Using (5), (3) becomes
$$C_2 = \int_0^{2\pi} \sin t (\lambda C_1 \sin t + \lambda C_2 \cos t) dt$$

or
$$C_2 = \frac{\lambda C_1}{2} \int_0^{2\pi} (1 - \cos 2t) dt + \frac{\lambda C_2}{2} \int_0^{2\pi} \sin 2t dt \quad \text{or} \quad C_2 = \frac{\lambda C_1}{2} \left[t - \frac{\sin 2t}{2} \right]_0^{2\pi} + \frac{\lambda C_2}{2} \left[-\frac{\cos 2t}{2} \right]_0^{2\pi}$$

or
$$C_2 = \lambda C_1 \pi \quad \text{or} \quad \pi \lambda C_1 - C_2 = 0. \quad \dots (7)$$

Thus, we have a system of homogeneous linear equations (6) and (7) for determining C_1 and C_2 . For non-zero solution of this system of equations, we must have

$$\begin{vmatrix} 1 & -\lambda \pi \\ \lambda \pi & -1 \end{vmatrix} = 0 \quad \text{or} \quad -1 + \lambda^2 \pi^2 = 0 \quad \text{so that} \quad \lambda = \pm \frac{1}{\pi}.$$

Hence the eigenvalues are given by $\lambda_1 = 1/\pi$ and $\lambda_2 = -1/\pi. \quad \dots (8)$

To determine eigenfunction corresponding to $\lambda = \lambda_1 = 1/\pi$.

Putting $\lambda = \lambda_1 = 1/\pi$ in (6) and (7), we get

$$C_1 - C_2 = 0 \quad \dots (9)$$

and
$$C_1 - C_2 = 0 \quad \dots (10)$$

Both (9) and (10) give $C_2 = C_1$. Hence from (4), we have

$$y(x) = (1/\pi) \times C_1 \sin x + (1/\pi) \times C_1 \cos x \quad \text{or} \quad y(x) = (C_1/\pi) \times (\sin x + \cos x).$$

Taking $(C_1 / \pi) = 1$, the required eigenfunction $y_1(x)$ is given by

$$y_1(x) = \sin x + \cos x. \quad \dots (11)$$

To determine eigenfunction corresponding to $\lambda = \lambda_2 = -1/\pi$.

Putting $\lambda = \lambda_2 = -1/\pi$ in (6) and (7), we get

$$C_1 + C_2 = 0 \quad \dots (12)$$

and

$$C_1 + C_2 = 0. \quad \dots (13)$$

Both (12) and (13) give $C_2 = -C_1$. Hence from (4), we have

$$y(x) = (-1/\pi) \times C_1 \sin x + (-1/\pi) \times (-C_1) \cos x \quad \text{or} \quad y(x) = (-C/\pi) \times (\sin x - \cos x).$$

Taking $(-C_1 / \pi) = 1$, the required eigenfunction $y_2(x)$ is given by

$$y_2(x) = \sin x - \cos x \quad \dots (14)$$

From (8), (11) and (14), the required eigenvalues and eigenfunctions are given by

$$\lambda_1 = 1/\pi, \quad y_1(x) = \sin x + \cos x \quad \text{and} \quad \lambda_2 = -1/\pi, \quad y_2(x) = \sin x - \cos x.$$

Ex.5. (b) The eigenvalues of the integral equation $y(x) = \lambda \int_0^{2\pi} \sin(x+t) y(t) dt$ are

$$(a) 1/2\pi, -1/2\pi \quad (b) 1/\pi, -1/\pi \quad (c) \pi, -\pi \quad (d) 2\pi, -2\pi \quad [GATE 2005]$$

Sol. Ans. (b). Refer Ex. 5(a)

Ex.5. (c) The integral equation $y(x) = \lambda \int_0^{2\pi} \sin(x+t) y(t) dt$ has

(a) two solutions for any value of λ

(b) unique solution for every value of λ

(c) infinitely many solutions for only one values of λ

(d) infinitely many solutions for two values of λ [GATE 2003]

Sol. (d). Refer Ex. 5(a).

Ex. 6. Find the eigenvalues and eigenfunctions of the homogeneous integral equation

$$y(x) = \lambda \int_0^\pi (\cos^2 x \cos 2t + \cos 3x \cos^3 t) y(t) dt.$$

[Meerut 2000, 01, 03, 08, 09, 10; Kanpur 2005, 06]

Sol. Given
$$y(x) = \lambda \int_0^\pi (\cos^2 x \cos 2t + \cos 3x \cos^3 t) y(t) dt$$

or
$$y(x) = \lambda \cos^2 x \int_0^\pi \cos 2t y(t) dt + \lambda \cos 3x \int_0^\pi \cos^3 t y(t) dt \quad \dots (1)$$

Let
$$C_1 = \int_0^\pi \cos 2t y(t) dt \quad \dots (2)$$

and
$$C_2 = \int_0^\pi \cos^3 t y(t) dt. \quad \dots (3)$$

Then (1) reduces to
$$y(x) = \lambda C_1 \cos^2 x + \lambda C_2 \cos 3x. \quad \dots (4)$$

From (4), $y(t) = \lambda C_1 \cos^2 t + \lambda C_2 \cos 3t.$... (5)

Using (5), (2) becomes $C_1 = \int_0^\pi \cos 2t (\lambda C_1 \cos^2 t + \lambda C_2 \cos 3t) dt.$

or $C_1 [1 - \lambda \int_0^\pi \cos 2t \cos^2 t dt] + \lambda C_2 \int_0^\pi \cos 2t \cos 3t dt = 0.$... (6)

Now, $\int_0^\pi \cos 2t \cos^2 t dt = \int_0^\pi \cos 2t \frac{(1 + \cos 2t)}{2} dt = \frac{1}{2} \int_0^\pi \cos 2t dt + \frac{1}{2} \int_0^\pi \cos^2 2t dt$
 $= \frac{1}{2} \left[\frac{\sin 2t}{2} \right]_0^\pi + \frac{1}{2} \int_0^\pi \left(\frac{1 + \cos 4t}{2} \right) dt = 0 + \frac{1}{4} \left[t + \frac{\sin 4t}{4} \right]_0^\pi = \frac{\pi}{4}.$... (7)

Again, $\int_0^\pi \cos 2t \cos 3t dt = \frac{1}{2} \int_0^\pi [\cos 5t + \cos t] dt = \frac{1}{2} \left[\frac{\sin 5t}{5} + \sin t \right]_0^\pi = 0.$... (8)

Using (7) and (8), (6) reduces to $C_1 (1 - \lambda\pi/4) + 0 \cdot C_2 = 0$... (9)

Again, using (5), (3) becomes $C_2 = \int_0^\pi \cos^3 t (\lambda C_1 \cos^2 t + \lambda C_2 \cos 3t) dt$

or $\lambda C_1 \int_0^\pi \cos^5 t dt + C_2 [\lambda \int_0^\pi \cos^3 t \cos 3t dt - 1] = 0.$... (10)

Now, $\int_0^\pi \cos^5 t dt = 0,$ as $\cos^5(\pi - t) = -\cos^5 t$... (11)

and $\int_0^\pi \cos^3 t \cos 3t dt = \frac{1}{4} \int_0^\pi \cos 3t (\cos 3t + 3 \cos t) dt$
 $[\because \cos 3t = 4 \cos^3 t - 3 \cos t \Rightarrow \cos^3 t = (1/4) \times (\cos 3t + 3 \cos t)]$
 $= \frac{1}{4} \int_0^\pi \cos^2 3t dt + \frac{3}{4} \int_0^\pi \cos 3t \cos t dt$
 $= \frac{1}{4} \int_0^\pi \frac{1 + \cos 6t}{2} dt + \frac{3}{4} \int_0^\pi \frac{\cos 4t + \cos t}{2} dt = \frac{1}{8} \left[t + \frac{\sin 6t}{6} \right]_0^\pi + \frac{3}{8} \left[\frac{\sin 4t}{4} + \sin t \right]_0^\pi = \frac{\pi}{8}.$
 $\therefore \int_0^\pi \cos^3 t \cos 3t dt = \frac{\pi}{8}.$... (12)

Using (11) and (12), (10) reduces to

$0 \cdot C_1 + C_2 (\lambda\pi/8 - 1) = 0$ or $0 \cdot C_1 + C_2 (1 - \lambda\pi/8) = 0$... (13)

For non-zero solution of the system of equations (9) and (13), we must have

$\begin{vmatrix} 1 - (\lambda\pi/4) & 0 \\ 0 & 1 - (\lambda\pi/8) \end{vmatrix} = 0$ or $\left(1 - \frac{\lambda\pi}{4}\right) \left(1 - \frac{\lambda\pi}{8}\right) = 0$
 $\therefore \lambda = 4/\pi$ or $\lambda = 8/\pi.$

Hence the eigenvalues of (1) are $\lambda_1 = 4/\pi$ and $\lambda_2 = 8/\pi.$... (14)

Determination of eigenfunction corresponding to the eigenvalue $\lambda = \lambda_1 = 4/\pi.$

Putting $\lambda = \lambda_1 = 4/\pi$ in (9) and (13), we have

$$0 \cdot C_1 + 0 \cdot C_2 = 0 \quad \dots (15)$$

and

$$0 \cdot C_1 + (1/2) \times C_2 = 0 \quad \dots (16)$$

Solving (15) and (16), $C_2 = 0$ and C_1 is arbitrary. Putting these values in (4), we get

$$y(x) = \lambda C_1 \cos^2 x = (4/\pi) \times C_1 \cos^2 x.$$

Setting $(4/\pi) \times C_1 = 1$, the eigenfunction = $y_1(x) = \cos^2 x$.

Determination of eigenfunction corresponding to the eigenvalue $\lambda = \lambda_2 = 8/\pi$.

Putting $\lambda = \lambda_2 = 8/\pi$ in (9) and (13), we have

$$-C_1 + 0 \cdot C_2 = 0 \quad \dots (17)$$

and

$$0 \cdot C_1 + 0 \cdot C_2 = 0 \quad \dots (18)$$

Solving (17) and (18), $C_1 = 0$ and C_2 is arbitrary. Putting these values in (4), we get

$$y(x) = \lambda C_2 \cos 3x = (8/\pi) \times C_2 \cos 3x$$

Setting $(8/\pi) \times C_2 = 1$, the eigenfunction $y_2(x) = \cos 3x$.

Hence $y_1(x) = \cos^2 x$ and $y_2(x) = \cos 3x$ are the required eigenfunctions corresponding to the eigenvalues $\lambda_1 = 4/\pi$ and $\lambda_2 = 8/\pi$ respectively.

Ex. 7. Find the eigenvalues and eigenfunctions of the homogeneous integral equation

$$y(x) = \lambda \int_1^2 \left(xt + \frac{1}{xt} \right) y(t) dt. \quad [\text{Merrut 2006}]$$

Sol. Given

$$y(x) = \lambda \int_1^2 \left(xt + \frac{1}{xt} \right) y(t) dt$$

or

$$y(x) = \lambda x \int_1^2 t g(t) dt + \frac{\lambda}{x} \int_1^2 \frac{1}{t} g(t) dt. \quad \dots (1)$$

Let

$$C_1 = \int_1^2 t y(t) dt \quad \dots (2)$$

and

$$C_2 = \int_1^2 \frac{1}{t} y(t) dt. \quad \dots (3)$$

Then (1) reduces to

$$y(x) = \lambda C_1 x + (\lambda C_2 / x) \quad \dots (4)$$

From (4),

$$y(t) = \lambda C_1 t + (\lambda C_2 / t) \quad \dots (5)$$

Using (5), (2) becomes

$$C_1 = \int_1^2 t \left(\lambda C_1 t + \frac{\lambda C_2}{t} \right) dt = \lambda C_1 \left[\frac{t^3}{3} \right]_1^2 + \lambda C_2 [t]_1^2 = \lambda C_1 \left(\frac{8}{3} - \frac{1}{3} \right) + \lambda C_2 (2-1)$$

or

$$(1 - 7\lambda/3) C_1 - \lambda C_2 = 0. \quad \dots (6)$$

Using (5), (3) becomes

$$C_2 = \int_1^2 \frac{1}{t} \left(\lambda C_1 t + \frac{\lambda C_2}{t} \right) dt = \lambda C_1 [t]_1^2 + \lambda C_2 \left[\frac{t^{-1}}{-1} \right]_1^2 = \lambda C_1 (2-1) + \lambda C_2 \left(-\frac{1}{2} + 1 \right)$$

or
$$-\lambda C_1 + (1 - \lambda/2) \times C_2 = 0. \quad \dots (7)$$

For non-zero solution of the system of equations (6) and (7), we must have

$$\begin{vmatrix} 1 - (7\lambda/3) & -\lambda \\ -\lambda & 1 - (\lambda/2) \end{vmatrix} = 0$$

or
$$\left(1 - \frac{7}{3}\lambda\right)\left(1 - \frac{1}{2}\lambda\right) - \lambda^2 = 0 \quad \text{or} \quad 1 - \frac{17}{6}\lambda + \frac{1}{6}\lambda^2 = 0$$

or
$$\lambda^2 - 17\lambda + 6 = 0 \quad \text{so that} \quad \lambda = [17 \pm \{(17)^2 - 24\}^{1/2}] / 2 = (17 \pm \sqrt{265}) / 2$$

Hence the required eigenvalues are

$$\lambda_1 = (17 + \sqrt{265}) / 2 \approx 16.6394, \quad \lambda_2 = (17 - \sqrt{265}) / 2 \approx 0.3606. \quad \dots (8)$$

Here the symbol $\approx$ stands for approximately

Determination of eigenfunction corresponding to the eigenvalue $\lambda = \lambda_1 = 16.6394$.

Putting $\lambda = \lambda_1 = 16.6394$ in (6) and (7), we get

$$[1 - (7/3) \times (16.6394)] C_1 - 16.6394 C_2 = 0 \quad \dots (9)$$

and
$$-16.6394 C_1 + [1 - (1/2) \times (16.6394)] C_2 = 0. \quad \dots (10)$$

Both (9) and (10) reduce to
$$C_2 \approx -2.2732 C_1. \quad \dots (11)$$

Using (11) in (4), the eigenfunction $y_1(x)$ corresponding to eigenvalue $\lambda = \lambda_1 = 16.6394$ is given by

$$y_1(x) \approx \lambda_1 C_1 x + (\lambda_1 / x) \times (-2.2732 C_1) \approx \lambda_1 C_1 [x - 2.2732 \times (1/x)]$$

or
$$y_1(x) \approx [x - 2.2732 \times (1/x)]. \quad [\text{Taking } \lambda_1 C_1 = 16.6394 C_1 = 1]$$

Determination of eigenfunction corresponding to the eigenvalue $\lambda = \lambda_2 = 0.3606$.

Putting $\lambda = \lambda_2 = 0.3606$ in (6) and (7), we get

$$[1 - (7/3) \times (0.3606)] C_1 - 0.3606 C_2 = 0 \quad \dots (12)$$

and
$$-0.3606 C_1 + [1 - (1/2) \times (0.3606)] C_2 = 0. \quad \dots (13)$$

Both (12) and (13) reduce to
$$C_2 \approx 0.4399 C_1. \quad \dots (14)$$

Using (14) in (4), the eigenfunction $y_2(x)$ corresponding to eigenvalue $\lambda = \lambda_2 = 0.3606$ is given by

$$y_2(x) \approx \lambda_2 C_1 x + (\lambda_2 / x) \times (0.4399 C_1) \quad \text{or} \quad y_2(x) \approx \lambda C_1 [x + 0.4399 \times (1/x)]$$

or
$$y_2(x) \approx [x + 0.4399 \times (1/x)]. \quad [\text{Taking } \lambda_2 C_1 = 0.4399 C_1 = 1]$$

Thus eigenvalues are $\lambda_1 = 16.6394$ and $\lambda_2 = 0.3606$ and the corresponding eigenfunctions are
$$y_1(x) = [x - 2.2732 \times (1/x)] \quad \text{and} \quad y_2(x) = [x + 0.4399 \times (1/x)].$$

Ex. 8. Show that the homogeneous integral equation
$$y(x) = \lambda \int_0^1 (t\sqrt{x} - x\sqrt{t}) y(t)$$

does not have real eigenvalues and eigenfunctions.

Sol. Given
$$y(x) = \lambda \int_0^1 (t\sqrt{x} - x\sqrt{t}) y(t) dt$$

or
$$y(x) = \lambda \sqrt{x} \int_0^1 t y(t) dt - \lambda x \int_0^1 \sqrt{t} y(t) dt. \quad \dots (1)$$

Let
$$C_1 = \int_0^1 t y(t) dt \quad \dots (2)$$

and
$$C_2 = \int_0^1 \sqrt{t} y(t) dt. \quad \dots (3)$$

Then (1) reduces to
$$y(x) = \lambda C_1 \sqrt{x} - \lambda C_2 x. \quad \dots (4)$$

From (4),
$$y(t) = \lambda C_1 \sqrt{t} - \lambda C_2 t. \quad \dots (5)$$

Using (5), (2) becomes

$$C_1 = \int_0^1 t (\lambda C_1 \sqrt{t} - \lambda C_2 t) dt \quad \text{or} \quad C_1 = \lambda C_1 \left[\frac{t^{5/2}}{5/2} \right]_0^1 - \lambda C_2 \left[\frac{t^3}{3} \right]_0^1$$

or
$$\left(1 - \frac{2\lambda}{5}\right) C_1 + \frac{\lambda}{3} C_2 = 0. \quad \dots (6)$$

Using (5), (3) becomes
$$C_2 = \int_0^1 \sqrt{t} (\lambda C_1 \sqrt{t} - \lambda C_2 t) dt$$

or
$$C_2 = \lambda C_1 \left[\frac{t^2}{2} \right]_0^1 - \lambda C_2 \left[\frac{t^{5/2}}{5/2} \right]_0^1 \quad \text{or} \quad -\frac{\lambda}{2} C_1 + \left(1 + \frac{2\lambda}{5}\right) C_2 = 0 \quad \dots (7)$$

For non-zero solution of the system of equation (6) and (7), we must have

$$D(\lambda) = \begin{vmatrix} 1 - (2\lambda/5) & \lambda/3 \\ -\lambda/2 & 1 + (2\lambda/5) \end{vmatrix} = 0$$

or
$$\left(1 - \frac{2}{5}\lambda\right)\left(1 + \frac{2}{5}\lambda\right) + \frac{\lambda^2}{6} = 0 \quad \text{or} \quad 1 + \frac{\lambda^2}{150} = 0$$

or
$$\lambda^2 + 150 = 0 \quad \text{so that} \quad \lambda = \pm i \sqrt{150},$$

showing that $D(\lambda) \neq 0$ for any real value of λ . Hence the system of equations (6) and (7) has unique solution $C_1 = C_2 = 0$ for all real λ . Hence, from (4), $y(x) = 0$, which is zero solution. Hence, the given equation does not have real eigenvalues and eigenfunctions.

Ex. 9. Find the eigenvalues and eigenfunctions of the homogeneous integral equation

$$y(x) = \lambda \int_{-1}^1 (5x t^3 + 4x^2 t + 3x t) y(t) dt. \quad (\text{Meerut 2012})$$

Sol. Given
$$y(x) = \lambda \int_{-1}^1 (5x t^3 + 4x^2 t + 3x t) y(t) dt.$$

or
$$y(x) = 5\lambda x \int_{-1}^1 t^3 y(t) dt + \lambda (4x^2 + 3x) \int_{-1}^1 t y(t) dt. \quad \dots (1)$$

Let
$$C_1 = \int_{-1}^1 t^3 y(t) dt, \quad \dots (2)$$

and
$$C_2 = \int_{-1}^1 t y(t) dt. \quad \dots (3)$$

Then (1) reduces to
$$y(x) = 5\lambda C_1 x + \lambda C_2 (4x^2 + 3x). \quad \dots (4)$$

From (4),
$$y(t) = 5\lambda C_1 t + \lambda C_2 (4t^2 + 3t). \quad \dots (5)$$

Using (5), (2) becomes

$$C_1 = \int_{-1}^1 t^3 [5\lambda C_1 t + \lambda C_2 (4t^2 + 3t)] dt = 5\lambda C_1 \left[\frac{t^5}{5} \right]_{-1}^1 + \lambda C_2 \left[4 \left(\frac{t^6}{6} \right) + 3 \left(\frac{t^5}{5} \right) \right]_{-1}^1$$

or
$$C_1 = 2\lambda C_1 + (6\lambda/5) \times C_2 \quad \text{or} \quad C_1 (1 - 2\lambda) - (6\lambda/5) \times C_2 = 0. \quad \dots (6)$$

Using (5), (3) becomes

$$C_2 = \int_{-1}^1 t [5\lambda C_1 t + \lambda C_2 (4t^2 + 3t)] dt = 5\lambda C_1 \left[\frac{t^3}{3} \right]_{-1}^1 + \lambda C_2 \left[4 \left(\frac{t^4}{4} \right) + 3 \left(\frac{t^3}{3} \right) \right]_{-1}^1.$$

or
$$C_2 = (10\lambda/3) \times C_1 + 2\lambda C_2 \quad \text{or} \quad -(10\lambda/3) \times C_1 + C_2 (1 - 2\lambda) = 0. \quad \dots (7)$$

For non-zero solution of the system of equations (6) and (7), we must have

$$\begin{vmatrix} 1 - 2\lambda & -(6\lambda/5) \\ -(10\lambda/3) & 1 - 2\lambda \end{vmatrix} = 0$$

or
$$(1 - 2\lambda)^2 - 4\lambda^2 = 0 \quad \text{or} \quad 1 - 4\lambda = 0.$$

Hence the only eigenvalue is $\lambda = 1/4$. To determine the corresponding eigenfunction, we proceed as follows :

Putting $\lambda = 1/4$ in (6) and (7), we have

$$(1/2) \times C_1 - (3/10) \times C_2 = 0 \quad \dots (8)$$

and
$$-(5/6) \times C_1 + (1/2) \times C_2 = 0 \quad \dots (9)$$

Both (8) and (9) lead us to
$$C_1 = (3/5) \times C_2. \quad \dots (10)$$

Putting $\lambda = 1/4$ and using (10), (4) gives

$$y(x) = 5 \times \frac{1}{4} \times \left(\frac{3}{5} C_2 \right) x + \frac{1}{4} C_2 (4x^2 + 3x) \quad \text{or} \quad y(x) = C_2 \left(x^2 + \frac{3}{2} x \right)$$

Setting $C_2 = 1$,
$$y(x) = x^2 + (3/2) \times x.$$

Hence the required eigenvalue is $\lambda = 1/4$ and the corresponding eigenfunction is

$$y(x) = x^2 + (3/2) \times x.$$

Ex. 10. Find the eigenvalues and eigenfunctions of the homogeneous equation

$$y(x) = \lambda \int_0^\pi K(x, t) y(t) dt, \text{ where} \quad K(x, t) = \begin{cases} \cos x \sin t, & 0 \leq x \leq t \\ \cos t \sin x, & t \leq x \leq \pi. \end{cases}$$

[**Note :** Whenever homogeneous equation is given in the above special form of $K(x, t)$, we reduce the given integral equation into differential equation together with two boundary conditions. Then, we solve the resulting boundary value problem (*Strum-Liouville problem) to determine eigenvalue and eigenfunctions.]

Sol. Given
$$y(x) = \lambda \int_0^\pi K(x, t) y(t) dt. \quad \dots (1)$$

where
$$K(x, t) = \begin{cases} \cos x \sin t, & 0 \leq x \leq t \\ \cos t \sin x, & t \leq x \leq \pi. \end{cases} \quad \dots (2)$$

Re-writing (1), we have
$$y(x) = \lambda \int_0^x K(x, t) y(t) dt + \int_x^\pi K(x, t) y(t) dt$$

or
$$y(x) = \int_0^x (\lambda \cos t \sin x) y(t) dt + \int_x^\pi (\lambda \cos x \sin t) y(t) dt, \text{ using (2)} \quad \dots (3)$$

Differentiating both sides of (3) w.r.t. 'x', we get

$$y'(x) = \frac{d}{dx} \int_0^x (\lambda \cos t \sin x) y(t) dt + \frac{d}{dx} \int_x^\pi (\lambda \cos x \sin t) y(t) dt$$

or
$$y'(x) = \int_0^x \frac{\partial}{\partial x} \{ \lambda \cos t \sin x y(t) \} dt + \lambda \cos x \sin x y(x) \frac{dx}{dx} - \lambda \cos 0 \sin x y(0) \frac{d0}{dx} \\ + \int_x^\pi \frac{\partial}{\partial x} \{ \lambda \cos x \sin t y(t) \} dt + \lambda \cos x \sin \pi y(\pi) \frac{d\pi}{dx} - \lambda \cos x \sin x y(x) \frac{dx}{dx}$$

[using Leibnitz-Rule of differentiation under the sign of integration see (Art. 1.13)]

or
$$y'(x) = \int_0^x (\lambda \cos t \cos x) y(t) dt + \lambda \cos x \sin x y(x) \\ + \int_x^\pi (-\lambda \sin x \sin t) y(t) dx - \lambda \cos x \sin x y(x)$$

$\therefore y'(x) = \int_0^x (\lambda \cos t \cos x) y(t) dt - \int_x^\pi (\lambda \sin x \sin t) y(t) dt. \quad \dots (4)$

Differentiating both sides of (4) w.r.t. 'x', we get

$$y''(x) = \frac{d}{dx} \int_0^x (\lambda \cos t \cos x) y(t) dt - \frac{d}{dx} \int_x^\pi (\lambda \sin x \sin t) y(t) dt$$

or
$$y''(x) = \int_0^x \frac{\partial}{\partial x} \{ \lambda \cos t \cos x y(t) \} dt + \lambda \cos^2 x y(x) \frac{dx}{dx} - \lambda \cos 0 \cos x y(0) \frac{d0}{dx} \\ - \left[\int_x^\pi \frac{\partial}{\partial x} \{ \lambda \sin x \sin t y(t) \} dt + \lambda \sin x \sin \pi y(\pi) \frac{d\pi}{dx} - \lambda \sin^2 x y(x) \frac{dx}{dx} \right] \\ = - \int_0^x (\lambda \cos t \sin x) y(t) dt + \lambda \cos^2 x y(x) - \int_x^\pi (\lambda \cos x \sin t) y(t) dt + \lambda \sin^2 x y(x) \\ = \lambda y(x) - \left[\int_0^x (\lambda \cos t \sin x) y(t) dt + \int_x^\pi (\lambda \cos x \sin t) y(t) dt \right] \\ = \lambda y(x) - y(x), \text{ using (3)}$$

$\therefore y''(x) - (\lambda - 1) y(x) = 0. \quad \dots (5)$

* For more details of Strum-Liouville problem, refer author's part II of Ordinary and partial differential equations, published by S.Chand & Co., New Delhi.

Putting $x = \pi$ in (3), we get $y(\pi) = 0$ (6)

Again putting $x = 0$ in (4), we get $y'(0) = 0$ (7)

Now, we shall solve *Strum-Liouville problem* given by (5), (6) and (7) by the usual procedure to get eigenvalues and the corresponding eigenfunctions.

Three cases arise :

Case I. Let $\lambda - 1 = 0$ so that $\lambda = 1$. Then (5) reduces to $y''(x) = 0$ whose solution is

$$y(x) = Ax + B. \quad \dots (8)$$

From (8), $y'(x) = A$ (9)

Putting $x = \pi$ in (8) and using (6), we get $0 = A\pi + B$ (10)

Putting $x = 0$ in (9) and using (7), we get $0 = A$... (11)

Solving (10) and (11), $A = 0, B = 0$.

Hence (8) gives $y(x) = 0$, which is not an eigenfunction and so $\lambda = 1$ is not an eigenvalue.

Case II. Let $\lambda - 1 = \mu^2$, where $\mu \neq 0$. Then (5) reduces to $y''(x) - \mu^2 y(x) = 0$ whose solution is

$$y(x) = A e^{\mu x} + B e^{-\mu x}. \quad \dots (12)$$

From (12), $y'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$... (13)

Putting $x = \pi$ in (12) and using (6), we get $0 = A e^{\mu \pi} + B e^{-\mu \pi}$ (14)

Putting $x = 0$ in (13) and using (7), we get $0 = A\mu - B\mu$

or $0 = A - B$, as $\mu \neq 0$... (15)

Solving (14) and (15). $A = B = 0$.

Hence (8) gives $y(x) = 0$, which is not an eigenfunction and so $\lambda = \mu^2 + 1$ is not an eigen value.

Case III. Let $\lambda - 1 = -\mu^2$, where $\mu \neq 0$. Then (5) reduces to $y''(x) + \mu^2 y(x) = 0$ whose solution is

$$y(x) = A \cos \mu x + B \sin \mu x. \quad \dots (16)$$

From (16), $y'(x) = -A\mu \sin \mu x + B\mu \cos \mu x$ (17)

Putting $x = \pi$ in (16) and using (6), we get $0 = A \cos \mu \pi + B \sin \mu \pi$ (18)

Again, putting $x = 0$ in (17) and using (7), we get

$$0 = B\mu \quad \text{or} \quad B = 0, \text{ as } \mu \neq 0 \quad \dots (19)$$

Using (19), (18) gives $A \cos \mu \pi = 0$ (20)

Now, we must take $A \neq 0$, otherwise $A = 0$ and $B = 0$ will give $y(x) = 0$ as before and hence we shall not get eigenfunction.

$\therefore$ (20) gives $\cos \mu \pi = 0$ so that $\mu \pi = (2n+1) \times (\pi/2)$, where n is an integer. Then $\mu = n + 1/2$ But $\lambda - 1 = -\mu^2$ so that $\lambda = 1 - \mu^2$.

Hence the eigenvalues are given by $\lambda_n = 1 - \mu^2 = 1 - (n + 1/2)^2$ (21)

Putting $B = 0$ and $\mu = (n + 1/2)$ in (16), corresponding eigenfunctions $y_n(x)$ are given by

$$y_n(x) = A \cos(n + 1/2)x \quad \text{or} \quad y_n(x) = \cos(n + 1/2)x, \quad \text{taking } A = 1.$$

Hence the required eigenfunctions $y_n(x)$ with the corresponding eigenvalues λ_n are given by

$$y_n(x) = \cos(n + 1/2)x, \quad \lambda_n = 1 - (n + 1/2)^2, \quad \text{where } n \text{ is an integer.}$$

Ex. 11. Determine the eigenvalues and eigenfunctions of the homogeneous integral equation

$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt, \quad \text{where} \quad K(x, t) = \begin{cases} x(t-1), & 0 \leq x \leq t, \\ t(x-1), & t \leq x \leq 1. \end{cases}$$

Sol. Given
$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt, \quad \dots (1)$$

where
$$K(x, t) = \begin{cases} x(t-1), & 0 \leq x \leq t, \\ t(x-1), & t \leq x \leq 1. \end{cases} \quad \dots (2)$$

Re-writing (1), we have
$$y(x) = \lambda \left[\int_0^x K(x, t) y(t) dt + \int_x^1 K(x, t) y(t) dt \right]$$

or
$$y(x) = \int_0^x \lambda t(x-1) y(t) dt + \int_x^1 \lambda x(t-1) y(t) dt, \text{ using (2)} \quad \dots (3)$$

Differentiating (3) w.r.t. 'x' and using Leibnitz's rule of differentiating under integral sign (refer Art. 1.13), we have

$$y'(x) = \int_0^x \lambda t y(t) dt + \lambda x(x-1) y(x) - 0 + \int_x^1 \lambda(t-1) y(t) dt + 0 - \lambda x(x-1) y(x)$$

or
$$y'(x) = \int_0^x \lambda t y(t) dt + \int_x^1 \lambda(t-1) y(t) dt. \quad \dots (4)$$

Differentiating (4) w.r.t. 'x' and using Leibnitz rule as before, we have

$$y''(x) = 0 + \lambda x y(x) - 0 + 0 + 0 - \lambda(x-1) y(x)$$

or
$$y''(x) - \lambda y(x) = 0. \quad \dots (5)$$

Putting $x = 0$ and $x = 1$ by turn in (3), we get

$$y(0) = 0 \quad \dots (6A)$$

and
$$y(1) = 1. \quad \dots (6B)$$

We shall solve (5) subject to boundary conditions (6A) and (6B) to determine the required eigenvalues and eigenfunctions.

Three cases arise :

Case I. Let $\lambda = 0$. Then (5) reduces to $y''(x) = 0$ whose general solution is

$$y(x) = Ax + B. \quad \dots (7)$$

Putting $x = 0$ in (7) and using (6A), we get $0 = B. \quad \dots (8)$

Again, putting $x = 1$ in (7) and using (6B), we get $0 = A + B \quad \dots (9)$

Solving (8) and (9), $A = B = 0$. Hence (7) gives $y(x) = 0$, which is not an eigenfunction and so $\lambda = 0$ is not an eigenvalue.

Case II. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (5) reduces to $y''(x) - \mu^2 y(x) = 0$ whose

general solution is
$$y(x) = Ae^{\mu x} + Be^{-\mu x}. \quad \dots (10)$$

Putting $x = 0$ in (10) and using (6A), we get $0 = A + B$ (11)

Again, putting $x = 1$ in (10) and using (6B), we get $0 = Ae^{\mu} + Be^{-\mu}$ (12)

Solving (11) and (12), $A = B = 0$. Hence (7) reduces to $y(x) = 0$, which is not an eigenfunction and hence $\lambda = \mu^2$ does not give eigenvalues.

Case III. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (5) reduces to $y''(x) + \mu^2 y(x) = 0$ whose general solution

$$y(x) = A \cos \mu x + B \sin \mu x. \quad \dots (13)$$

Putting $x = 0$ in (13) and using (6A), we get $0 = A$ (14)

Again, putting $x = 1$ in (13) and using (6B), we get

$$0 = A \cos \mu + B \sin \mu. \quad \text{or} \quad B \sin \mu = 0, \text{ using (14)} \quad \dots (15)$$

But $B \neq 0$, otherwise $B = 0$ and $A = 0$ will give $y(x) = 0$ by (13) and so we shall not get an eigenfunction. Hence (15) gives

$$\sin \mu = 0 \quad \text{so that} \quad \mu = n\pi, \quad n = 1, 2, 3, \dots$$

$\therefore$ The required eigenvalues are given by

$$\lambda_n = \lambda = -\mu^2 = -n^2\pi^2, \quad n = 1, 2, 3, \dots$$

From (13), the corresponding eigenfunctions $y_n(x)$ are given by

$$y_n(x) = B \sin n\pi x \quad (\because A = 0, \quad \mu = n\pi)$$

or $y_n(x) = \sin n\pi x$, taking $B = 1$.

Thus the required eigenvalues and eigenfunctions are given by

$$\lambda_n = -n^2\pi^2, \quad y_n(x) = \sin n\pi x, \quad n = 1, 2, 3, \dots$$

Ex. 12. Determine the eigenvalues and eigenfunctions of the homogeneous integral equation

$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt, \quad \text{where} \quad K(x, t) = \begin{cases} t(x+1), & 0 \leq x \leq t, \\ x(t+1), & t \leq x \leq 1. \end{cases}$$

Sol. Given $y(x) = \lambda \int_0^1 K(x, t) y(t) dt$, ... (1)

where $K(x, t) = \begin{cases} t(x+1), & 0 \leq x \leq t, \\ x(t+1), & t \leq x \leq 1. \end{cases}$... (2)

Re-writing (1), we have $y(x) = \lambda \left[\int_0^x K(x, t) y(t) dt + \int_x^1 K(x, t) y(t) dt \right]$

or $y(x) = \int_0^x \lambda x(t+1) y(t) dt + \int_x^1 \lambda t(x+1) y(t) dt$, using (2) ... (3)

Differentiating (3) w.r.t. 'x' and using Leibnitz's rule of differentiating under integral sign (refer Art. 1.13), we have

$$y'(x) = \int_0^x \lambda(t+1) y(t) dt + \lambda x(x+1) y(x) - 0 + \int_x^1 \lambda t y(t) dt + 0 - \lambda x(x+1) y(x)$$

or $y'(x) = \int_0^x \lambda(t+1) y(t) dt + \int_x^1 \lambda t y(t) dt$ (4)

Differentiating (4) w.r.t. 'x' and using Leibnitz's rule as before, we have

$$y''(x) = 0 + \lambda (x+1) y(x) - 0 + 0 + 0 - \lambda x y(x)$$

$$\text{or} \quad y''(x) - \lambda y(x) = 0. \quad \dots (5)$$

Putting $x = 0$ in (3) and (4), we get

$$y(0) = \int_0^1 \lambda t y(t) dt \quad \dots (6)$$

$$\text{and} \quad y'(0) = \int_0^1 \lambda t y(t) dt. \quad \dots (7)$$

Putting $x = 1$ in (3) and (4), we get

$$y(1) = \int_0^1 \lambda (t+1) y(t) dt. \quad \dots (8)$$

$$\text{and} \quad y'(1) = \int_0^1 \lambda (t+1) y(t) dt. \quad \dots (9)$$

$$\text{From (6) and (7),} \quad y(0) = y'(0). \quad \dots (10)$$

$$\text{From (8) and (9),} \quad y(1) = y'(1). \quad \dots (11)$$

We shall now solve (5) under boundary conditions (10) and (11) by usual method.

Three cases arise :

Case I. Let $\lambda = 0$. Then (5) reduces to $y''(x) = 0$ whose general solution is

$$y(x) = Ax + B. \quad \dots (12)$$

$$\text{From (12),} \quad y'(x) = A. \quad \dots (13)$$

Putting $x = 0$ in (12) and (13), we get

$$y(0) = B \quad \text{and} \quad y'(0) = A.$$

$$\therefore (10) \text{ reduces to} \quad B = A. \quad \dots (14)$$

Putting $x = 1$ in (12) and (13), we get

$$y(1) = A + B \quad \text{and} \quad y'(1) = A. \quad \dots (15)$$

$$\therefore (11) \text{ reduces to} \quad A + B = A \quad \text{so that} \quad B = 0. \quad \dots (16)$$

Solving (14) and (16), $A = B = 0$. Hence (12) gives $y(x) = 0$, which is not an eigenfunction and so $\lambda = 0$ is not an eigenvalue.

Case II. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (5) reduces to $y''(x) - \mu^2 y(x) = 0$, whose general solution is

$$y(x) = Ae^{\mu x} + Be^{-\mu x} \quad \dots (17)$$

$$\text{From (17)} \quad y'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}. \quad \dots (18)$$

Putting $x = 0$ in (17) and (18), we get

$$y(0) = A + B \quad \text{and} \quad y'(0) = A\mu - B\mu.$$

$$\therefore (10) \text{ reduces to} \quad A + B = A\mu - B\mu$$

$$\text{or} \quad A(1 - \mu) + B(1 + \mu) = 0. \quad \dots (19)$$

Putting $x = 1$ in (17) and (18), we get

$$y(1) = Ae^{\mu} + Be^{-\mu} \quad \text{and} \quad y'(1) = A\mu e^{\mu} - B\mu e^{-\mu}.$$

$$\therefore (11) \text{ reduces to } Ae^{\mu} + Be^{-\mu} = A\mu e^{\mu} - B\mu e^{-\mu}$$

$$\text{or } Ae^{\mu} (1 - \mu) + Be^{-\mu} (1 + \mu) = 0. \quad \dots (20)$$

For non-trivial solution of (19) and (20), we must have

$$\begin{vmatrix} 1 - \mu & 1 + \mu \\ e^{\mu} (1 - \mu) & e^{-\mu} (1 + \mu) \end{vmatrix} = 0$$

$$\text{or } (1 - \mu) (1 + \mu) e^{-\mu} - (1 - \mu) (1 + \mu) e^{\mu} = 0$$

$$\text{or } (1 - \mu) (1 + \mu) (e^{\mu} - e^{-\mu}) = 0 \quad \text{or} \quad 2(1 - \mu) (1 + \mu) \sinh \mu = 0. \quad \dots (21)$$

Since $\mu \neq 0$ by assumption, $\sinh \mu \neq 0$. So (21) reduces to

$$(1 - \mu) (1 + \mu) = 0 \quad \text{so that} \quad \mu = 1 \quad \text{or} \quad \mu = -1.$$

When $\mu = 1$, (19) and (20) reduces to

$$A.0 + 2B = 0 \quad \text{and} \quad A.0 + 2Be^{-1} = 0 \quad \dots (22)$$

Solving (22), $B = 0$ and A is arbitrary constant.

$$\therefore (17) \text{ gives } y(x) = Ae^x. \quad \dots (23)$$

Next, when $\mu = -1$, (19) and (20) reduces to

$$2A + B.0 = 0 \quad \text{and} \quad 2Ae + B.0 = 0$$

Solving these, $A = 0$ and B is an arbitrary constant.

$$\therefore (17) \text{ gives } y(x) = Be^x. \quad \dots (24)$$

Setting $A = 1$ in (23) or $B = 1$ in (24), the required eigenfunction is e^x which correspond to eigenvalue $\lambda = \mu^2 = (1)^2 = (-1)^2 = 1$.

Case III. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (5) reduces to $y''(x) + \mu^2 y(x) = 0$, whose general solution is

$$y(x) = A \cos \mu x + B \sin \mu x. \quad \dots (25)$$

$$\text{From (25), } y'(x) = -A\mu \sin \mu x + B\mu \cos \mu x. \quad \dots (26)$$

Putting $x = 0$ in (25) and (26), we get

$$y(0) = A \quad \text{and} \quad y'(0) = B\mu.$$

$$\therefore (10) \text{ reduces to } A = B\mu. \quad \dots (27)$$

Putting $x = 1$ in (25) and (26), we get

$$y(1) = A \cos \mu + B \sin \mu \quad \text{and} \quad y'(1) = -A\mu \sin \mu + B\mu \cos \mu.$$

$$\therefore (11) \text{ reduces to } A \cos \mu + B \sin \mu = -A\mu \sin \mu + B\mu \cos \mu \quad \dots (28)$$

Putting value of A given by (27) in (28), we get

$$B\mu \cos \mu + B \sin \mu = -B\mu^2 \sin \mu + B\mu \cos \mu \quad \text{or} \quad B(1 + \mu^2) \sin \mu = 0. \quad \dots (29)$$

But $B \neq 0$ for otherwise from (27), $A = 0$ when $B = 0$ and it gives $y(x) = 0$ by (25). Thus, we do not get eigenfunction when $B = 0$.

Again $(1 + \mu^2) \neq 0$, for otherwise $1 + \mu^2 = 0$ would give $\mu^2 = -1$ which is not possible as μ is real and so μ^2 cannot be negative.

$$\therefore (29) \text{ reduces to } \sin \mu = 0, \quad \text{giving} \quad \mu = n\pi, n = 1, 2, 3, \dots$$

$$\therefore \lambda = -\mu^2 = -n^2 \pi^2, n = 1, 2, 3, \dots$$

Putting $\mu = n\pi$ and $A = B\mu$ in (25), we get

$$y(x) = B\mu \cos n\pi x + B \sin n\pi x = B(\mu \cos n\pi x + \sin n\pi x).$$

Setting $B = 1$, we have

$$y(x) = \mu \cos n\pi x + \sin n\pi x.$$

Hence the required eigenvalues λ_0, λ_n and the corresponding eigenfunctions $y_0(x), y_n(x)$ are given by $\lambda_0 = 1, y_0(x) = e^x, \lambda_n = -n^2 \pi^2, y_n(x) = \mu \cos n\pi x + \sin n\pi x, n = 1, 2, 3, \dots$

Ex. 13. Determine the eigenvalues and eigenfunctions of the homogeneous integral equation

$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt, \quad \text{where} \quad K(x, t) = \begin{cases} -e^{-t} \sinh x, & 0 \leq x \leq t \\ -e^{-x} \sinh t, & t \leq x \leq 1. \end{cases}$$

Sol. Given

$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt, \quad \dots (1)$$

where

$$K(x, t) = \begin{cases} -e^{-t} \sinh x, & 0 \leq x \leq t \\ -e^{-x} \sinh t, & t \leq x \leq 1. \end{cases} \quad \dots (2)$$

Re-writing (1), we have
$$y(x) = \lambda \left[\int_0^x K(x, t) y(t) dt + \int_x^1 K(x, t) y(t) dt \right]$$

or
$$y(x) = - \int_0^x \lambda e^{-x} \sinh t y(t) dt - \int_x^1 \lambda e^{-t} \sinh x y(t) dt, \text{ using (2)} \quad \dots (3)$$

Differentiating (3) w.r.t. 'x' and using Leibnitz's rule of differentiating under integral sign (refer Art. 1.13), we have

$$\begin{aligned} y'(x) = & - \left[\int_0^x (-\lambda e^{-x} \sinh t) y(t) dt + \lambda e^{-x} \sinh x y(x) - 0 \right] \\ & - \left[\int_x^1 (\lambda e^{-t} \cosh x) y(t) dt + 0 - \lambda e^{-x} \sinh x y(x) \right] \end{aligned}$$

or
$$y'(x) = \int_0^x \lambda e^{-x} \sinh t y(t) dt - \int_x^1 \lambda e^{-t} \cosh x y(t) dt. \quad \dots (4)$$

Differentiating (4) w.r.t. 'x' and using Leibnitz rule of differentiating under integral sign as before, we have

$$\begin{aligned} y''(x) = & \int_0^x (-\lambda e^{-x} \sinh t) y(t) dt + \lambda e^{-x} \sinh x y(x) - 0 \\ & - \left[\int_x^1 (\lambda e^{-t} \sinh x) y(t) dt + 0 - \lambda e^{-x} \cosh x y(x) \right] \end{aligned}$$

or
$$y'' = - \int_0^x \lambda e^{-x} \sinh t y(t) dt - \int_x^1 \lambda e^{-t} \sinh x y(t) dt + \lambda e^{-x} y(x) (\sinh x + \cosh x)$$

$$= y(x) + \lambda e^{-x} y(x) \left[\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} \right], \text{ by (3)}$$

$$= y(x) + \lambda e^{-x} y(x) e^x = y(x) + \lambda y(x)$$

or $y''(x) - (1 + \lambda) y(x) = 0.$... (5)

Putting $x = 0$ in (3), $y(0) = 0.$... (6)

Putting $x = 1$ in (3) and (4), we get

$$y(1) = - \int_0^1 \lambda e^{-1} \sinh t y(t) dt \quad \dots (7)$$

and $y'(1) = \int_0^1 \lambda e^{-1} \sinh t y(t) dt.$... (8)

Adding (7) and (8), $y(1) + y'(1) = 0.$... (9)

We shall now solve (5) under boundary conditions (6) and (9). Three cases arise :

Case I. Let $\lambda = 0$. Then (5) reduces to $y''(x) = 0$, whose general solution is

$$y(x) = Ax + B. \quad \dots (10)$$

From (10) $y'(x) = A.$... (11)

Putting $x = 0$ in (10) and using B.C. (6), we get $0 = B$... (12)

Putting $x = 1$ in (10) and (11), we get $y(1) = A + B$ and $y'(1) = A.$... (13)

$\therefore$ B.C. (9) becomes $A + B + A = 0$... (14)

Solving (12) and (14), $A = B = 0$. So by (10), $y(x) = 0$, which is not an eigenfunction.

Case II. Let $1 + \lambda = \mu^2$, where $\mu \neq 0$. Then (5) reduces to $y''(x) - \mu^2 y(x) = 0$, whose

general solution is $y(x) = Ae^{\mu x} + Be^{-\mu x}$... (15)

From (15), $y'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$... (16)

Putting $x = 0$ in (15) and using B.C. (6), we get $0 = A + B$ or $B = -A$... (17)

Putting $x = 1$ in (15) and (16), we get

$$y(1) = Ae^{\mu} + Be^{-\mu} \quad \text{and} \quad y'(1) = A\mu e^{\mu} - B\mu e^{-\mu}.$$

$\therefore$ B.C. (9) becomes

$$Ae^{\mu} + Be^{-\mu} + A\mu e^{\mu} - B\mu e^{-\mu} = 0 \quad \text{or} \quad Ae^{\mu} - Ae^{-\mu} + A\mu e^{\mu} + A\mu e^{-\mu} = 0, \text{ by (17)}$$

or $A[e^{\mu} - e^{-\mu} + \mu(e^{\mu} + e^{-\mu})] = 0$ or $A[2 \sinh \mu + 2\mu \cosh \mu] = 0.$

$\therefore A = 0$ and so from (17), $B = 0$. With $A = B = 0$, (15) gives $y(x) = 0$, which is not an eigenfunction.

Case III. Let $1 + \lambda = -\mu^2$, where $\mu \neq 0$. Then (5) reduces to $y''(x) + \mu^2 y(x) = 0$, whose

general solution is $y(x) = A \cos \mu x + B \sin \mu x.$... (18)

From (18) $y'(x) = -A\mu \sin \mu x + B\mu \cos \mu x.$... (19)

Putting $x = 0$ in (18) and using B.C. (6), we get $0 = A.$... (20)

Putting $x = 1$ and $A = 0$ in (18) and (19), we get

$$y(1) = B \sin \mu \quad \text{and} \quad y'(1) = B \mu \cos \mu.$$

$\therefore$ B.C. (9) becomes $B \sin \mu + B \mu \cos \mu = 0$ or $B(\sin \mu + \mu \cos \mu) = 0.$... (21)

If $B = 0$, then with $A = 0$, (18) reduces to $y(x) = 0$, which is not an eigenfunction. So we take $B \neq 0$ and hence (21) gives

$$\sin \mu + \mu \cos \mu = 0 \quad \text{or} \quad \tan \mu = -\mu, \quad \dots (22)$$

which is a trigonometrical equation in μ . Let μ_n ($n = 1, 2, 3, \dots$) be the positive roots of (22). With $A = 0$, (18) reduces to

$$y(x) = B \sin \mu x \quad \text{or} \quad y(x) = \sin \mu x, \text{ taking } B = 1.$$

Again, $\lambda = -1 - \mu^2$. Hence the required eigenvalues λ_n and the corresponding eigenfunctions $y_n(x)$ are given by $\lambda_n^2 = -1 - \mu_n^2$ and $y_n(x) = \sin \mu_n x$, $n = 1, 2, 3, \dots$, where μ_n are the positive roots of (22).

Ex. 14. The eigenvalue λ of the Fredholm integral equation $y(x) = \lambda \int_0^1 x^2 t y(t) dt$ is

$$(a) -2 \quad (b) 2 \quad (c) 4 \quad (d) -4 \quad \text{[GATE 2011]}$$

Solution. Ans. (c). Given $y(x) = \lambda x^2 \int_0^1 t y(t) dt \quad \dots (1)$

Let $c = \int_0^1 t y(t) dt \quad \dots (2)$

Then, (1) yields $y(x) = \lambda c x^2$ so that $y(t) = \lambda c t^2 \quad \dots (3)$

Using (3), (2) reduces to $c = \int_0^1 \lambda c t^3 dt = (\lambda c) / 4$ so that $c(4 - \lambda) = 0 \quad \dots (4)$

If $c = 0$, then (4) gives $y(x) = 0$. We, therefore, assume that for nonzero solution of (1), $c \neq 0$. Hence, (4) reduce to $4 - \lambda = 0$ or $\lambda = 4$, which is the required eigenvalue.

EXERCISE

1. Solve the following homogeneous integral equations :

$$(i) \quad y(x) = - \int_0^1 y(t) dt.$$

$$(ii) \quad y(x) = \frac{1}{2} \int_0^\pi \sin x y(t) dt.$$

$$(iii) \quad y(x) = \frac{1}{50} \int_0^{10} t y(t) dt.$$

$$(iv) \quad y(x) = \frac{1}{e^2 - 1} \int_0^1 2e^x e^t y(t) dt.$$

2. Determine the eigenvalues and the eigenfunctions of the following homogeneous integral equations :

$$(i) \quad y(x) = \lambda \int_0^{\pi/4} \sin^2 x y(t) dt.$$

$$(ii) \quad y(x) = \lambda \int_0^{2\pi} \sin x \cos t y(t) dt.$$

$$(iii) \quad y(x) = \lambda \int_0^{2\pi} \sin x \sin t y(t) dt.$$

$$(iv) \quad y(x) = \lambda \int_{-1}^1 (5xt^3 + 4x^2t) y(t) dt.$$

$$(v) \quad y(x) = \lambda \int_0^1 (45x^2 \ln t - 9t^2 \ln x) y(t) dt. \text{ where } \ln x = \log_e x$$

$$(vi) \quad y(x) = \lambda \int_0^\pi \cos(x+t) y(t) dt.$$

$$(vii) \quad y(x) = \lambda \int_{-1}^1 (x \cosh t - t \sinh x) y(t) dt$$

$$(viii) \quad y(x) = \lambda \int_{-1}^1 (x \cosh t - t^2 \sinh x) y(t) dt \quad (x) \quad \phi(x) = \lambda \int_0^1 e^{x-\xi} \phi(\xi) d\xi \quad (\text{Kanpur 2008})$$

$$(ix) \quad y(x) = \lambda \int_{-1}^1 (x \cosh t - t \cosh x) y(t) dt.$$

3. Show that the integral equation $y(x) = \lambda \int_0^\pi (\sin x \sin 2t) y(t) dt$ has no eigenvalues.
4. Find the eigenvalues and eigenfunctions of the following homogeneous integral equations

$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt, \text{ where}$$

$$(i) \quad K(x, t) = \begin{cases} \sin x \cos t, & 0 \leq x \leq t, \\ \sin t \cos x, & t \leq x \leq \pi. \end{cases} \quad (ii) \quad K(x, t) = \begin{cases} \sin x \cos t, & 0 \leq x \leq t, \\ \sin t \cos x, & t \leq x \leq \pi/2. \end{cases}$$

$$(iii) \quad K(x, t) = \begin{cases} (x+1)(t-2), & 0 \leq x \leq t, \\ (t+1)(x-2), & t \leq x \leq 1 \end{cases} \quad (iv) \quad K(x, t) = \begin{cases} \sin x \sin(t-1), & -\pi \leq x \leq t, \\ \sin t \sin(x-1), & t \leq x \leq \pi. \end{cases}$$

$$(v) \quad K(x, t) = \begin{cases} \sin(x + \pi/4) \sin(t - \pi/4), & 0 \leq x \leq t \\ \sin(t + \pi/4) \sin(x - \pi/4), & t \leq x \leq \pi. \end{cases}$$

$$(vi) \quad K(x, t) = e^{-|x-t|}, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1.$$

5. What do you understand by eigenvalues and eigenfunctions of the integral equation

$$y(x) = \lambda \int_a^b K(x, t) y(t) dt. \text{ Prove that the integral equation with degenerate kernel}$$

$$K(x, t) = \sum_{r=1}^n a_r(x) b_r(t) \text{ has at most } n \text{ eigenvalues.}$$

ANSWERS

1. (i) $y(x) = 0$ (ii) $y(x) = 0$ (iii) $y(x) = 0$ (iv) $y(x) = 0$.
2. (i) $\lambda = 8/(\pi - 2)$, $y(x) = \sin^2 x$.
(ii) Eigenvalues and eigenfunctions do not exist.
(iii) $\lambda = 1/\pi$, $y(x) = \sin x$. (iv) $\lambda = 1/2$, $y(x) = (5/2) \times x + (10/3) \times x^2$
(v) There are no real eigenvalues and real eigenfunctions.
(vi) $\lambda_1 = -2/\pi$, $y_1(x) = \sin x$; $\lambda_2 = 2/\pi$, $y_2(x) = \cos x$.
(vii) $\lambda = -e/2$, $y(x) = \sinh x$.
(viii) Eigenvalues and eigenfunctions do not exist.
(ix) There are no real eigenvalues and real eigenfunctions.
(x) $\lambda = 1$, $\phi(x) = e^x$

4. (i) $\lambda_n = (n + 1/2)^2 - 1$; $y_n(x) = \sin(n + 1/2)x$, $n = 1, 2, 3, \dots$

(ii) $\lambda_n = 4n^2 - 1$; $y_n(x) = \sin 2nx$, $n = 1, 2, 3, \dots$

(iii) $\lambda_n = -(1/3) \times \mu_n^2$; $y_n(x) = \sin \mu_n x + \mu_n \cos \mu_n x$, $n = 1, 2, 3, \dots$

where μ_n is a root of the equation $\mu - (1/\mu) = 2 \cot \mu$.

(iv) $\lambda_n = (1 - \mu_n^2) \operatorname{cosec} 1$; $y_n(x) = \sin \{\mu_n (\pi + x)\}$, $n = 1, 2, 3, \dots$

where μ_n are roots of the equation $\tan 2\pi\mu = -\mu \tan 1$.

(v) $\lambda_n = 1 - \mu_n^2$; $y_n(x) = \sin \mu_n x + \mu_n \cos \mu_n x$, $n = 1, 2, 3, \dots$

where μ_n are the roots of the equation $2 \cot \pi\mu = \mu - (1/\mu)$.

(vi) $\lambda_n = (1 + \mu_n^2)/2$; $y_n(x) = \sin \mu_n x + \cos \mu_n x$, $n = 1, 2, 3, \dots$

where μ_n are roots of the equation $2 \cot \mu = \mu - (1/\mu)$.

CHAPTER 4

Fredholm Integral Equations of the Second Kind With Separable (or Degenerate) Kernels

4.1 SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND WITH SEPARABLE (OR DEGENERATE) KERNEL.

[Meerut 2000, 01, 05]

Consider a Fredholm integral equation of the second kind :

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt. \quad \dots (1)$$

Since kernel $K(x, t)$ is separable, we take

$$K(x, t) = \sum_{i=1}^n f_i(x) g_i(t). \quad \dots (2)$$

where the functions $f_i(x)$ are assumed to be linearly independent.

Using (2), (1) reduces to
$$y(x) = f(x) + \lambda \int_a^b \left[\sum_{i=1}^n f_i(x) g_i(t) \right] y(t) dt$$

or
$$y(x) = f(x) + \lambda \sum_{i=1}^n f_i(x) \int_a^b g_i(t) y(t) dt. \quad \dots (3)$$

[Interchanging the order of summation and integration]

Let
$$\int_a^b g_i(t) y(t) dt = C_i, (i = 1, 2, \dots, n) \quad \dots (4)$$

Using (4), (3) reduces to
$$y(x) = f(x) + \lambda \sum_{i=1}^n C_i f_i(x), \quad \dots (5)$$

where constants $C_i (i = 1, 2, 3, \dots, n)$ are to be determined in order to find solution of (1) in the form given by (5). We now proceed to evaluate C_i 's as follows :

Re-writing (5),
$$y(x) = f(x) + \lambda \sum_{j=1}^n C_j f_j(x)$$

so that
$$y(t) = f(t) + \lambda \sum_{j=1}^n C_j f_j(t) \quad \dots (6)$$

Substituting the values of $y(x)$ and $y(t)$ given by (5) and (6) respectively in (3), we have

$$f(x) + \lambda \sum_{i=1}^n C_i f_i(x) = f(x) + \lambda \sum_{i=1}^n f_i(x) \int_a^b g_i(t) \left\{ f(t) + \lambda \sum_{j=1}^n C_j f_j(t) \right\} dt$$

or
$$\sum_{i=1}^n C_i f_i(x) = \sum_{i=1}^n f_i(x) \left\{ \int_a^b g_i(t) f(t) dt + \lambda \sum_{j=1}^n C_j \int_a^b g_i(t) f_j(t) dt \right\} \quad \dots (7)$$

4.2 Fredholm Integral Equations of the Second Kind With Separable (or Degenerate) Kernels

We now introduce the following useful notations :

$$\int_a^b g_i(t) f(t) dt = \beta_i \quad \text{and} \quad \int_a^b g_i(t) f_j(t) dt = \alpha_{ij}, \quad i, j = 1, 2, \dots, n \quad \dots (8)$$

where β_i and α_{ij} are known constants. Then (7) may be simplified as

$$\sum_{i=1}^n C_i f_i(x) = \sum_{i=1}^n f_i(x) \left\{ \beta_i + \lambda \sum_{j=1}^n \alpha_{ij} C_j \right\} \quad \text{or} \quad \sum f_i(x) \left\{ C_i - \beta_i - \lambda \sum_{j=1}^n \alpha_{ij} C_j \right\} = 0$$

But the functions $f_i(x)$ are linearly independent, therefore

$$C_i - \beta_i - \lambda \sum_{j=1}^n \alpha_{ij} C_j = 0, \quad i = 1, 2, 3, \dots, n$$

$$\text{or} \quad C_i - \lambda \sum_{j=1}^n \alpha_{ij} C_j = \beta_i, \quad i = 1, 2, 3, \dots, n \quad \dots (9)$$

Taking $i = 1$ in (9), we have

$$C_1 - \lambda \sum_{j=1}^n \alpha_{1j} C_j = \beta_1 \quad \text{or} \quad C_1 - \lambda (\alpha_{11} C_1 + \alpha_{12} C_2 + \dots + \alpha_{1n} C_n) = \beta_1$$

$$\text{or} \quad (1 - \lambda \alpha_{11}) C_1 - \lambda \alpha_{12} C_2 - \dots - \lambda \alpha_{1n} C_n = \beta_1.$$

Similarly, we may simplify $(A_2), (A_3), \dots (A_n)$. Thus, we obtain the following system of linear equations to determine $C_1, C_2, \dots, C_n$.

$$(1 - \lambda \alpha_{11}) C_1 - \lambda \alpha_{12} C_2 - \dots - \lambda \alpha_{1n} C_n = \beta_1. \quad \dots (B_1)$$

$$-\lambda \alpha_{21} C_1 + (1 - \lambda \alpha_{22}) C_2 - \dots - \lambda \alpha_{2n} C_n = \beta_2. \quad \dots (B_2)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\text{and} \quad -\lambda \alpha_{n1} C_1 - \lambda \alpha_{n2} C_2 - \dots + (1 - \lambda \alpha_{nn}) C_n = \beta_n. \quad \dots (B_n)$$

The determinant $D(\lambda)$ of the system $(B_1), (B_2), \dots, (B_n)$ of linear equations is given by

$$D(\lambda) = \begin{vmatrix} 1 - \alpha_{11} & -\lambda \alpha_{12} & \dots & -\lambda \alpha_{1n} \\ -\lambda \alpha_{21} & 1 - \lambda \alpha_{22} & \dots & -\lambda \alpha_{2n} \\ \vdots & \vdots & \dots & \vdots \\ -\lambda \alpha_{n1} & -\lambda \alpha_{n2} & \dots & 1 - \lambda \alpha_{nn} \end{vmatrix}, \quad \dots (10)$$

which is a polynomial in λ of degree at most n . Again, $D(\lambda)$ is not identically zero, since, when $\lambda = 0$, $D(\lambda) = 1$.

To discuss the solution of (1), the following situations arise :

Situation I. When at least one right member of the system $(B_1), \dots, (B_n)$ is nonzero.

The following two cases arise under this situation :

(i) If $D(\lambda) \neq 0$, then a unique nonzero solution of the system $(B_1), \dots, (B_n)$ exists and so (1) has a unique nonzero solution given by (5).

(ii) If $D(\lambda) = 0$, then the equations $(B_1), \dots (B_n)$ have either no solution or they possess infinite solutions and hence (1) has either no solution or infinite solutions.

Situation II. When $f(x) = 0$. Then (8), shows that $\beta_j = 0$ for $j = 1, 2, \dots, n$. Hence the equations $(B_1), \dots (B_n)$ reduce to a system of homogeneous linear equations.

The following two cases arise under this situation :

(i) If $D(\lambda) \neq 0$, then a unique zero solution $C_1 = C_2 = \dots = C_n = 0$ of the system $(B_1), \dots, (B_n)$ exists and so from (5), we see that (1) has only unique zero solution $y(x) = 0$.

(ii) If $D(\lambda) = 0$, then the system $(B_1), \dots (B_n)$ possess infinite nonzero solutions and so (1) has infinite nonzero solutions. Those values of λ for which $D(\lambda) = 0$ are known as the eigenvalues (or characteristic constants or values) and any nonzero solution of the homogeneous Fredholm integral equation $y(x) = \lambda \int_a^b K(x, t) y(t) dt$ (with a convenient choice of the arbitrary constant or constants) is known as a corresponding eigenfunction (or characteristic function) of integral equation.

For more discussion of theory and problems based on situation II, refer chapter 2.

Situation III. When $f(x) \neq 0$, but

$$\int_a^b g_1(x) f(x) dx = 0, \quad \int_a^b g_2(x) f(x) dx = 0, \quad \dots, \quad \int_a^b g_n(x) f(x) dx = 0,$$

i.e., $f(x)$ is orthogonal to all the functions $g_1(t), g_2(t), \dots, g_n(t)$, then (8) shows that $\beta_1 = 0, \beta_2 = 0, \dots, \beta_n = 0$ and hence the equations $(B_1), \dots, (B_n)$ reduce to a system of homogeneous linear equations.

The following two cases arise under this situation :

(i) If $D(\lambda) \neq 0$, then a unique zero solution $C_1 = C_2 = \dots = C_n = 0$ of the system $(B_1), \dots, (B_n)$ exists and so from (5), we see that (1) has only unique solution $y(x) = f(x)$.

(ii) If $D(\lambda) = 0$, then the system $(B_1), \dots, (B_n)$ possess infinite nonzero solutions and so (1) has infinite nonzero solutions. The solutions corresponding to the eigenvalues of λ are now expressed as the sum of $f(x)$ and arbitrary multiples of eigenfunctions.

4.2. SOLVED EXAMPLES BASES ON ART. 4.1.

Ex. 1. The solution of the integral equation $g(s) = s + \int_0^1 s u^2 g(u) du$ is given by

(a) $g(t) = 3t/4$ (b) $g(t) = 4t/3$ (c) $g(t) = 2t/3$ (d) $g(t) = 3t/2$ [GATE 1999]

Sol. Ans (b) Given $g(s) = s + s \int_0^1 u^2 g(u) du$... (1)

Let $C = \int_0^1 u^2 g(u) du$... (2)

Using (2), (1) yields $g(s) = s + Cs = s(1+C)$... (3)

From (3), $g(u) = u(1+C)$... (4)

Using (4), (2) yields $C = \int_0^1 u^3 (1+C) du = (1+C) \times \left[\frac{u^4}{4} \right]_0^1$

or $C = (1+C) \times (1/4)$ so that $C = 1/3$

Hence, (3) $\Rightarrow g(s) = s(1+1/3) = 4s/3$ and so $g(t) = 4t/3$.

Ex. 2(a). Solve : $y(x) = e^x + \lambda \int_0^1 2e^x e^t y(t) dt$. [Meerut 2009]

Sol. Given $y(x) = e^x + \lambda \int_0^1 2e^x e^t y(t) dt$. or $y(x) = e^x + 2\lambda e^x \int_0^1 e^t y(t) dt$ (1)

Let $C = \int_0^1 e^t y(t) dt$ (2)

Using (2), (1) reduces to $y(x) = e^x + 2C\lambda e^x = e^x (1 + 2C\lambda)$ (3)

From (3), $y(t) = e^t (1 + 2C\lambda)$ (4)

Using (4), (2) becomes

$$C = \int_0^1 [e^t \cdot e^t (1 + 2C\lambda)] dt = (1 + 2C\lambda) \left[\frac{e^{2t}}{2} \right]_0^1 = (1 + 2C\lambda) \times \frac{1}{2} (e^2 - 1)$$

$$\text{or } C [1 - \lambda (e^2 - 1)] = \frac{1}{2} (e^2 - 1) \quad \text{or } C = \frac{e^2 - 1}{2 [1 - \lambda (e^2 - 1)]}, \text{ where } \lambda \neq \frac{1}{e^2 - 1}.$$

Putting this value of C in (3), we get

$$y(x) = e^x \left[1 + 2\lambda \cdot \frac{e^2 - 1}{2 \{1 - \lambda (e^2 - 1)\}} \right] \quad \text{or } y(x) = \frac{e^x}{1 - \lambda (e^2 - 1)}, \text{ where } \lambda \neq \frac{1}{e^2 - 1}.$$

which is the required solution of given integral equation.

Ex. 2(b). Solve : $y(x) = \cos x + \lambda \int_0^\pi \sin x y(t) dt$. [Kanpur 2008]

Sol. Given $y(x) = \cos x + \lambda \int_0^\pi \sin x y(t) dt$. or $y(x) = \cos x + \lambda \sin x \int_0^\pi y(t) dt$... (1)

Let $C = \int_0^\pi y(t) dt$... (2)

Using (2), (1) becomes $y(x) = \cos x + \lambda C \sin x$ (3)

From (3), $y(t) = \cos t + \lambda C \sin t$ (4)

Using (4), (2) reduces to $C = \int_0^\pi (\cos t + \lambda C \sin t) dt = [\sin t]_0^\pi + \lambda C [-\cos t]_0^\pi$

$$\text{or } C = 0 + \lambda C [-\cos \pi + \cos 0] \quad \text{or } C = 2\lambda C$$

$$\text{or } C(1 - 2\lambda) = 0 \quad \text{so that } C = 0, \text{ if } \lambda \neq 1/2$$

Hence by (3), the required solution is $y(x) = \cos x$, provided $\lambda \neq 1/2$.

Ex.3. Solve : $y(x) = 2x - \pi + 4 \int_0^{\pi/2} \sin^2 x y(t) dt$. [Kanpur 2007]

Sol. Given $y(x) = 2x - \pi + 4 \int_0^{\pi/2} \sin^2 x y(t) dt$

$$\text{or } y(x) = 2x - \pi + 4 \sin^2 x \int_0^{\pi/2} y(t) dt \quad \dots (1)$$

Let
$$C = \int_0^{\pi/2} y(t) dt \quad \dots (2)$$

Using (2), (1) becomes
$$y(x) = 2x - \pi + 4C \sin^2 x \quad \dots (3)$$

From (3),
$$y(t) = 2t - \pi + 4C \sin^2 t \quad \dots (4)$$

Using (4), (2) becomes

$$C = \int_0^{\pi/2} (2t - \pi + 4C \sin^2 t) dt = \left[t^2 - \pi t \right]_0^{\pi/2} + 4C \int_0^{\pi/2} \frac{1 - \cos 2t}{2} dt$$

or
$$C = \frac{\pi^2}{4} - \frac{\pi^2}{2} + 2C \left[t - \frac{\sin 2t}{2} \right]_0^{\pi/2} \quad \text{or} \quad C = -\frac{\pi^2}{4} + 2C \left[\frac{\pi}{2} \right] \quad \text{or} \quad C = \frac{\pi^2}{4(\pi - 1)}$$

Putting this value of C in (3), the required solution of given integral equation is

$$y(x) = 2x - \pi + \frac{\pi^2 \sin^2 x}{\pi - 1}.$$

Ex. 4. Solve : $y(x) = f(x) + \lambda \int_0^1 xt y(t) dt.$

Sol. Given
$$y(x) = f(x) + \lambda \int_0^1 xt y(t) dt.$$

or
$$y(x) = f(x) + \lambda x \int_0^1 t y(t) dt \quad \dots (1)$$

Let
$$C = \int_0^1 t y(t) dt. \quad \dots (2)$$

Then (1) reduces to
$$y(x) = f(x) + \lambda C x \quad \dots (3)$$

From (3),
$$y(t) = f(t) + \lambda C t \quad \dots (4)$$

Using (4), (2) reduces to

$$C = \int_0^1 t [f(t) + \lambda C t] dt \quad \text{or} \quad C = \int_0^1 t f(t) dt + \lambda C \left[\frac{t^3}{3} \right]_0^1 \quad \text{or} \quad C = \int_0^1 t f(t) dt + \frac{\lambda C}{3}$$

or
$$C \left(1 - \frac{\lambda}{3} \right) = \int_0^1 t f(t) dt \quad \text{or} \quad C = \frac{3}{3 - \lambda} \int_0^1 t f(t) dt, \text{ where } \lambda \neq 3.$$

Putting this value of C in (3), the required solution is

$$y(x) = f(x) + \frac{3\lambda x}{3 - \lambda} \int_0^1 t f(t) dt, \text{ where } \lambda \neq 3$$

Ex. 5. Invert the integral equation : $y(x) = f(x) + \lambda \int_0^{2\pi} (\sin x \cos t) y(t) dt.$

OR

Find the solution of the integral equation $y(x) = f(x) + \lambda \int_0^{2\pi} (\sin x \cos t) y(t) dt.$

[Kanpur 2006]

Sol. Given
$$y(x) = f(x) + \lambda \int_0^{2\pi} (\sin x \cos t) y(t) dt.$$

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or
$$y(x) = f(x) + \lambda \sin x \int_0^{2\pi} \cos t y(t) dt. \quad \dots (1)$$

Let
$$C = \int_0^{2\pi} \cos t y(t) dt. \quad \dots (2)$$

Then (1) reduces to
$$y(x) = f(x) + \lambda C \sin x \quad \dots (3)$$

From (3),
$$y(t) = f(t) + \lambda C \sin t \quad \dots (4)$$

Using (4), (2) reduces to
$$C = \int_0^{2\pi} \cos t [f(t) + \lambda C \sin t] dt$$

or
$$C = \int_0^{2\pi} \cos t f(t) dt + \frac{\lambda C}{2} \int_0^{2\pi} \sin 2t dt \quad \text{or} \quad C = \int_0^{2\pi} \cos t f(t) dt + \frac{\lambda C}{2} \left[-\frac{\cos 2t}{2} \right]_0^{2\pi}$$

or
$$C = \int_0^{2\pi} \cos t f(t) dt + \frac{\lambda C}{2} \left[-\frac{1}{2} + \frac{1}{2} \right] \quad \text{or} \quad C = \int_0^{2\pi} \cos t f(t) dt.$$

Putting this value of C in (3), the required solution is

$$y(x) = f(x) + \lambda \sin x \int_0^{2\pi} f(t) \cos t dt$$

or
$$y(x) = f(x) + \lambda \int_0^{2\pi} (\sin x \cos t) f(t) dt.$$

Ex. 6. Solve the Fredholm integral equation of the second kind :

$$y(x) = x + \lambda \int_0^1 (xt^2 + x^2t) y(t) dt.$$

Sol. Given
$$y(x) = x + \lambda \int_0^1 (xt^2 + x^2t) y(t) dt.$$

or
$$y(x) = x + \lambda x \int_0^1 t^2 y(t) dt + \lambda x^2 \int_0^1 t y(t) dt. \quad \dots (1)$$

Let
$$C_1 = \int_0^1 t^2 y(t) dt \quad \dots (2)$$

and
$$C_2 = \int_0^1 t y(t) dt \quad \dots (3)$$

Using (2) and (3), (1) reduces to
$$y(x) = x + \lambda C_1 x + \lambda C_2 x^2. \quad \dots (4)$$

From (4),
$$y(t) = t + \lambda C_1 t + \lambda C_2 t^2. \quad \dots (5)$$

Using (5), (2) reduces to

$$C_1 = \int_0^1 t^2 (t + \lambda C_1 t + \lambda C_2 t^2) dt = \left[\frac{t^4}{4} + \frac{\lambda C_1 t^4}{4} + \frac{\lambda C_2 t^5}{5} \right]_0^1 = \frac{1}{4} + \frac{\lambda C_1}{4} + \frac{\lambda C_2}{5}$$

or
$$(20 - 5\lambda) C_1 - 4\lambda C_2 = 5. \quad \dots (6)$$

Next, using (5), (2) reduces to

$$C_2 = \int_0^1 t(t + \lambda C_1 t + \lambda C_2 t^2) dt = \left[\frac{t^3}{3} + \frac{\lambda C_1 t^3}{3} + \frac{\lambda C_2 t^4}{4} \right]_0^1 = \frac{1}{3} + \frac{\lambda C_1}{3} + \frac{\lambda C_2}{4}$$

or $-4\lambda C_1 + (12 - 3\lambda) C_2 = 4. \quad \dots (7)$

Solving (6) and (7) for C_1 and C_2 , we get

$$C_1 = \frac{60 + \lambda}{240 - 120\lambda - \lambda^2} \quad \text{and} \quad C_2 = \frac{80}{240 - 120\lambda - \lambda^2}.$$

Putting these values of C_1 and C_2 in (4), the required solution is

$$y(x) = x + \frac{\lambda x (60 + \lambda)}{240 - 120\lambda - \lambda^2} + \frac{80 \lambda x^2}{240 - 120\lambda - \lambda^2} \quad \text{or} \quad y(x) = \frac{(240 - 60\lambda)x + 80\lambda x^2}{240 - 120\lambda - \lambda^2}$$

Ex. 7. Solve : $y(x) = 1 + \int_0^1 (1 + e^{x+t}) y(t) dt.$ [Kanpur 2007]

Sol. Given $y(x) = 1 + \int_0^1 (1 + e^{x+t}) y(t) dt$ or $y(x) = 1 + \int_0^1 y(t) dt + e^x \int_0^1 e^t y(t) dt \quad \dots (1)$

Let $C_1 = \int_0^1 y(t) dt \quad \dots (2)$

and $C_2 = \int_0^1 e^t y(t) dt. \quad \dots (3)$

Using (2) and (3), (1) reduces to $y(x) = 1 + C_1 + C_2 e^x. \quad \dots (4)$

From (4), we have $y(t) = 1 + C_1 + C_2 e^t \quad \dots (5)$

Using (5), (2) reduces to

$$C_1 = \int_0^1 (1 + C_1 + C_2 e^t) dt \quad \text{or} \quad C_1 = \left[t + C_1 t + C_2 e^t \right]_0^1$$

or $C_1 = 1 + C_1 + C_2 (e - 1) \quad \text{or} \quad C_2 = -1 / (e - 1). \quad \dots (6)$

Using (5), (3) reduces to

$$C_2 = \int_0^1 e^t (1 + C_1 + C_2 e^t) dt = \left[e^t + C_1 e^t + C_2 \frac{e^{2t}}{2} \right]_0^1 = e - 1 + C_1 (e - 1) + \frac{C_2}{2} (e^2 - 1)$$

or $-\frac{1}{e-1} = e - 1 + C_1 (e - 1) - \frac{e^2 - 1}{2(e-1)}, \text{ using (6)}$

or $C_1 (e - 1) = -\frac{1}{e-1} - \frac{(e-1)}{2} \quad \text{or} \quad C_1 = -\frac{e^2 - 2e + 3}{2(e-1)^2}. \quad \dots (7)$

Using (6) and (7) in (4), the required solution is

$$y(x) = 1 - \frac{e^2 - 2e + 3}{2(e-1)^2} - \frac{e^x}{e-1} \quad \text{or} \quad y(x) = \frac{e^2 - 2e - 1}{2(e-1)^2} - \frac{e^x}{e-1}.$$

or
$$y(x) = \frac{e^2 - 2e - 1 - 2e^x(e-1)}{2(e-1)^2}.$$

Ex. 8. Solve : $y(x) = (1+x)^2 + \int_{-1}^1 (xt + x^2 t^2) y(t) dt.$

Sol. Given
$$y(x) = (1+x)^2 + \int_{-1}^1 (xt + x^2 t^2) y(t) dt$$

or
$$y(x) = (1+x)^2 + x \int_{-1}^1 t y(t) dt + x^2 \int_{-1}^1 t^2 y(t) dt. \quad \dots (1)$$

Let
$$C_1 = \int_{-1}^1 t y(t) dt \quad \dots (2)$$

and
$$C_2 = \int_{-1}^1 t^2 y(t) dt. \quad \dots (3)$$

Using (2) and (3), (1) reduces to
$$y(x) = (1+x)^2 + C_1 x + C_2 x^2. \quad \dots (4)$$

From (4),
$$y(t) = (1+t)^2 + C_1 t + C_2 t^2. \quad \dots (5)$$

Using (5), (2) reduces to

$$C_1 = \int_{-1}^1 t[(1+t)^2 + C_1 t + C_2 t^2] dt \quad \text{or} \quad C_1 = \int_{-1}^1 t[1 + (2+C_1)t + (1+C_2)t^2] dt$$

or
$$C_1 = \left[\frac{t^2}{2} \right]_{-1}^1 + (2+C_1) \left[\frac{t^3}{3} \right]_{-1}^1 + (1+C_2) \left[\frac{t^4}{4} \right]_{-1}^1$$

or
$$C_1 = (2/3) \times (2+C_1) \quad \text{so that} \quad C_1 = 4 \quad \dots (6)$$

Using (5), (3) reduces to

$$C_2 = \int_{-1}^1 t^2[(1+t)^2 + C_1 t + C_2 t^2] dt = \int_{-1}^1 t^2[1 + (2+C_1)t + (1+C_2)t^2] dt$$

or
$$C_2 = \left[\frac{t^3}{3} \right]_{-1}^1 + (2+C_1) \left[\frac{t^4}{4} \right]_{-1}^1 + (1+C_2) \left[\frac{t^5}{5} \right]_{-1}^1$$

or
$$C_2 = 2/3 + (1+C_2) \times (2/5) \quad \text{or} \quad C_2 = 16/9. \quad \dots (7)$$

Using (6) and (7), (4) gives the required solution

$$y(x) = (1+x)^2 + 4x + (16/9) \times x^2 \quad \text{or} \quad y(x) = 1 + 6x + (25/9) \times x^2.$$

Ex.9. Solve : $y(x) = \cos x + \lambda \int_0^\pi \sin(x-t) y(t) dt.$ [Merrut 2007]

Sol. Given
$$y(x) = \cos x + \lambda \int_0^\pi \sin(x-t) y(t) dt.$$

or
$$y(x) = \cos x + \lambda \int_0^\pi (\sin x \cos t - \cos x \sin t) y(t) dt$$

or
$$y(x) = \cos x + \lambda \sin x \int_0^\pi \cos t y(t) dt - \lambda \cos x \int_0^\pi \sin t y(t) dt. \quad \dots (1)$$

Let
$$C_1 = \int_0^\pi \cos t \, y(t) \, dt \quad \dots (2)$$

and
$$C_2 = \int_0^\pi \sin t \, y(t) \, dt. \quad \dots (3)$$

Using (2) and (3), (1) reduces to
$$y(x) = \cos x + \lambda C_1 \sin x - \lambda C_2 \cos x. \quad \dots (4)$$

From (4),
$$y(t) = \cos t + \lambda C_1 \sin t - \lambda C_2 \cos t. \quad \dots (5)$$

Using (5), (2) reduces to

$$C_1 = \int_0^\pi \cos t (\cos t + \lambda C_1 \sin t - \lambda C_2 \cos t) \, dt = \int_0^\pi [(1 - \lambda C_2) \cos^2 t \, dt + \frac{1}{2} \lambda C_1 \sin 2t] \, dt$$

or
$$C_1 = (1 - \lambda C_2) \int_0^\pi \frac{1 + \cos 2t}{2} \, dt + \frac{\lambda C_1}{2} \int_0^\pi \sin 2t \, dt = \frac{1 - \lambda C_2}{2} \left[t + \frac{\sin 2t}{2} \right]_0^\pi + \frac{\lambda C_1}{2} \left[-\frac{\cos 2t}{2} \right]_0^\pi$$

or
$$C_1 = (1 - \lambda C_2) \times (\pi/2) \quad \text{or} \quad 2C_1 + \lambda \pi C_2 = \pi. \quad \dots (6)$$

Using (5), (3) reduces to

$$C_2 = \int_0^\pi \sin t (\cos t + \lambda C_1 \sin t - \lambda C_2 \cos t) \, dt = \frac{1 - \lambda C_2}{2} \int_0^\pi \sin 2t \, dt + \frac{\lambda C_1}{2} \int_0^\pi (1 - \cos 2t) \, dt$$

or
$$C_2 = \frac{1 - \lambda C_2}{2} \left[-\frac{\cos 2t}{2} \right]_0^\pi + \frac{\lambda C_1}{2} \left[t - \frac{\sin 2t}{2} \right]_0^\pi$$

or
$$C_2 = (\lambda C_1 \pi) / 2 \quad \dots (7)$$

Solving (6) and (7) for C_1 and C_2 , we get

$$C_1 = (2\pi) / (4 + \lambda^2 \pi^2) \quad \text{and} \quad C_2 = (\lambda \pi^2) / (4 + \lambda^2 \pi^2).$$

Putting these values of C_1 and C_2 in (4), the required solution is

$$y(x) = \cos x + \frac{2\pi\lambda \sin x}{4 + \lambda^2 \pi^2} - \frac{\lambda^2 \pi^2 \cos x}{4 + \lambda^2 \pi^2} \quad \text{or} \quad y(x) = \cos x \left[1 - \frac{\lambda^2 \pi^2}{4 + \lambda^2 \pi^2} \right] + \frac{2\pi\lambda \sin x}{4 + \lambda^2 \pi^2}$$

$$y(x) = (4 \cos x + 2\pi\lambda \sin x) / (4 + \lambda^2 \pi^2)$$

Ex. 10. Solve : $y(x) = f(x) + \lambda \int_0^1 (x+t) y(t) \, dt.$

Sol. Given
$$y(x) = f(x) + \lambda \int_0^1 (x+t) y(t) \, dt.$$

or
$$y(x) = f(x) + \lambda x \int_0^1 y(t) \, dt + \lambda \int_0^1 t y(t) \, dt. \quad \dots (1)$$

Let
$$C_1 = \int_0^1 y(t) \, dt \quad \dots (2)$$

and
$$C_2 = \int_0^1 t y(t) \, dt. \quad \dots (3)$$

Using (2) and (3), (1) reduces to
$$y(x) = f(x) + \lambda x C_1 + \lambda C_2. \quad \dots (4)$$

From (4),
$$y(t) = f(t) + \lambda t C_1 + \lambda C_2. \quad \dots (5)$$

4.10 Fredholm Integral Equations of the Second Kind With Separable (or Degenerate) Kernels

Using (4), (2) reduces to

$$C_1 = \int_0^1 [f(t) + \lambda t C_1 + \lambda C_2] dt \quad \text{or} \quad C_1 = \int_0^1 f(t) dt + \lambda C_1 \left[\frac{t^2}{2} \right]_0^1 + \lambda C_2 [t]_0^1$$

$$\text{or} \quad C_1 = f_1 + (\lambda/2) \times C_1 + \lambda C_2, \quad \dots (6)$$

$$\text{where} \quad f_1 = \int_0^1 f(t) dt. \quad \dots (7)$$

Using (4), (3) reduces to

$$C_2 = \int_0^1 t [f(t) + \lambda t C_1 + \lambda C_2] dt \quad \text{or} \quad C_2 = \int_0^1 t f(t) dt + \lambda C_1 \left[\frac{t^3}{3} \right]_0^1 + \lambda C_2 \left[\frac{t^2}{2} \right]_0^1$$

$$\text{or} \quad C_2 = f_2 + (\lambda/3) \times C_1 + (\lambda/2) \times C_2 \quad \dots (8)$$

$$\text{where} \quad f_2 = \int_0^1 t f(t) dt. \quad \dots (9)$$

Re-writing (6) and (8), we have

$$(2 - \lambda) C_1 - 2\lambda C_2 = 2f_1 \quad \dots (10)$$

$$\text{and} \quad -2\lambda C_1 + 3(2 - \lambda) C_2 = 6f_2. \quad \dots (11)$$

Solving (10) and (11) for C_1 and C_2 , we get

$$C_1 = \frac{6(\lambda - 2)f_1 - 12\lambda f_2}{\lambda^2 + 12\lambda - 12} \quad \text{and} \quad C_2 = \frac{-4\lambda f_1 + 6(\lambda - 2)f_2}{\lambda^2 + 12\lambda - 12}$$

Putting these values of C_1 and C_2 in (4), the required solution is

$$y(x) = f(x) + \frac{\lambda x \{6(\lambda - 2)f_1 - 12\lambda f_2\}}{\lambda^2 + 12\lambda - 12} + \lambda \frac{-4\lambda f_1 + 6(\lambda - 2)f_2}{\lambda^2 + 12\lambda - 12}$$

$$\text{or} \quad y(x) = f(x) + \lambda \frac{f_1 \{6x(\lambda - 2) - 4\lambda\} + f_2 \{6(\lambda - 2) - 12\lambda x\}}{\lambda^2 + 12\lambda - 12}$$

$$\text{or} \quad y(x) = f(x) + \frac{\lambda}{\lambda^2 + 12\lambda - 12} \left[\{6x(\lambda - 2) - 4\lambda\} \int_0^1 f(t) dt + \{6(\lambda - 2) - 12\lambda x\} \int_0^1 t f(t) dt \right]$$

$$\text{or} \quad y(x) = f(x) + \frac{\lambda}{\lambda^2 + 12\lambda - 12} \left[\int_0^1 \{6x(\lambda - 2) - 4\lambda\} f(t) dt + \int_0^1 \{6(\lambda - 2) - 12\lambda x\} t f(t) dt \right]$$

$$\text{or} \quad y(x) = f(x) + \frac{\lambda}{\lambda^2 + 12\lambda - 12} \int_0^1 \{6(\lambda - 2)(x + t) - 12\lambda xt - 4\lambda\} f(t) dt$$

$$\text{or} \quad y(x) = f(x) + \lambda \int_0^1 \frac{6(\lambda - 2)(x + t) - 12\lambda xt - 4\lambda}{\lambda^2 + 12\lambda - 12} f(t) dt.$$

Ex. 11. Solve : $y(x) = f(x) + \lambda \int_{-1}^1 (xt + x^2 t^2) y(t) dt$. Find its resolvent kernel also.

[Meerut 2009; Kanpur 2006]

Sol. Given
$$y(x) = f(x) + \lambda \int_{-1}^1 (xt + x^2 t^2) y(t) dt$$

or
$$y(x) = f(x) + \lambda x \int_{-1}^1 t y(t) dt + \lambda x^2 \int_{-1}^1 t^2 y(t) dt. \quad \dots (1)$$

Let
$$C_1 = \int_{-1}^1 t y(t) dt \quad \dots (2)$$

and
$$C_2 = \int_{-1}^1 t^2 y(t) dt. \quad \dots (3)$$

Using (2) and (3), (1) reduces to
$$y(x) = f(x) + \lambda C_1 x + \lambda C_2 x^2. \quad \dots (4)$$

From (4),
$$y(t) = f(t) + \lambda C_1 t + \lambda C_2 t^2. \quad \dots (5)$$

Using (5), (2) reduces to

$$C_1 = \int_{-1}^1 t [f(t) + \lambda C_1 t + \lambda C_2 t^2] dt = \int_{-1}^1 t f(t) dt + \lambda C_1 \left[\frac{t^3}{3} \right]_{-1}^1 + \lambda C_2 \left[\frac{t^4}{4} \right]_{-1}^1$$

or
$$C_1 = \int_{-1}^1 t f(t) dt + \frac{2\lambda C_1}{3} \quad \text{or} \quad C_1 \left(1 - \frac{2\lambda}{3} \right) = \int_{-1}^1 t f(t) dt$$

or
$$C_1 = \frac{3}{3 - 2\lambda} \int_{-1}^1 t f(t) dt. \quad \dots (5)$$

Using (5), (3) reduces to

$$C_2 = \int_{-1}^1 t^2 [f(t) + \lambda C_1 t + \lambda C_2 t^2] dt = \int_{-1}^1 t^2 f(t) dt + \lambda C_1 \left[\frac{t^4}{4} \right]_{-1}^1 + \lambda C_2 \left[\frac{t^5}{5} \right]_{-1}^1$$

or
$$C_2 = \int_{-1}^1 t^2 f(t) dt + \frac{2\lambda C_2}{5}$$

or
$$C_2 \left(1 - \frac{2\lambda}{5} \right) = \int_{-1}^1 t^2 f(t) dt \quad \text{or} \quad C_2 = \frac{5}{5 - 2\lambda} \int_{-1}^1 t^2 f(t) dt. \quad \dots (7)$$

Using (6) and (7) in (4), the required solution is

$$y(x) = f(x) + \frac{3\lambda x}{3 - 2\lambda} \int_{-1}^1 t f(t) dt + \frac{5\lambda x^2}{5 - 2\lambda} \int_{-1}^1 t^2 f(t) dt$$

or
$$y(x) = f(x) + \lambda \int_{-1}^1 \left\{ \frac{3xt}{3 - 2\lambda} + \frac{5x^2 t^2}{5 - 2\lambda} \right\} f(t) dt. \quad \dots (8)$$

The required resolvent kernel $R(x, t; \lambda)$ is given by
$$R(x, t; \lambda) = \frac{3xt}{3 - 2\lambda} + \frac{5x^2 t^2}{5 - 2\lambda}.$$

Ex. 12. Solve the integral equation $y(x) - \lambda \int_{-\pi}^{\pi} (x \cos t + t^2 \sin x + \cos x \sin t) y(t) dt = x$.

Sol. Given
$$y(x) = x + \lambda \int_{-\pi}^{\pi} (x \cos t + t^2 \sin x + \cos x \sin t) y(t) dt$$

or
$$y(x) = x + \lambda x \int_{-\pi}^{\pi} \cos t y(t) dt + \lambda \sin x \int_{-\pi}^{\pi} t^2 y(t) dt + \lambda \cos x \int_{-\pi}^{\pi} \sin t y(t) dt. \quad \dots (1)$$

4.12 *Fredholm Integral Equations of the Second Kind With Separable (or Degenerate) Kernels*

Let
$$C_1 = \int_{-\pi}^{\pi} \cos t \, y(t) \, dt, \quad \dots (2)$$

$$C_2 = \int_{-\pi}^{\pi} t^2 \, y(t) \, dt, \quad \dots (3)$$

and
$$C_3 = \int_{-\pi}^{\pi} \sin t \, y(t) \, dt. \quad \dots (4)$$

Using (2), (3) and (4), (1) reduces to

$$y(x) = x + \lambda C_1 x + \lambda C_2 \sin x + \lambda C_3 \cos x. \quad \dots (5)$$

From (5),
$$y(t) = t + \lambda C_1 t + \lambda C_2 \sin t + \lambda C_3 \cos t. \quad \dots (6)$$

Using (6), (2) reduces to
$$C_1 = \int_{-\pi}^{\pi} \cos t (t + \lambda C_1 t + \lambda C_2 \sin t + \lambda C_3 \cos t) \, dt$$

or
$$C_1 = (1 + \lambda C_1) \int_{-\pi}^{\pi} t \cos t \, dt + \lambda C_2 \int_{-\pi}^{\pi} \sin t \cos t \, dt + \lambda C_3 \int_{-\pi}^{\pi} \cos^2 t \, dt$$

or
$$C_1 = 0 + 0 + 2\lambda C_3 \int_0^{\pi} \cos^2 t \, dt$$

[$\because t \cos t$ and $\sin t \cos t$ are odd functions whereas $\cos^2 t$ is an even function]

or
$$C_1 = 2\lambda C_3 \int_0^{\pi} \frac{(1 + \cos 2t)}{2} \, dt \quad \text{or} \quad C_1 = \lambda C_3 \left[t + \frac{\sin 2t}{2} \right]_0^{\pi}$$

$\therefore C_1 - \lambda \pi C_3 = 0. \quad \dots (7)$

Using (6), (3) reduces to
$$C_2 = \int_{-\pi}^{\pi} t^2 [t + \lambda C_1 t + \lambda C_2 \sin t + \lambda C_3 \cos t] \, dt$$

or
$$C_2 = (1 + \lambda C_1) \int_{-\pi}^{\pi} t^3 \, dt + \lambda C_2 \int_{-\pi}^{\pi} t^2 \sin t \, dt + \lambda C_3 \int_{-\pi}^{\pi} t^2 \cos t \, dt$$

or
$$C_2 = 2\lambda C_3 \int_0^{\pi} t^2 \cos t \, dt = 2\lambda C_3 \left[\left[t^2 \sin t \right]_0^{\pi} - \int_0^{\pi} 2t \sin t \, dt \right], \text{ integrating by parts}$$

or
$$C_2 = -4\lambda C_3 \int_0^{\pi} t \sin t \, dt = -4\lambda C_3 \left[\left[(t)(-\cos t) \right]_0^{\pi} - \int_0^{\pi} (-\cos t) \, dt \right]$$

or
$$C_2 = -4\lambda C_3 \left[\pi + \int_0^{\pi} \cos t \, dt \right] \quad \text{or} \quad C_2 = -4\lambda \pi C_3 - 4\lambda \pi C_3 [\sin t]_0^{\pi}$$

or
$$C_2 + 4\lambda \pi C_3 = 0. \quad \dots (8)$$

Using (6), (4) reduces to
$$C_3 = \int_{-\pi}^{\pi} \sin t (t + \lambda C_1 t + \lambda C_2 \sin t + \lambda C_3 \cos t) \, dt$$

or
$$C_3 = (1 + \lambda C_1) \int_{-\pi}^{\pi} t \sin t \, dt + \lambda C_2 \int_{-\pi}^{\pi} \sin^2 t \, dt + \lambda C_3 \int_{-\pi}^{\pi} \sin t \cos t \, dt$$

or
$$C_3 = 2(1 + \lambda C_1) \int_0^{\pi} t \sin t \, dt + 2\lambda C_2 \int_0^{\pi} \sin^2 t \, dt + 0$$

or
$$C_3 = 2(1 + \lambda C_1) \left[[t(-\cos t)]_0^\pi - \int_0^\pi (-\cos t) dt \right] + 2\lambda C_2 \int_0^\pi \frac{1 - \cos 2t}{2} dt$$

or
$$C_3 = 2(1 + \lambda C_1) \left[\pi + [\sin t]_0^\pi \right] + \lambda C_2 \left[t - \frac{\sin 2t}{2} \right]_0^\pi$$

or
$$C_3 = 2(1 + \lambda C_1)\pi + \lambda C_2\pi \quad \text{or} \quad -2\lambda\pi C_1 - \lambda\pi C_2 + C_3 = 2\pi \quad \dots (9)$$

Solving (7), (8) and (9) for C_1 , C_2 and C_3 , we have

$$C_1 = \frac{2\pi^2\lambda}{1 + 2\lambda^2\pi^2}, \quad C_2 = \frac{-8\pi^2\lambda}{1 + 2\lambda^2\pi^2} \quad \text{and} \quad C_3 = \frac{2\pi}{1 + 2\lambda^2\pi^2}.$$

Putting these values of C_1 , C_2 and C_3 in (5), the required solution is

$$y(x) = x + \frac{2\pi^2\lambda^2x}{1 + 2\lambda^2\pi^2} - \frac{8\pi^2\lambda^2\sin x}{1 + 2\lambda^2\pi^2} + \frac{2\pi\lambda\cos x}{1 + 2\lambda^2\pi^2}$$

or
$$y(x) = x + \frac{2\pi\lambda}{1 + 2\lambda^2\pi^2} (\lambda\pi x - 4\lambda\pi\sin x + \cos x).$$

Ex. 13. Show that the integral equation

$$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x+t) y(t) dt$$

possesses no solution for $f(x) = x$, but that it possesses infinitely many solutions when $f(x) = 1$.

Sol. Given

$$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x+t) y(t) dt$$

or
$$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} (\sin x \cos t + \cos x \sin t) y(t) dt$$

or
$$y(x) = f(x) + \frac{\sin x}{\pi} \int_0^{2\pi} \cos t y(t) dt + \frac{\cos x}{\pi} \int_0^{2\pi} \sin t y(t) dt. \quad \dots (1)$$

Let
$$C_1 = \int_0^{2\pi} \cos t y(t) dt \quad \dots (2)$$

and
$$C_2 = \int_0^{2\pi} \sin t y(t) dt. \quad \dots (3)$$

Using (2) and (3), (1) reduces to
$$y(x) = f(x) + (C_1/\pi) \times \sin x + (C_2/\pi) \times \cos x \quad \dots (4)$$

We now discuss two particular cases as mentioned in the problem.

Case I. Let $f(x) = x$. Then (4) reduces to

$$y(x) = x + (C_1/\pi) \times \sin x + (C_2/\pi) \times \cos x \quad \dots (5)$$

From (5),
$$y(t) = t + (C_1/\pi) \sin t + (C_2/\pi) \cos t \quad \dots (6)$$

Using (6), (2) becomes

$$C_1 = \int_0^{2\pi} \cos t \left(t + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi} \right) dt$$

$$= \int_0^{2\pi} t \cos t \, dt + \frac{C_1}{2\pi} \int_0^{2\pi} \sin 2t \, dt + \frac{C_2}{2\pi} \int_0^{2\pi} (1 + \cos 2t) \, dt$$

$$\text{or} \quad C_1 = [t \sin t]_0^{2\pi} - \int_0^{2\pi} \sin t \, dt + \frac{C_1}{2\pi} \left[-\frac{\cos 2t}{2} \right]_0^{2\pi} + \frac{C_2}{2\pi} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi}$$

$$\text{or} \quad C_1 = -[-\cos t]_0^{2\pi} + \frac{C_2}{2\pi} [2\pi + 0] \quad \text{or} \quad C_1 - C_2 = 0 \quad \dots (7)$$

Again using (6), (3) becomes

$$C_2 = \int_0^{2\pi} \sin t \left(t + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi} \right) dt = \int_0^{2\pi} t \sin t \, dt + \frac{C_1}{2\pi} \int_0^{2\pi} (1 - \cos 2t) \, dt + \frac{C_2}{2\pi} \int_0^{2\pi} \sin 2t \, dt$$

$$\text{or} \quad C_2 = [-t \cos t]_0^{2\pi} - \int_0^{2\pi} (-\cos t) \, dt + \frac{C_1}{2\pi} \left[t - \frac{\sin 2t}{2} \right]_0^{2\pi} + \frac{C_2}{2\pi} \left[-\frac{\cos 2t}{2} \right]_0^{2\pi}$$

$$\text{or} \quad C_2 = -2\pi + [\sin t]_0^{2\pi} + (C_1 / 2\pi) \times (2\pi + 0) \quad \text{or} \quad C_1 - C_2 = 2\pi. \quad \dots (8)$$

The system of equations (7) and (8) is inconsistent and so it possesses no solution.

Hence C_1 and C_2 cannot be determined and so (5) shows that the given integral equation possesses no solution when $f(x) = x$.

Case II. Let $f(x) = 1$. Then (4) reduces to

$$y(x) = 1 + (C_1 / \pi) \times \sin x + (C_2 / \pi) \times \cos x \quad \dots (9)$$

$$\text{From (9),} \quad y(t) = 1 + (C_1 / \pi) \times \sin t + (C_2 / \pi) \times \cos t \quad \dots (10)$$

Using (6), (2) becomes

$$C_1 = \int_0^{2\pi} \cos t \left(1 + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi} \right) dt = \int_0^{2\pi} \cos t \, dt + \frac{C_1}{2\pi} \int_0^{2\pi} \sin 2t \, dt + \frac{C_2}{2\pi} \int_0^{2\pi} (1 + \cos 2t) \, dt$$

$$\text{or} \quad C_1 = [\sin t]_0^{2\pi} + \frac{C_1}{2\pi} \left[-\frac{\cos 2t}{2} \right]_0^{2\pi} + \frac{C_2}{2\pi} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi}$$

$$\text{or} \quad C_1 = 0 + 0 + (C_2 / 2\pi) \times (2\pi + 0) \quad \text{or} \quad C_1 = C_2 \quad \dots (11)$$

Again using (6), (3) becomes

$$C_2 = \int_0^{2\pi} \sin t \left(1 + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi} \right) dt = \int_0^{2\pi} \sin t \, dt + \frac{C_1}{2\pi} \int_0^{2\pi} (1 - \cos 2t) \, dt + \frac{C_2}{2\pi} \int_0^{2\pi} \sin 2t \, dt$$

$$\text{or} \quad C_2 = [-\cos t]_0^{2\pi} + \frac{C_1}{2\pi} \left[t - \frac{\sin 2t}{2} \right]_0^{2\pi} + \frac{C_2}{2\pi} \left[-\frac{\cos 2t}{2} \right]_0^{2\pi}$$

$$\text{or} \quad C_2 = 0 + (C_1 / 2\pi) \times (2\pi + 0) + 0 \quad \text{or} \quad C_1 = C_2. \quad \dots (12)$$

From (11) and (12), we see that $C_1 = C_2 = C'$, (say). Here C' is an arbitrary constant. Thus, the system (11) – (12) has infinite number of solutions $C_1 = C'$ and $C_2 = C'$. Putting these values in (9), the required solution of given integral equation is

$$y(x) = 1 + (C' / \pi) \times (\sin x + \cos x) \quad \text{or} \quad y(x) = 1 + C (\sin x + \cos x).$$

where $C = (C' / \pi)$ is another arbitrary constant. Since C is an arbitrary constant, we have infinitely many solutions of (1) when $f(x) = 1$.

Ex. 14. Solve the following integral equation and discuss all its possible cases

$$y(x) = f(x) + \lambda \int_0^1 (1 - 3xt) y(t) dt. \quad [\text{Meerut 2002, 04, 06, 08, 10, 11, 12}]$$

Sol. Given
$$y(x) = f(x) + \lambda \int_0^1 (1 - 3xt) y(t) dt. \quad \dots (1)$$

or
$$y(x) = f(x) + \lambda \int_0^1 y(t) dt - 3x\lambda \int_0^1 t y(t) dt \quad \dots (2)$$

Let
$$C_1 = \int_0^1 y(t) dt \quad \dots (3)$$

and
$$C_2 = \int_0^1 t y(t) dt. \quad \dots (4)$$

Using (3) and (4), (2) reduces to

$$y(x) = f(x) + C_1\lambda - 3x C_2\lambda. \quad \dots (5)$$

From (5),
$$y(t) = f(t) + C_1\lambda - 3t C_2\lambda. \quad \dots (6)$$

Using (6) and (3) becomes

$$C_1 = \int_0^1 [f(t) + C_1\lambda - 3t C_2\lambda] dt \quad \text{or} \quad C_1 = \int_0^1 f(t) dt + C_1\lambda [t]_0^1 - 3C_2\lambda \left[\frac{t^2}{2} \right]_0^1$$

or
$$C_1 = \int_0^1 f(x) dx + C_1\lambda - \frac{3}{2} C_2\lambda \quad \dots (7)$$

Using (6), (4) becomes

$$C_2 = \int_0^1 t [f(t) + C_1\lambda - 3t C_2\lambda] dt \quad \text{or} \quad C_2 = \int_0^1 t f(t) dt + C_1\lambda \left[\frac{t^2}{2} \right]_0^1 - 3C_2\lambda \left[\frac{t^3}{3} \right]_0^1$$

or
$$C_2 = \int_0^1 x f(x) dx + \frac{1}{2} C_1\lambda - C_2\lambda. \quad \dots (8)$$

Re-writing (7) and (8), we have

$$(1 - \lambda) C_1 + \frac{3\lambda}{2} C_2 = \int_0^1 f(x) dx \quad \dots (9)$$

and
$$-\frac{1}{2} \lambda C_1 + (1 + \lambda) C_2 = \int_0^1 x f(x) dx. \quad \dots (10)$$

The determinant of coefficients of the system of equations (9) and (10) is given by

$$D(\lambda) = \begin{vmatrix} 1 - \lambda & \frac{3\lambda}{2} \\ -\frac{1}{2} \lambda & 1 + \lambda \end{vmatrix} = \frac{1}{4} (4 - \lambda^2). \quad \dots (11)$$

Hence a unique solution of the system (9) and (10) exists if and only if $D(\lambda) \neq 0$ i.e., $\lambda \neq \pm 2$. When $D(\lambda) \neq 0$, C_1 and C_2 can be determined by solving (9) and (10). By putting the values of C_1 and C_2 so obtained in (5), the required unique solution of (1) can be obtained. In

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particular, if $f(x) = 0$ and $\lambda \neq \pm 2$, the only zero solution $C_1 = C_2 = 0$ is obtained from (9) and (10) and hence we get trivial solution $\phi(x) = 0$ for (1). The numbers $\lambda = \pm 2$ are the eigenvalues of the problem.

If $\lambda = 2$, equations (9) and (10) become

$$-C_1 + 3C_2 = \int_0^1 f(x) dx \quad \dots (12)$$

and
$$-C_1 + 3C_2 = \int_0^1 x f(x) dx. \quad \dots (13)$$

While if $\lambda = -2$, equations (9) and (10) become

$$C_1 - C_2 = \frac{1}{3} \int_0^1 f(x) dx \quad \dots (14)$$

and
$$C_1 - C_2 = \int_0^1 x f(x) dx. \quad \dots (15)$$

Equations (12) and (13) are incompatible (*i.e.*, possess no solution) unless the given function $f(x)$ satisfies the condition

$$\int_0^1 f(x) dx = \int_0^1 x f(x) dx \quad \text{or} \quad \int_0^1 (1-x) f(x) dx. \quad \dots (16)$$

When condition (16) is satisfied, the equations (12) and (13) are redundant (*i.e.*, identical and hence possess infinitely many solutions.)

Similarly, equations (14) and (15) are incompatible unless

$$\frac{1}{3} \int_0^1 f(x) dx = \int_0^1 x f(x) dx \quad \text{or} \quad \int_0^1 (1-3x) f(x) dx = 0. \quad \dots (17)$$

When condition (17) is satisfied, the equations (14) and (15) are again redundant.

We now discuss solution of (1). Two cases arise :

Case I. When $f(x) = 0$. Then (1) reduces to homogeneous equation

$$y(x) = \lambda \int_0^1 (1-3xt) y(t) dt. \quad \dots (18)$$

Then if $\lambda \neq \pm 2$, (1) has only trivial solution $y(x) = 0$, as mentioned above.

For non-trivial solution of (18), we have $\lambda = \pm 2$. Hence the eigenvalues are $\lambda = \pm 2$.

To find eigenfunction corresponding to $\lambda = 2$, we use (12) and (13) with $f(x) = 0$. These give $C_1 = 3C_2$ and so (5) becomes

$$y(x) = 2(3C_2 - 3x C_2) = 6C_2(1-x) = A(1-x),$$

where $A (= 6C_2)$ is an arbitrary constant.

Thus the function $1-x$ (or any convenient non-zero multiple of that function) is the eigenfunction corresponding to the eigenvalue $\lambda = 2$.

Next, to find eigenfunction corresponding to $\lambda = -2$, we use (14) and (15). With $f(x) = 0$, these give $C_1 = C_2$ and so (5) becomes

$$y(x) = -2C_1(1-3x) = B(1-3x),$$

where $B (= -2C_1)$ is an arbitrary constant.

Thus the function $1-3x$ (or any convenient non-zero multiple of that function) is the eigenfunction corresponding to the eigenvalue $\lambda = -2$.

Case II. Let $f(x) \neq 0$. Then (1) is non-homogeneous integral equation. Three cases arise.

(i) **When $\lambda \neq \pm 2$.** (1) possesses a unique solution as explained above.

(ii) **When $\lambda = 2$.** Equations (12) and (13) show that no solution exists unless $f(x)$ is orthogonal to $1 - x$ over the relevant interval $(0, 1)$, that is, unless $f(x)$ is orthogonal to the eigenfunction corresponding to $\lambda = 2$. When $f(x)$ satisfies this restriction, equations (12) and (13) are identical and these give us

$$C_1 = 3C_2 - \int_0^1 f(x) dx.$$

Putting this value of C_1 in (5), we get

$$y(x) = f(x) + \lambda [3C_2 - \int_0^1 f(x) dx] - 3x C_2 \lambda$$

or
$$y(x) = f(x) - 2 \int_0^1 f(x) dx + 6C_2 (1-x), \text{ as } \lambda = 2$$

or
$$y(x) = f(x) - 2 \int_0^1 f(x) dx A(1-x), \quad \dots (19)$$

where $A = (6C_2)$ is an arbitrary constant.

Thus, if $\lambda = 2$ and $\int_0^1 (1-x) f(x) dx = 0$, the given equation (1) possesses infinitely many solutions given by (19).

(iii) **When $\lambda = -2$.** Equation (14) and (15) show that no solution exists unless $f(x)$ is orthogonal to $1 - 3x$ over the relevant interval $(0, 1)$, that is, unless $f(x)$ is orthogonal to the eigenfunction corresponding to $\lambda = -2$. When $f(x)$ satisfies this restriction, equations (14) and (15) are identical and these give us

$$C_1 = C_2 + \frac{1}{3} \int_0^1 f(x) dx$$

Putting this value of C_1 in (5), we get

$$y(x) = f(x) + \lambda [C_2 + \frac{1}{3} \int_0^1 f(x) dx] - 3xC_2 \lambda$$

or
$$y(x) = f(x) - \frac{2}{3} \int_0^1 f(x) dx - 2C_2 (1-3x), \text{ as } \lambda = -2$$

or
$$y(x) = f(x) - \frac{2}{3} \int_0^1 f(x) dx + B (1-3x), \quad \dots (20)$$

where $B = (-2C_2)$ is an arbitrary constant.

Thus, if $\lambda = -2$ and $\int_0^1 (1-3x) f(x) dx = 0$, the given equation (1) possesses infinitely many solutions given by (20).

EXERCISE 4 (a)

1. Solve the following integral equations :

$$(i) \quad y(x) = \tan x + \int_{-1}^1 e^{\sin^{-1} x} y(t) dt. \quad (ii) \quad y(x) = \sin x + \lambda \int_0^{\pi/2} \sin x \cos t y(t) dt.$$

$$(iii) \quad y(x) = \sec x \tan x - \lambda \int_0^1 y(t) dt \quad (iv) \quad y(x) = \frac{1}{\sqrt{1-x^2}} + \lambda \int_0^1 \cos^{-1} t y(t) dt.$$

$$(v) \quad y(x) = \sec^2 x + \lambda \int_0^1 y(t) dt. \quad (vi) \quad y(x) - \lambda \int_{-\pi/4}^{\pi/4} \tan t y(t) dt = \cot x$$

[Kanpur 2007]

$$(vii) \quad y(x) - \lambda \int_0^1 \cos(q \ln t) y(t) dt = 1. \quad (viii) \quad y(x) - \lambda \int_0^1 \left(\ln \frac{1}{t} \right)^p y(t) dt = 1. \quad (p > -1).$$

2. Solve the following integral equations :

$$(i) \quad y(x) - \lambda \int_0^1 (x \ln t - t \ln x) y(t) dt = \frac{6}{5}(1-4x), \text{ where } \ln t = \log_e t$$

$$(ii) \quad y(x) - \lambda \int_0^{2\pi} |\pi - t| \sin x y(t) dt = x.$$

$$(iii) \quad y(x) = x + \lambda \int_0^1 (1+x+t) y(t) dt. \quad (iv) \quad y(x) = x + \lambda \int_0^{\pi} (1 + \sin x \sin t) y(t) dt.$$

[Meerut 2006, 10, 11; Kanpur 2011]

[Kanpur 2008]

$$(v) \quad y(x) - \lambda \int_0^1 (4xt - x^3) y(t) dt = x. \quad (vi) \quad y(x) - \lambda \int_0^1 (4xt - x^2) y(t) dt = x$$

3. Express the solution of the integral equation $y(x) = f(x) + \lambda \int_0^1 (1-3xt) y(t) dt$ in the form

$$y(x) = f(x) + \lambda \int_0^1 \Gamma(x, t; \lambda) f(t) dt, \text{ when } \lambda \neq \pm 2.$$

4. (a) Show that the characteristic values of λ for the equation $y(x) = \lambda \int_0^{2\pi} \sin(x+t) y(t) dt$

are $\lambda_1 = 1/\pi$ and $\lambda_2 = -1/\pi$, with corresponding characteristic functions of the form $y_1(x) = \sin x + \cos x$ and $y_2(x) = \sin x - \cos x$.

(b) Obtain the most general solution of the equation $y(x) = f(x) + \lambda \int_0^{2\pi} \sin(x+t) y(t) dt$

when $f(x) = x$ and when $f(x) = 1$, under the assumption that $\lambda \neq \pm 1/\pi$.

(c) Prove that the equation $y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x+t) y(t) dt$

possesses no solution when $f(x) = x$, but that it possesses infinitely many solutions when $f(x) = 1$. Determine all such solutions.

5. Consider the equation $y(x) = f(x) + \lambda \int_0^{2\pi} \cos(x+t) y(t) dt.$

(a) Determine the characteristic values of λ and the characteristic functions.

(b) Express the solution in the form $y(x) = f(x) + \lambda \int_0^{2\pi} \Gamma(x, t; \lambda) f(t) dt,$

when λ is not characteristic value.

- (c) Obtain the general solution (when it exists) if $f(x) = \sin x$, considering all possible cases.
6. Solve the equation $y(x) = 1 + \lambda \int_{-\pi}^{\pi} e^{i\omega(x-t)} y(t) dt$, considering separately all exceptional cases.
7. Obtain an approximate solution of the integral equation $y(x) = x^2 + \int_0^1 \sin(xt) y(t) dt$, by replacing $\sin(xt)$ by the first two terms of its power series development
- $$\sin(xt) = (xt) - \frac{(xt)^3}{3!} + \dots$$
8. Solve the integral equation $y(x) = f(x) + \lambda \int_0^{2\pi} \cos(x+t) y(t) dt$ and find the condition that $f(x)$ must satisfy in order that this equation has a solution when λ is an eigenvalue. Obtain the general solution if $f(x) = \sin x$ considering all possible cases.
9. Solve the integral equation $\phi(x) = \cos 3x + \lambda \int_0^1 \cos(x+t) \phi(t) dt$. Discuss all the cases.
- [Meerut 2000]
10. Find the eigenvalues of the equation $u(x) = f(x) + \lambda \int_0^{2\pi} \sin(x+t) u(t) dt$ (Kanpur 2011)

ANSWERS

1. (i) $y(x) = \tan x$. (ii) $y(x) = \{2/(2-\lambda)\} \sin x$, $\lambda \neq 2$.
 (iii) $y(x) = \sec x \tan x - \{\lambda/(1+\lambda)\} \sec 1$, $\lambda \neq -1$.
- (iv) $y(x) = \frac{1}{\sqrt{1-x^2}} - \frac{\lambda\pi^2}{8(\lambda-1)}$, $\lambda \neq 1$. (v) $y(x) = \sec^2 x + \{\lambda/(1-\lambda)\} \tan 1$, $\lambda \neq 1$.
- (vi) $y(x) = \cot x + (\pi\lambda)/2$. (vii) $y(x) = (1+q^2)/(1+q^2-\lambda)$.
 (viii) $y(x) = 1/\{1-\lambda \Gamma(p+1)\}$.
2. (i) $y(x) = \frac{6}{5}(1-4x) + \frac{2\lambda^2 x + (\lambda + \lambda^2/4) \ln x}{1 + (29/48)\lambda^2}$, where $\ln x = \log_e x$
- (ii) $y(x) = x + \lambda\pi^3 \sin x$. (iii) $y(x) = x + \frac{\lambda}{12 - 24\lambda - \lambda^2} [10 + (6 + \lambda)x]$
- (iv) $y(x) = x + \frac{\lambda}{(1-\lambda\pi)(1-\lambda\pi/2) + 4\lambda^2} \left[2\lambda\pi + \frac{1}{2}\pi^2(1-\lambda\pi/2) + \pi(1-2\lambda\pi)\sin x \right]$
- (v) $y(x) = [15(4+\lambda) - 30\lambda x^3]/(60 - 65\lambda + 4\lambda^2)$.
- (vi) $y(x) = [6(3+\lambda)x - 9\lambda x^2]/(18 - 18\lambda + \lambda^2)$.
3. $\Gamma(x, t; \lambda) = \frac{4}{4-\lambda^2} [1 + \lambda - \frac{3}{2}\lambda(x+t) - 3(1-\lambda)xt]$, $\lambda \neq \pm 2$.
4. (b) $f(x) = x$; $y(x) = \frac{2\pi^2\lambda^2}{\pi^2\lambda^2-1} \sin x + \frac{2\pi\lambda}{\pi^2\lambda^2-1} \cos x + x$; $f(x) = 1$; $y(x) = 1$.
 (c) $f(x) = 1$; $y(x) = 1 + C(\cos x + \sin x)$.

5. (a) $\lambda_1 = 1/\pi$, $y_1(x) = \cos x$; $\lambda_2 = -1/\pi$, $y_2(x) = \sin x$.

(b) $\Gamma(x, t; \lambda) = \frac{\cos(x+t) + \pi\lambda \cos(x-t)}{1 - \lambda^2 \pi^2}$, if $\lambda = \pm \frac{1}{\pi}$.

(c) $y(x) = \frac{\sin x}{1 + \pi\lambda}$ if $\lambda \neq \pm \frac{1}{\pi}$

$y(x) = \frac{1}{2} \sin x + A \cos x$, A arbitrary, if $\lambda = \frac{1}{\pi}$; No solution if $\lambda = -\frac{1}{\pi}$

6. $y(x) = 1 + \frac{2\lambda \sin \pi\omega}{(1 - 2\pi\lambda)\omega} e^{i\omega x}$, if $\lambda \neq \frac{1}{2\pi}$, $\omega \neq 0$; $y(x) = 1 + \frac{2\pi\lambda}{1 - 2\pi\lambda} e^{i\omega x}$, if $\lambda \neq \frac{1}{2\pi}$, $\omega = 0$;

No solution if $\lambda = 1/2\pi$.

7. $y(x) \approx 0.363x + x^2 - 0.039x^3$, when $\approx$ stands for approximately.

10. Required eigenvalues are $1/\pi$ and $-(1/\pi)$.

4.3 FREDHOLM ALTERNATIVE

In Art. 4.1, we have seen that, if the kernel is separable, the problem of solving an integral equation of the second kind reduces to that of solving an algebraic system of equations. Although the integral equation with separable kernel are not found frequently in practice, yet the results derived for such equations are essential to study integral equations of more general type. Furthermore, any reasonably well-behaved kernel can be expressed as an infinite series of degenerate kernels.

When an integral equation cannot be solved in closed form, then we have to use approximate methods to solve a given integral equation. However any approximate method can be employed with confidence only if the existence of the solution is known in advance. The Fredholm theorems proved in this chapter will provide an assurance for the existence of the solution of a given integral equation. The basic theorems of the general theory of integral equations given by Fredholm, correspond to the basic theorems of linear algebraic systems. Fredholm's classical theory will be explained in Chapter 6 for general kernels. In the present chapter we shall study integral equations with separable kernels and make use of the well known results of linear algebra.

Proceed exactly as in Art. 4.1 upto equation (10). *Then proceed as follows. We see that the required solution of (1) depends on the determinant $D(\lambda)$ given by (10).

Two case arises :

Case (i) If $D(\lambda) \neq 0$, then the system of equations $(B_1), (B_2) \dots, (B_n)$ has only one solution, given by Cramer's rule.

$$C_i = (D_{1i} \beta_1 + D_{2i} \beta_2 + \dots + D_{ni} \beta_n) / D(\lambda), \quad i = 1, 2, \dots, n. \quad \dots (11)$$

where D_{ki} denotes the cofactor of the (k, i) the element of the determinant $D(\lambda)$. Substituting the value of C_i given by (11) in (5), the unique solution of integral equation (1), is given by

$$y(x) = f(x) + \lambda \sum_{i=1}^n \frac{D_{1i} \beta_1 + D_{2i} \beta_2 + \dots + D_{ni} \beta_n}{D(\lambda)} f_i(x) \quad \dots (12)$$

Also observe that the corresponding homogeneous integral equation

$$y(x) = \lambda \int_a^b k(x, t) y(t) dt \quad \dots (13)$$

has only the trivial solution $y(x) = 0$

* In the entire discussion of the present article for equations (1) to (10), please refer Art. 4.1.

Substituting the values of $\beta_1, \beta_2, \dots, \beta_n$ given by (8) in (12), the unique solution of (1) can be re-written as

$$y(x) = f(x) + \frac{\lambda}{D(\lambda)} \int_a^b \left[\sum_{i=1}^n \{D_{1i}g_i(t) + D_{2i}g_2(t) + \dots + D_{ni}g_n(t)\} f_i(x) \right] f(t) dt \quad \dots (14)$$

Let us consider the following determinant of $(n+1)^{\text{th}}$ order

$$D(x, t : \lambda) = - \begin{vmatrix} 0 & f_1(x) & f_2(x) & \dots & f_n(x) \\ g_1(t) & 1 - \lambda\alpha_{11} & -\lambda\alpha_{12} & \dots & -\lambda\alpha_{1n} \\ g_2(t) & -\lambda\alpha_{21} & 1 - \lambda\alpha_{22} & \dots & -\lambda\alpha_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ g_n(t) & -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \dots & 1 - \lambda\alpha_{nn} \end{vmatrix} \quad \dots (15)$$

Now, by developing the determinant (15) by the elements of the first row and the corresponding minors of the first column, we have

$$\sum_{i=1}^n \{D_{1i}g_i(t) + D_{2i}g_2(t) + \dots + D_{ni}g_n(t)\} f_i(x) = D(x, t, \lambda)$$

Hence (14), can be re-written as

$$y(x) = f(x) + \lambda \int_a^b \frac{D(x, t : \lambda)}{D(\lambda)} f(t) dt$$

or

$$y(x) = f(x) + \lambda \int_a^b R(x, t : \lambda) f(t) dt, \quad \dots (16)$$

where

$$R(x, t : \lambda) = D(x, t : \lambda) / D(\lambda) \quad \dots (17)$$

The function $R(x, t : \lambda)$ is known as the *resolvent (or reciprocal) kernel* of the given integral equation (1). We note that the only possible singular points of $R(x, t : \lambda)$ in the λ -plane are the roots of the equation $D(\lambda) = 0$, i.e., the eigenvalues of the kernel $K(x, t)$.

In view of the above discussion, we have the following basic Fredholm theorem.

Fredholm Theorem. *The inhomogeneous Fredholm integral equation (1) with a separable kernel has unique solution, given by (16). The resolvent kernel $R(x, t : \lambda)$ is given by the quotient (17) of two polynomials.*

Case (ii) If $D(\lambda) \neq 0$, then (1) has no solution in general, because an algebraic system with $D(\lambda) = 0$ can be solved only for some particular values of $\beta_1, \beta_2, \dots, \beta_n$. In order to discuss this situation, we re-write the system of equations $(B_1), (B_2), \dots, (B_n)$ in matrix form as follows :

$$(I - \lambda A) C = B, \quad \dots (18)$$

$$\text{where } A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_n \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \dots \\ C_n \end{bmatrix}, \quad \dots (19)$$

and I is the unit (or identity) matrix of order n . Since $D(\lambda) = 0$, for each nontrivial solution of the homogeneous system

$$(I - \lambda A) C = 0 \quad \dots (20)$$

there corresponds a nontrivial solution (an eigenfunction) of the homogeneous integral equation (13). From linear algebra, we know that if λ is equal to certain eigenvalue λ_0 for which the determinant $D(\lambda_0) = |I - \lambda_0 A|$ has the rank $p, 1 \leq p \leq n$, then there are $r (= n - p)$ linearly independent solutions of the algebraic system $(B_1), (B_2), \dots, (B_n)$. r is known as the *index of the eigenvalue* λ_0 . The same result holds for the homogeneous integral equation (13). Let these r linearly independent solutions be also denoted by $y_{01}(x), y_{02}(x), \dots, y_{0r}(x)$ and let us assume that they have been normalized. Then, to each eigenvalue λ_0 of index $r (= n - p)$, there corresponds a solution $y_0(x)$ of (3) of the form

$$y_0(x) = \sum_{k=1}^r a_k y_{0k}(x),$$

where a_k are arbitrary constants.

Let m be the *multiplicity of the eigenvalue* λ_0 , i.e., $D(\lambda) = 0$ has m equal roots λ . Then from linear algebra, we know that by using the elementary transformations on the determinant $|I - \lambda A|$, we shall get at most $m + 1$ identical rows and this maximum is obtained only when A is symmetric. It follows that the rank p of $D(\lambda_0)$ is greater than or equal to $n - m$ and hence

$$r = n - p \leq n - (n - m), \quad \text{i.e.,} \quad r \leq m,$$

where the equality holds only when $\alpha_{ij} = \alpha_{ji}$

Thus we have proved the Fredholm theorem, namely, if $\lambda = \lambda_0$ is a root of multiplicity $m \geq 1$ of the equation $D(\lambda) = 0$, then the homogeneous equation (13) has r linearly independent solutions; r is the index of the eigenvalue such that $1 \leq r \leq m$.

The number r and m are known as the *geometric multiplicity* and *algebraic multiplicity* respectively. Since $r \leq m$, it follows that the algebraic multiplicity of an eigenvalue is always greater than or equal to its geometric multiplicity.

We now proceed with the situation when the inhomogeneous Fredholm integral equation (1) has solutions even when $D(\lambda) = 0$. To this end, we first define the *transpose* (or *adjoint*) of the equation (1). The integral equation

$$z(x) = f(x) + \lambda \int_a^b K(t, x) z(t) dt \quad \dots (21)$$

is known as the *transpose* (or *adjoint*) of the integral equation (1). Note that the relation between (1) and its transpose (21) is symmetric, since (1) is the transpose of (21).

Since the separable kernel $K(x, t)$ of (1) is given by (2), it follows that the kernel $K(t, x)$ of the transposed equation has the expansion

$$K(t, x) = \sum_{i=1}^n f_i(t) g_i(x) \quad \dots (22)$$

Proceeding as before in Art. 4.1 and the present article, we shall arrive at the algebraic system

$$(I - \lambda A^T) C = B, \quad \dots (23)$$

where A^T denotes the transpose of matrix A and where C_i and β_i are now given by the following relations

$$C_i = \int_a^b f_i(t) y(t) dt \quad \text{and} \quad \beta_i = \int_a^b f_i(t) f(t) dt \quad \dots (24)$$

Note carefully that the determinant $D(\lambda)$ for the system (23) is the same function (10) except that there has been interchange of rows and columns in view of the interchange in the functions f_i and g_r . Hence, the eigenvalues of the transposed equation (21) are the same as those of the original equation (1). It follows that the transposed equation (21) also possesses a unique solution whenever (1) does.

On the otherhand, from linear algebra, we know that the eigenfunctions of the homogeneous system

$$(I - \lambda A^T) C = 0,$$

are different from the corresponding eigenfunctions of the system (20). The same result is also applicable to the eigenfunctions of the transposed integral equation. Since the index r of eigenvalue λ_0 is the same in both these systems, the number of linearly independent eigenfunctions is also r for the transposed system. Let these r linearly independent solutions be denoted by $z_{01}, z_{02}, \dots, z_{0r}$ and let us also assume that they have been normalized. Then, any solution $z_0(x)$ of the transposed homogeneous integral equation

$$z(x) = \lambda \int_a^b K(t, x) z(t) dt \quad \dots (25)$$

corresponding to the eigenvalue λ_0 is of the form

$$z_0(x) = \sum_{i=1}^r b_i z_{0i}(x),$$

where b_i are arbitrary constants.

We now prove that eigenfunctions $y(x)$ and $z(x)$ corresponding to distinct eigenvalues λ_1 and λ_2 , respectively, of (13) and its transpose (25) are orthogonal.

Indeed, by definition of eigenfunction, we have

$$y(x) = \lambda_1 \int_a^b K(x, t) y(t) dt \quad \dots (26)$$

and

$$z(x) = \lambda_2 \int_a^b K(t, x) z(t) dt \quad \dots (27)$$

Multiplying both sides of (26) by $\lambda_2 z(x)$ and then integrating w.r.t. 'x' over the interval (a, b) , we get

$$\lambda_2 \int_a^b y(x) z(x) dx = \lambda_1 \lambda_2 \int_a^b z(x) \left\{ \int_a^b K(x, t) y(t) dt \right\} dx$$

or
$$\lambda_2 \int_a^b y(x) z(x) dx = \lambda_1 \lambda_2 \int_a^b y(t) \left\{ \int_a^b K(x, t) z(x) dx \right\} dt$$
 [on interchanging the order of integration]

or
$$\lambda_2 \int_a^b y(x) z(x) dx = \lambda_1 \lambda_2 \int_a^b y(x) \left\{ \int_a^b K(t, x) z(t) dt \right\} dx \quad \dots (28)$$

Again, multiplying both sides of (27) by $\lambda y(x)$ and then integrating w.r.t. 'x' over the interval (a, b) , we get

$$\lambda_1 \int_a^b y(x) z(x) dx = \lambda_1 \lambda_2 \int_a^b y(x) \left\{ \int_a^b K(t, x) z(t) dt \right\} dx \quad \dots (29)$$

From (28) and (29), we obtain

$$\lambda_2 \int_a^b y(x) z(x) dx = \lambda_1 \int_a^b y(x) z(x) dx \quad \text{or} \quad (\lambda_2 - \lambda_1) \int_a^b y(x) z(x) dx = 0$$

Since $\lambda_2 \neq \lambda_1$, we have
$$\int_a^b y(x) z(x) dx = 0,$$

showing that the eigenfunction $y(x)$ of (13) and eigenfunction $z(x)$ of (25) are orthogonal.

We now proceed to discuss the solution of (1) for the case $D(\lambda) = 0$. In what follows, we shall prove that *the necessary and sufficient condition for (1) to possess a solution for $\lambda = \lambda_0$, a root of $D(\lambda) = 0$, is that $f(x)$ be orthogonal to the r eigenfunctions z_{0i} of the transposed equation (25).*

Proof. The condition is necessary. Suppose that (1) for $\lambda = \lambda_0$ admits a certain solution $y(x)$. Then, we have

$$y(x) = f(x) + \lambda_0 \int_a^b K(x, t) y(t) dt \quad \text{or} \quad f(x) = y(x) - \lambda_0 \int_a^b K(x, t) y(t) dt.$$

Multiplying both sides of the above equation by $z_{0i}(x)$ and then integrating w.r.t. 'x' over the interval (a, b) , we get

$$\begin{aligned} \int_a^b f(x) z_{0i}(x) dx &= \int_a^b y(x) z_{0i}(x) dx - \lambda_0 \int_a^b z_{0i}(x) \left\{ \int_a^b K(x, t) y(t) dt \right\} dx \\ &= \int_a^b y(x) z_{0i}(x) dx - \lambda_0 \int_a^b y(t) \left\{ \int_a^b K(x, t) z_{0i}(x) dx \right\} dt \\ &\quad \text{[on interchanging the order of integration]} \\ \therefore \int_a^b f(x) z_{0i}(x) dx &= \int_a^b y(x) z_{0i}(x) dx - \lambda_0 \int_a^b y(x) \left\{ \int_a^b K(t, x) z_{0i}(t) dt \right\} dx = 0, \quad \dots (30) \end{aligned}$$

because λ_0 and $z_{0i}(x)$ are eigenvalues and corresponding eigenfunctions of the transposed equation.

The condition is sufficient. In what follows we shall make use of well known results of linear algebra. Actually, the corresponding condition of orthogonality for linear-algebraic system shows that the inhomogeneous system (18) reduces to only $n - r$ independent equations. This

implies that the rank of the matrix $(I - \lambda A)$ is exactly $p (= n - r)$ and hence the system $(B_1), (B_2), \dots, (B_n)$ or (18) is soluble. Substituting this solution in (5), we arrive at the required solution for (1).

Finally, we note that the difference of any two solutions of (1) is a solution of the homogeneous equation (13). Hence, the most general solution of (1) has the form.

$$y(x) = Y(x) + k_1 y_{01}(x) + k_2 y_{02}(x) + \dots + k_r y_{or}(x),$$

where $Y(x)$ is a suitable linear combination of the functions $f_1(x), f_2(x), \dots, f_n(x)$.

We have thus proved the theorem that, if $\lambda = \lambda_0$ is a root of multiplicity $m \geq 1$ of the equation

$D(\lambda) = 0$, then the inhomogeneous equation (1) has a solution if and only if $f(x)$ is orthogonal to all the eigenfunctions of the transposed equation.

Combining all the results of this article, we have the following theorem.

Fredholm Alternative Theorem. *Either the integral equation*

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt \quad \dots (i)$$

with fixed λ possesses one and only one solution $y(x)$ for arbitrary $\mathcal{L}_2$ -functions $f(x)$ and $K(x, t)$, in particular the solution $y = 0$ for $f = 0$; or the homogeneous equation

$$y(x) = \lambda \int_a^b K(x, t) y(t) dt \quad \dots (ii)$$

possesses a finite number r of linearly independent solutions y_{0i} , $i = 1, 2, \dots, r$. In the first case, the transposed inhomogeneous equation

$$z(x) = f(x) + \lambda \int_a^b K(t, x) z(t) dt \quad \dots (iii)$$

also possesses a unique solution. In the second case, the transposed homogeneous equation

$$z(x) = \lambda \int_a^b K(t, x) z(t) dt \quad \dots (iv)$$

also has r linearly independent solutions z_{0i} , $i = 1, 2, \dots, r$; the inhomogeneous integral equation (i) has a solution if and only if the given function $f(x)$ satisfies the r conditions

$$\int_a^b f(x) z_{0i}(x) dx = 0, \quad i = 1, 2, \dots, r \quad \dots (v)$$

In this case the solution of (i) is determined only upto an additive linear combination $\sum_{i=1}^r k_i y_{0i}$.

We now illustrate the above theorem with help of the following solved examples.

4.4 SOLVED EXAMPLES BASED ON ART. 4.3.

Ex. 1. *Show that the integral equation*

$$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x+t) y(t) dt$$

possesses no solution for $f(x) = x$, but that it possesses infinitely many solutions when $f(x) = 1$.

Sol. Given $y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x+t) y(t) dt \quad \dots (1)$

Comparing (1) with $y(x) = f(x) + \lambda \int_0^{2\pi} K(x, t) y(t) dt, \quad \dots (2)$

we have $\lambda = 1/\pi$ and $K(x, t) = \sin(x+t) = \sin x \cos t + \cos x \sin t$

4.26 Fredholm Integral Equations of the Second Kind With Separable (or Degenerate) Kernels

Let $K(x, t) = f_1(x) g_1(t) + f_2(x) g_2(t)$. So, for the given problem, we have
 $f_1(x) = \sin x$, $f_2(x) = \cos x$, $g_1(t) = \cos t$, $g_2(t) = \sin t$.

But
$$\alpha_{ij} = \int_0^{2\pi} g_i(t) f_j(t) dt, i, j = 1, 2$$

$$\therefore \alpha_{11} = \int_0^{2\pi} g_1(t) f_1(t) dt = \int_0^{2\pi} \cos t \sin t dt = \left| \frac{\sin^2 t}{2} \right|_0^{2\pi} = 0$$

$$\alpha_{12} = \int_0^{2\pi} g_1(t) f_2(t) dt = \int_0^{2\pi} \cos^2 t dt = \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt = \pi$$

Similarly, $\alpha_{21} = \pi$ and $\alpha_{22} = 0$.

$$\therefore D(\lambda) = \begin{vmatrix} 1 - \lambda \alpha_{11} & -\lambda \alpha_{12} \\ -\lambda \alpha_{21} & 1 - \lambda \alpha_{22} \end{vmatrix} = \begin{vmatrix} 1 & -\lambda \pi \\ -\lambda \pi & 1 \end{vmatrix} = 1 - \lambda^2 \pi^2.$$

The eigenvalues are given by $D(\lambda) = 0$ i.e., $1 - \lambda^2 \pi^2 = 0$

$\Rightarrow$ the eigenvalues are $\lambda_1 = 1/\pi$ and $\lambda_2 = -1/\pi$.

But here (1) contains $\lambda_1 = 1/\pi$. Thus $D(\lambda) = 0$ and hence (1) will not possess a unique solution.

It follows that (1) will possess either no solution or infinite number of solutions. We now proceed to examine these facts.

Let us find the eigenfunctions of the transposed equation (note that the kernel is symmetric)

$$y(x) = \frac{1}{\pi} \int_0^{2\pi} \sin(x+t) y(t) dt \quad \dots (3)$$

The algebraic system corresponding to (3) (given by (β_1) and (β_2) of Art. 4.1) is

$$\begin{aligned} (1 - \lambda \alpha_{11}) C_1 - \lambda \alpha_{12} C_2 &= 0 & \text{and} & & -\lambda \alpha_{21} C_1 + (1 - \lambda \alpha_{22}) C_2 &= 0 \\ \text{i.e., } C_1 - \lambda \pi C_2 &= 0 & \text{and} & & -\lambda \pi C_1 + C_2 &= 0 \end{aligned} \quad \dots (4)$$

Here $\lambda = 1/\pi$. Hence (4) gives $C_1 = C_2$.

Then the corresponding eigenfunction $z_1(x)$ is given by

$$z_1(x) = \lambda \sum_{i=1}^2 C_i f_i(x) = (1/\pi) \times [C_1 f_1(x) + C_2 f_2(x)] \quad \dots (5)$$

$$\therefore z_1(x) = (1/\pi) \times (C_1 \sin x + C_1 \cos x) = C(\sin x + \cos x), \text{ where } C = C_1/\pi$$

When $\lambda = -1/\pi$. Hence (4) gives $C_1 = -C_2$.

As, before the corresponding eigenfunction $z_2(x)$ is given by

$$z_2(x) = C'(\sin x - \cos x), \text{ when } C' = -C_1/\pi$$

Discussion of solution of (1) when $f(x) = x$. Then, we have

$$\int_0^{2\pi} f(x) z_1(x) dx = \int_0^{2\pi} Cx(\sin x + \cos x) dx \neq 0$$

and
$$\int_0^{2\pi} f(x) z_2(x) dx = \int_0^{2\pi} C'x(\sin x - \cos x) dx \neq 0.$$

Hence (1) will possess infinite solution given by

$$y(x) = f(x) + C_1 z_1(x) + C_2 z_2(x)$$

$$i.e., \quad y(x) = x + C_1 C (\sin x + \cos x) + C_2 C' (\sin x - \cos x)$$

$$i.e., \quad y(x) = x + A \cos x + B \sin x,$$

where $A(= C_1 C + C_2 C')$ and $B(= C_1 C - C_2 C')$ are arbitrary constants.

Discussion of solution of (1) when $f(x) = 1$. Then, we have

$$\int_0^{2\pi} f(x) z_1(x) dx = \int_0^{2\pi} C(\sin x + \cos x) dx = 0,$$

showing that $f(x)$ is not orthogonal to $z_1(x)$ and hence (1) will possess no solution.

Remark. For an alternative solution, refer Ex. 13 on page 4.13.

Ex. 2. Solve $y(x) = f(x) + \lambda \int_0^1 (1-3xt) y(t) dt$.

$$\text{Sol. Given} \quad y(x) = f(x) + \lambda \int_0^1 y(t) dt - 3x\lambda \int_0^1 t y(t) dt \quad \dots (1)$$

$$\text{Let} \quad C_1 = \int_0^1 y(t) dt \quad \dots (2)$$

$$\text{and} \quad C_2 = \int_0^1 t y(t) dt. \quad \dots (3)$$

Using (2) and (3), (1) reduces to

$$y(x) = f(x) + C_1 \lambda - 3x C_2 \lambda \quad \dots (4)$$

$$\text{From (4)} \quad y(t) = f(t) + C_1 \lambda - 3t C_2 \lambda \quad \dots (5)$$

$$\text{Let} \quad \int_0^1 f(t) dt = f_1 \quad \text{and} \quad \int_0^1 t f(t) dt = f_2 \quad \dots (6)$$

Using (5), (2) reduces to

$$C_1 = \int_0^1 \{f(t) + C_1 \lambda - 3t C_2 \lambda\} dt \quad \text{or} \quad C_1 = f_1 + C_1 \lambda - (3/2) \times C_2 \lambda, \text{ by (6)}$$

$$\text{or} \quad (1-\lambda)C_1 + (3/2) \times C_2 \lambda = f_1 \quad \dots (7)$$

Using (5), (3) reduces to

$$C_2 = \int_0^1 t \{f(t) + C_1 \lambda - 3t C_2 \lambda\} dt \quad \text{or} \quad C_2 = f_2 + (1/2) \times C_1 \lambda - C_2 \lambda, \text{ by (6)}$$

$$\text{or} \quad -(1/2) \times \lambda C_1 + (1+\lambda)C_2 = f_2 \quad \dots (8)$$

$$\therefore \text{ Here} \quad D(\lambda) = \begin{vmatrix} 1-\lambda & 3\lambda/2 \\ -\lambda/2 & 1+\lambda \end{vmatrix} = \frac{1}{4} (4 - \lambda^2) \quad \dots (9)$$

Therefore, the inhomogeneous equation (5) will have a unique solution if and only if $\lambda \neq \pm 2$. The required solution can be obtained by solving (7) and (8) for C_1 & C_2 and substituting the values of C_1 and C_2 so obtained in (4).

Then the homogeneous equation

$$y(x) = \lambda \int_0^1 (1-3xt) y(t) dt$$

has only the trivial solution.

Let us now examine the case when λ is equal to one of the eigenvalues and examine the eigenfunctions of the transposed homogeneous equation (note that the kernel is symmetric)

$$y(x) = \lambda \int_0^1 (1-3xt) y(t) dt \quad \dots (10)$$

The algebraic system corresponding to (10) is given by (7) and (8) with $f_1 = f_2 = 0$, i.e., by $(1-\lambda) C_1 + (3/2) \times C_2 \lambda = 0$ and $-(1/2) \times \lambda C_1 + (1+\lambda) C_2 = 0 \quad \dots (11)$

Case (i) When $\lambda = 2$, then the algebraic system (11) gives $C_1 = 3C_2$. Then, the corresponding eigenfunction $z_1(x)$ is given by (5) with $f(x) = 0$ and $\lambda = 2$.

$$\therefore z_1(x) = 2C_1 - 6xC_2 = 6C_1(1-x) = C(1-x),$$

where $C(= 6C_1)$ is an arbitrary constant. It follows from the Fredholm alternative theorem that the integral equation

$$y(x) = f(x) + 2 \int_0^1 (1-3xt) y(t) dt$$

will possess a solution if $f(x)$ satisfies the condition

$$\int_0^1 f(x) z_1(x) dx = 0, \quad \text{i.e.,} \quad \int_0^1 (1-x) f(x) dx = 0$$

Case (ii) When $\lambda = -2$, then the algebraic system (11) given $C_1 = C_2$. Then, the corresponding eigenfunction $z_2(x)$ is given by (5) with $f(x) = 0$ and $\lambda = -2$.

$$\therefore z_2(x) = -2C_1 + 6xC_1 = -2C_1(1-3x),$$

where $C' = (-2C_1)$ is an arbitrary constant. It follows from the Fredholm alternative theorem that the integral equation

$$y(x) = f(x) - 2 \int_0^1 (1-3xt) y(t) dt$$

will possess a solution if $f(x)$ satisfies the condition

$$\int_0^1 f(x) z_2(x) dx = 0, \quad \text{i.e.,} \quad \int_0^1 (1-3x) f(x) dx = 0$$

Remark. For an alternative solution, refer Ex. 14 on page 4.15

EXERCISE- 4 (b)

1. Solve the integral equation $y(x) = f(x) + \lambda \int_0^{2\pi} \cos(x+t) y(t) dt$ and find the condition that $f(x)$ must satisfy in order that this equation has a solution when λ is an eigenvalue. Obtain the general solution if $f(x) = \sin x$, considering all possible cases.

2. Solve the integral equation $y(x) = 1 + \lambda \int_{-\pi}^{\pi} e^{iw(x-t)} y(t) dt$, considering separately all the exceptional cases.

4.5 AN APPROXIMATE METHOD.

We propose to describe a useful method for finding approximate solutions of some special type of integral equations. We shall explain this method with help of the following examples.

Example 1. Solve the integral equation $y(x) = e^x - x - \int_0^1 x(e^{xt} - 1)y(t) dt$

Solution. Let us approximate the kernel by the sum of the first three terms in its Taylor series :

$$K(x, t) = x(e^{xt} - 1) \approx x \left\{ 1 + \frac{xt}{1!} + \frac{(xt)^2}{2!} + \frac{(xt)^3}{3!} - 1 \right\}$$

i.e., $K(x, t) \approx x^2 t + (1/2) \times x^3 t^2 + (1/6) \times x^4 t^3$, which is a separable kernel.

Then, the given integral equation takes the form

$$y(x) = e^x - x - \int_0^1 \left(x^2 t + \frac{1}{2} x^3 t^2 + \frac{1}{6} x^4 t^3 \right) y(t) dt \quad \dots (1)$$

$$\text{or} \quad y(x) = e^x - x - x^2 \int_0^1 t y(t) dt - \frac{x^3}{2} \int_0^1 t^2 y(t) dt - \frac{x^4}{6} \int_0^1 t^3 y(t) dt \quad \dots (2)$$

$$\text{Let} \quad C_1 = - \int_0^1 t y(t) dt \quad \dots (3)$$

$$C_2 = - \frac{1}{2} \int_0^1 t^2 y(t) dt \quad \dots (4)$$

$$C_3 = - \frac{1}{6} \int_0^1 t^3 y(t) dt \quad \dots (5)$$

$$\text{Then (2) gives} \quad y(x) = e^x - x + C_1 x^2 + C_2 x^3 + C_3 x^4 \quad \dots (6)$$

$$\text{From (6),} \quad y(t) = e^t - t + C_1 t^2 + C_2 t^3 + C_3 t^4 \quad \dots (7)$$

Substituting the value of $y(t)$ given by (7) in (3), we get

$$C_1 = - \int_0^1 t (e^t - t + C_1 t^2 + C_2 t^3 + C_3 t^4) dt$$

$$\text{or} \quad C_1 = - \int_0^1 t e^t dt + \int_0^1 t^2 dt - C_1 \int_0^1 t^3 dt - C_2 \int_0^1 t^4 dt - C_3 \int_0^1 t^5 dt$$

$$\text{or} \quad C_1 = - \left[t e^t - e^t \right]_0^1 + (1/3) - (C_1/4) - (C_2/5) - (C_3/6)$$

$$\text{or} \quad (5/4) \times C_1 + (1/5) \times C_2 + (1/6) \times C_3 = -2/3 \quad \dots (8)$$

Substituting the value of $y(t)$ given by (7) in (4), we get

$$C_2 = - \frac{1}{2} \int_0^1 t^2 (e^t - t + C_1 t^2 + C_2 t^3 + C_3 t^4) dt$$

$$\text{or} \quad C_2 = - \frac{1}{2} \int_0^1 t^2 e^t dt + \frac{1}{2} \int_0^1 t^3 dt - \frac{C_1}{2} \int_0^1 t^4 dt - \frac{C_2}{2} \int_0^1 t^5 dt - \frac{C_3}{2} \int_0^1 t^6 dt$$

$$\text{or} \quad C_2 = - \frac{1}{2} \left[(t^2) (e^t) - (2t) (e^t) + (2) (e^t) \right]_0^1 + \frac{1}{8} - \frac{C_1}{10} - \frac{C_2}{12} - \frac{C_3}{14}$$

[using chain rule of integration by parts to evaluate the first integral]

4.30 Fredholm Integral Equations of the Second Kind With Separable (or Degenerate) Kernels

$$\begin{aligned} \text{or} \quad & C_1/10 + (13/12) \times C_2 + C_3/14 = -(1/2) \times (e - 2e + 2e - 2) + 1/8 \\ \text{or} \quad & (1/5) \times C_1 + (13/6) \times C_2 + (1/7) \times C_3 = (9/4) - e \quad \dots (9) \end{aligned}$$

Substituting the value of $y(t)$ given by (7) in (5), we get

$$C_3 = -\frac{1}{6} \int_0^1 t^3 (e^t - t + C_1 t^2 + C_2 t^3 + C_3 t^4) dt$$

$$\text{or} \quad C_3 = -\frac{1}{6} \int_0^1 t^3 e^t dt + \frac{1}{6} \int_0^1 t^4 dt - \frac{C_1}{6} \int_0^1 t^5 dt - \frac{C_2}{6} \int_0^1 t^6 dt - \frac{C_3}{6} \int_0^1 t^7 dt$$

$$\begin{aligned} \text{or} \quad C_3 = & -(1/6) \times \left[(t^3)(e^t) - (3t^2)(e^t) + (6t)(e^t) - (6)(e^t) \right]_0^1 + 1/30 \\ & - (1/36) \times C_1 - (1/42) \times C_2 - (1/48) \times C_3 \\ & \text{[using chain rule of integration by parts to evaluate the first integral]} \end{aligned}$$

$$\text{or} \quad C_3 = -(1/6) \times (e - 3e + 6e - 6e + 6) + 1/30 - (1/36) \times C_1 - (1/42) \times C_2 - (1/48) \times C_3$$

$$\text{or} \quad (1/36) \times C_1 + (1/42) \times C_2 + (49/48) \times C_3 = e/3 - (29/36)$$

$$\text{or} \quad (1/6) \times C_1 + (1/7) \times C_2 + (49/8) \times C_3 = 2e - (29/5) \quad \dots (10)$$

Solving (8), (9) and (10) leads to (after simplifications)

$$C_1 = -0.5010, \quad C_2 = -0.1671, \quad \text{and} \quad C_3 = -0.0422 \quad \dots (11)$$

With these values, (6) gives the required approximate solution of (1) as

$$y(x) = e^x - x - 0.5010 x^2 - 0.1671 x^3 - 0.0422 x^4 \quad \dots (12)$$

Now as usual, we prove that the exact solution of given equation

$$y(x) = e^x - x - \int_0^1 x (e^{xt} - 1) y(t) dt \quad \dots (13)$$

$$\text{is given by} \quad y(x) = 1 \quad \dots (14)$$

From (14), $y(t) = 1$. Then, we have

$$\begin{aligned} \text{R. H.S. of (13)} &= e^x - x - \int_0^1 x (e^{xt} - 1) dt = e^x - x - x \left[\frac{e^{xt}}{x} \right]_0^1 + [xt]_0^1 = e^x - x - (e^x - 1) + x \\ &= 1 = y(x) = \text{L.H.S. of (13)} \end{aligned}$$

Hence $y(x) = 1$ is the exact solution of (13).

Using the approximate solution (12) for $x = 0$, $x = 0.5$ and $x = 1.0$, the values of $y(x)$ are

$$y(0) = 1.0000, \quad y(0.5) = 1.0000 \quad \text{and} \quad y(1) = 1.0080 \quad \dots (15)$$

which agrees with exact solution (14) rather closely

Example 2. Consider the integral equation
$$y(x) = x^2 + \int_0^1 (\sin xt) y(t) dt;$$

Replacing $\sin xt$ by the first two terms of its power-series development, namely,

$\sin xt = xt - (xt)^3/3! + \dots$, obtain an approximate solution.

Solution. Proceed as in example 1.

Method of successive approximations

5.1 INTRODUCTION

We already know that ordinary first-order differential equation are solved by the well-known *Picard method* of successive approximations. In this chapter we shall study an iterative scheme based on the same principle for linear integral equations of the second order. Throughout our discussion we shall assume that the functions $f(x)$ and $K(x, t)$ involved in an integral equation are $\mathcal{L}_2$ - functions.

5.2 ITERATED KERNELS OR FUNCTIONS. DEFINITION.

(i) Consider Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt. \quad \dots (1)$$

Then, the iterated kernels $K_n(x, t)$, $n = 1, 2, 3, \dots$ are defined as follows :

$$K_1(x, t) = K(x, t), \quad \dots (2a)$$

$$\left. \begin{array}{l} \text{and} \quad K_n(x, t) = \int_a^b K(x, z) K_{n-1}(z, t) dz, n = 2, 3, \dots \\ \text{or} \quad K_n(x, t) = \int_a^b K_{n-1}(x, z) K(z, t) dz, n = 2, 3, \dots \end{array} \right\} \quad \dots (2b)$$

(ii) Consider Volterra integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt \quad \dots (3)$$

Then the iterated kernels $K_n(x, t)$, $n = 1, 2, 3, \dots$ are defined as follows :

$$K_1(x, t) = K(x, t) \quad \dots (3a)$$

$$\left. \begin{array}{l} \text{and} \quad K_n(x, t) = \int_t^x K(x, z) K_{n-1}(z, t) dz, n = 2, 3, \dots \\ \text{or} \quad K_n(x, t) = \int_t^x K_{n-1}(x, z) K(z, t) dz, n = 2, 3, \dots \end{array} \right\} \quad \dots (3b)$$

5.3 RESOLVENT KERNEL OR RECIPROCAL KERNEL.

(i) Suppose solution of Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt \quad \dots (1)$$

takes the form
$$y(x) = f(x) + \lambda \int_a^b R(x, t; \lambda) f(t) dt, \quad \dots (2a)$$

or
$$y(x) = f(x) + \lambda \int_a^b \Gamma(x, t; \lambda) f(t) dt, \quad \dots (2b)$$

then $R(x, t; \lambda)$ or $\Gamma(x, t; \lambda)$ is known as the *resolvent kernel of (1)*.

(ii) Suppose solution of Volterra integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt \quad \dots (3)$$

takes the form
$$y(x) = f(x) + \lambda \int_a^x R(x, t; \lambda) f(t) dt \quad \dots (4a)$$

or
$$y(x) = f(x) + \lambda \int_a^x \Gamma(x, t; \lambda) f(t) dt \quad \dots (4b)$$

then $R(x, t; \lambda)$ or $\Gamma(x, t; \lambda)$ is known as the *resolvent kernel of (3)*

5.4. Theorem. The m th iterated kernel $K_m(x, t)$ satisfies the relation

$$K_m(x, t) = \int_a^b K_r(x, y) K_{m-r}(y, t) dy,$$

where r is any positive integer less than m .

Proof. The m th iterated kernel $K_m(x, t)$ is defined as

$$K_1(x, t) = K(x, t), \quad \dots (1)$$

and
$$K_m(x, t) = \int_a^b K(x, s) K_{m-1}(s, t) ds, \quad m = 2, 3, \dots \quad \dots (2)$$

Re-writing (2), we have
$$K_m(x, t) = \int_a^b K(x, s_1) K_{m-1}(s_1, t) ds_1. \quad \dots (3)$$

Replacing m by $m - 1$ in (2), we have

$$K_{m-1}(x, t) = \int_a^b K(x, s) K_{m-2}(s, t) ds = \int_a^b K(x, s_2) K_{m-2}(s_2, t) ds_2$$

or
$$K_{m-1}(s_1, t) = \int_a^b K(s_1, s_2) K_{m-2}(s_2, t) ds_2. \quad \dots (4)$$

Using (4), (3) reduces to

$$K_m(x, t) = \int_a^b K(x, s_1) \left\{ \int_a^b K(s_1, s_2) K_{m-2}(s_2, t) ds_2 \right\} ds_1$$

or
$$K_m(x, t) = \int_a^b \int_a^b K(x, s_1) K(s_1, s_2) K_{m-2}(s_2, t) ds_2 ds_1$$

Proceeding likewise, we obtain

$$K_m(x, t) = \int_a^b \int_a^b \dots \int_a^b K(x, s_1) K(s_1, s_2) K(s_2, s_3) \dots K_1(s_{m-1}, t) ds_{m-1} \dots ds_2 ds_1$$

or
$$K_m(x, t) = \underbrace{\int_a^b \int_a^b \dots \int_a^b}_{(m-1)\text{th order integral}} K(x, s_1) K(s_1, s_2) \dots K(s_{r-1}, s_r) K(s_r, s_{r+1}) \dots \dots K(s_{m-1}, t) ds_{m-1} \dots ds_2 ds_1 \quad \dots (5)$$

Note that R.H.S. of (5) is a multiple integral of order $m - 1$.

Proceeding as above, we may also write.

$$K_r(x, y) = \underbrace{\int_a^b \int_a^b \dots \int_a^b}_{(r-1)\text{th order integral}} K(x, u_1) K(u_1, u_2) \dots K(u_{r-1}, y) du_{r-1} \dots du_2 du_1 \quad \dots (6)$$

and
$$K_{m-r}(y, t) = \underbrace{\int_a^b \int_a^b \dots \int_a^b}_{(m-r-1)\text{th order integral}} K(y, v_1) K(v_1, v_2) \dots K(v_{m-r-1}, t) dv_{m-r-1} \dots dv_2 dv_1. \quad \dots (7)$$

$$\begin{aligned} \text{Now, } & \int_a^b K_r(x, y) K_{m-r}(y, t) dy \\ &= \int_a^b \left[\left\{ \int_a^b \int_a^b \dots \int_a^b K(x, u_1) K(u_1, u_2) \dots K(u_{r-1}, y) du_{r-1} \dots du_2 du_1 \right\} \right. \\ & \quad \times \left. \left\{ \int_a^b \int_a^b \dots \int_a^b K(y, v_1) K(v_1, v_2) \dots K(v_{m-r-1}, t) dv_{m-r-1} \dots dv_2 dv_1 \right\} \right] dy, \text{ by (6) and (7)} \end{aligned}$$

$$\begin{aligned} \text{or } & \int_a^b K_r(x, y) K_{m-r}(y, t) dy = \int_a^b \int_a^b \dots \int_a^b K(x, u_1) K(u_1, u_2) \dots K(u_{r-1}, y) \\ & \quad \times K(y, v_1) K(v_1, v_2) \dots K(v_{m-r-1}, t) dv_{m-r-1} \dots dv_2 dv_1 dy du_{r-1} \dots du_2 du_1 \dots (8) \\ & \quad \text{[on changing the order of integration]} \end{aligned}$$

Note that the order of the multiple integral on R.H.S. of (8) is $1 + (r - 1) + (m - r - 1)$, that is, $m - 1$. We have already proved that the order of the multiple integral on R.H.S. of (5) is also $m - 1$. Thus, multiple integrals involved in $K_m(x, t)$ and $\int_a^b K_r(x, y) K_{m-r}(y, t) dy$ are both of the same order, namely, $(m - 1)$ th.

Now, changing the variables of integrations in (8) without changing the limits of integration according to the following scheme

$$\begin{array}{cccccccc} u_1 & u_2 & \dots & u_{r-1} & y & v_1 & v_2 & \dots & v_{m-r-1} \\ \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ s_1 & s_2 & & s_{r-1} & s_r & s_{r+1} & s_{r+2} & & s_{m-1} \end{array}$$

we obtain

$$\begin{aligned} \int_a^b K_r(x, y) K_{m-r}(y, t) dy &= \underbrace{\int_a^b \int_a^b \dots \int_a^b}_{(m-1)\text{th order integral}} K(x, s_1) K(s_1, s_2) \dots K(s_{r-1}, s_r) \\ & \quad \times K(s_r, s_{r+1}) \dots K(s_{m-1}, t) ds_{m-1} \dots ds_2 ds_1 \dots (9) \end{aligned}$$

$$\text{From (5) and (9), we obtain } K_m(x, t) = \int_a^b K_r(x, y) K_{m-r}(y, t) dy.$$

5.5. SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND BY SUCCESSIVE SUBSTITUTIONS. [Garhwal 1996; Meerut 2000, 02]

Theorem. Let
$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt \quad \dots (1)$$

be given Fredholm integral equation of the second kind. Suppose that

(i) Kernel $K(x, t) \equiv 0$, is real and continuous in the rectangle R , for which
 $a \leq x \leq b, a \leq t \leq b$. Also let $|K(x, t)| \leq M$, in R (2)

(ii) $f(x) \equiv 0$, is real and continuous in the interval I , for which $a \leq x \leq b$.
 Also, let $|f(x)| \leq N$, in I (3)

(iii) λ is a constant such that $|\lambda| < 1/M(b-a)$... (4)

Then (1) has a unique continuous solution in I and this solution is given by the absolutely and uniformly convergent series

$$y(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1) f(t_1) dt_1 dt + \dots \quad \dots (5)$$

Proof. Re-writing (1), we have

$$y(x) = f(x) + \lambda \int_a^b K(x, t_1) y(t_1) dt_1. \quad \dots (6)$$

Replacing x by t in (6), we get $y(t) = f(t) + \lambda \int_a^b K(t, t_1) y(t_1) dt_1$ (7)

Substituting the above value of $y(t)$ in (1), we get

$$y(x) = f(x) + \lambda \int_a^b K(x, t) \left[f(t) + \lambda \int_a^b K(t, t_1) y(t_1) dt_1 \right] dt.$$

or
$$y(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1) y(t_1) dt_1 dt. \quad \dots (8)$$

Re-writing (7), we have
$$y(t) = f(t) + \lambda \int_a^b K(t, t_2) y(t_2) dt_2. \quad \dots (9)$$

Replacing t by t_1 in (9), we have
$$y(t_1) = f(t_1) + \lambda \int_a^b K(t_1, t_2) y(t_2) dt_2. \quad \dots (10)$$

Substituting the above value of $y(t_1)$ in (8), we get

$$y(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1) \left[f(t_1) + \lambda \int_a^b K(t_1, t_2) y(t_2) dt_2 \right] dt_1 dt$$

or
$$y(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1) f(t_1) dt_1 dt + \lambda^3 \int_a^b K(x, t) \int_a^b K(t, t_1) \int_a^b K(t_1, t_2) y(t_2) dt_2 dt_1 dt. \quad \dots (11)$$

Proceeding likewise, we have

$$y(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1) f(t_1) dt_1 dt + \dots + \lambda^n \int_a^b K(x, t) \int_a^b K(t, t_1) \dots \int_a^b K(t_{n-2}, t_{n-1}) f(t_{n-1}) dt_{n-1} \dots dt_1 dt + R_{n+1}(x), \dots (12)$$

where
$$R_{n+1}(x) = \lambda^{n+1} \int_a^b K(x, t) \int_a^b K(t, t_1) \dots \int_a^b K(t_{n-1}, t_n) y(t_n) dt_n \dots dt_1 dt. \quad \dots (13)$$

Now, let us consider the following infinite series

$$f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1) f(t_1) dt_1 dt + \dots \quad \dots (14)$$

In view of the assumptions (i) and (ii), each term of the series (14) is continuous in I . It follows that the series (14) is also continuous in I , provided it converges uniformly in I .

Let $U_n(x)$ denote the general term of the series (14) i.e., let

$$U_n(x) = \lambda^n \int_a^b K(x, t) \int_a^b K(t, t_1) \dots \int_a^b K(t_{n-2}, t_{n-1}) f(t_{n-1}) dt_{n-1} \dots dt_1 dt \quad \dots (15)$$

From (15), we have

$$|U_n(x)| = \left| \lambda^n \int_a^b K(x, t) \int_a^b K(t, t_1) \dots \int_a^b K(t_{n-2}, t_{n-1}) f(t_{n-1}) dt_{n-1} \dots dt_1 dt \right|$$

$$\therefore |U_n(x)| \leq |\lambda|^n M^n (b-a)^n, \text{ using (2) and (3)} \quad \dots (16)$$

The series of which this is a general term converges only when

$$|\lambda| M (b-a) < 1 \quad \text{or} \quad |\lambda| < 1/M (b-a)$$

which holds in view of assumption (iii).

It follows that the series (14) converges absolutely and uniformly when condition (4) holds.

If (1) has a continuous solutions, clearly it must be expressed by (12). If $y(x)$ is continuous in I , $|y(x)|$ must have a maximum value Y . Thus,

$$|y(x)| \leq Y. \quad \dots (17)$$

Now, from (13), we have

$$|R_{n+1}(x)| = \left| \lambda^{n+1} \int_a^b K(x, t) \int_a^b K(t, t_1) \dots \int_a^b K(t_{n-1}, t_n) y(t_n) dt_n \dots dt_1 dt \right|$$

$$\therefore |R_{n+1}(x)| \leq |\lambda|^{n+1} Y M^{n+1} (b-a)^{n+1}, \text{ using (2) and (17)}$$

Since (4) holds, so $\lim_{n \rightarrow \infty} R_{n+1}(x) = 0$.

It follows that the function $y(x)$ satisfying (12) is the continuous function given by the series (14). This proves what we wished to prove.

5.6. SOLUTION OF VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND BY SUCCESSIVE SUBSTITUTIONS. [Garhwal 1996; Meerut 2000, 01]

Theorem. Let $y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt \quad \dots (1)$

be given Volterra integral equation of the second kind. Suppose that

(i) kernel $K(x, t) \neq 0$, is real and continuous in the rectangle R , for which $a \leq x \leq b$, $a \leq t \leq b$.

Also, let $|K(x, t)| \leq M$, in R (2)

(ii) $f(x) \neq 0$, real and continuous in the interval I , for which $a \leq x \leq b$.

Also, let $|f(x)| \leq N$, in I (3)

(iii) λ is a constant. ... (4)

Then (1) has a unique continuous solution in I and this solution is given by the absolutely and uniformly convergent series

$$y(x) = f(x) + \lambda \int_a^x K(x, t) f(t) dt + \lambda^2 \int_a^x K(x, t) \int_a^t K(t, t_1) f(t_1) dt_1 dt + \dots \quad \dots (5)$$

Proof. Re-writing (1), we have

$$y(x) = f(x) + \lambda \int_a^x K(x, t_1) y(t_1) dt_1. \quad \dots (6)$$

Replacing x by t in (6), we get $y(t) = f(t) + \lambda \int_a^t K(t, t_1) y(t_1) dt_1. \quad \dots (7)$

Substituting the above value of $y(t)$ in (1), we get

$$y(x) = f(x) + \lambda \int_a^x K(x, t) \left[f(t) + \lambda \int_a^t K(t, t_1) y(t_1) dt_1 \right] dt.$$

or $y(x) = f(x) + \lambda \int_a^x K(x, t) f(t) dt + \lambda^2 \int_a^x K(x, t) \int_a^t K(t, t_1) y(t_1) dt_1 dt \quad \dots (8)$

Re-writing (7), we have $y(t) = f(t) + \lambda \int_a^t K(t, t_2) y(t_2) dt_2. \quad \dots (9)$

Replacing t by t_1 in (9), we have $y(t_1) = f(t_1) + \lambda \int_a^{t_1} K(t_1, t_2) y(t_2) dt_2. \quad \dots (10)$

Substituting the above value of $y(t_1)$ in (8), we get

$$y(x) = f(x) + \lambda \int_a^x K(x, t) f(t) dt + \lambda^2 \int_a^x K(x, t) \int_a^t K(t, t_1) \left[f(t_1) + \lambda \int_a^{t_1} K(t_1, t_2) y(t_2) dt_2 \right] dt_1 dt$$

or $y(x) = f(x) + \lambda \int_a^x K(x, t) f(t) dt + \lambda^2 \int_a^x K(x, t) \int_a^t K(t, t_1) f(t_1) dt_1 dt$
 $+ \lambda^3 \int_a^x K(x, t) \int_a^t K(t, t_1) \int_a^{t_1} K(t_1, t_2) y(t_2) dt_2 dt_1 dt. \quad \dots (11)$

Proceeding likewise, we have

$$y(x) = f(x) + \lambda \int_a^x K(x, t) f(t) dt + \lambda^2 \int_a^x K(x, t) \int_a^t K(t, t_1) f(t_1) dt_1 dt + \dots$$

$$+ \lambda^n \int_a^x K(x, t) \int_a^t K(t, t_1) \dots \int_a^{t_{n-2}} K(t_{n-2}, t_{n-1}) f(t_{n-1}) dt_{n-1} \dots dt_1 dt + R_{n+1}(x), \quad \dots (12)$$

where $R_{n+1}(x) = \lambda^{n+1} \int_a^x K(x, t) \int_a^t K(t, t_1) \dots \int_a^{t_{n-1}} K(t_{n-1}, t_n) y(t_n) dt_n \dots dt_1 dt. \quad \dots (13)$

Now, let us consider the following infinite series

$$f(x) + \lambda \int_a^x K(x, t) f(t) dt + \lambda^2 \int_a^x K(x, t) \int_a^t K(t, t_1) f(t_1) dt_1 + \dots \quad \dots (14)$$

In view of the assumptions (i) and (ii), each term of the series (14) is continuous in I . It follows that the series (14) is also continuous in I , provided it converges uniformly in I . Let $U_n(x)$ denote the general term of the series (14), i.e., let

$$U_n(x) = \lambda^n \int_a^x K(x, t) \int_a^t K(t, t_1) \dots \int_a^{t_{n-2}} K(t_{n-2}, t_{n-1}) f(t_{n-1}) dt_{n-1} \dots dt_1 dt. \quad \dots (15)$$

From (15) we have

$$|U_n(x)| = |\lambda|^n \int_a^x K(x,t) \int_a^t K(t,t_1) \dots \int_a^{t_{n-2}} K(t_{n-2}, t_{n-1}) f(t_{n-1}) dt_{n-1} \dots dt_1 dt |$$

$$\therefore |U_n(x)| \leq |\lambda|^n NM^n \frac{(x-a)^n}{n!}, \text{ using (2) and (3)}$$

$$\text{or } |U_n(x)| \leq |\lambda|^n NM^n \frac{(b-a)^n}{n!}, a \leq x \leq b$$

$$\text{or } |U_n(x)| \leq |\lambda|^n N \frac{[M(b-a)]^n}{n!}, a \leq x \leq b \quad \dots (16)$$

Clearly, the series, for which the positive constant $\frac{|\lambda|^n N[M(b-a)]^n}{n!}$ is the general expression for the n th term, is convergent for all values of $\lambda, N, M, (b-a)$. So from (16), it follows that the series (14) converges absolutely and uniformly.

If (1) has a continuous solution, clearly it must be expressed by (12). If $y(x)$ is continuous in I , $|y(x)|$ must have a maximum value Y . Thus,

$$|y(x)| \leq Y. \quad \dots (17)$$

Now, from (13), we have

$$|R_{n+1}(x)| = |\lambda|^{n+1} \int_a^x K(x,t) \int_a^t K(t,t_1) \dots \int_a^{t_{n-1}} K(t_{n-1}, t_n) y(t_n) dt_n \dots dt_1 dt |$$

$$\therefore |R_{n+1}(x)| \leq |\lambda|^{n+1} YM^{n+1} \frac{(x-a)^{n+1}}{(n+1)!} \leq |\lambda|^{n+1} YM^{n+1} \frac{(b-a)^{n+1}}{(n+1)!}, (a \leq x \leq b)$$

$$\text{Hence } \lim_{n \rightarrow \infty} R_{n+1}(x) = 0.$$

It follows that the function $y(x)$ satisfying (12) is the continuous function given by the series (14). This proves the desired result.

5.7. SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND BY SUCCESSIVE APPROXIMATIONS. ITERATIVE METHOD (ITERATIVE SCHEME). NEUMANN'S SERIES. [Meerut 2006]

Consider Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) dt. \quad \dots (1)$$

As a zero-order approximation to the required solution $y(x)$, let us take

$$y_0(x) = f(x) \quad \dots (2)$$

Further, if $y_n(x)$ and $y_{n-1}(x)$ are the n th order and $(n-1)$ th-order approximations respectively, then these are connected by

$$y_n(x) = f(x) + \lambda \int_a^b K(x,t) y_{n-1}(t) dt. \quad \dots (3)$$

We know that the iterated kernels (or iterated functions) $K_n(x,t)$, $(n = 1, 2, 3, \dots)$ are defined by

$$K_1(x,t) = K(x,t) \quad \dots (4A)$$

and

$$K_n(x,t) = \int_a^b K(x,z) K_{n-1}(z,t) dz. \quad \dots (4B)$$

Putting $n = 1$ in (3), the first-order approximation $y_1(x)$ is given by

$$y_1(x) = f(x) + \lambda \int_a^b K(x, t) y_0(t) dt. \quad \dots (5)$$

But from (2), $y_0(t) = f(t)$... (6)
 Substituting the above value of $y_0(t)$ in (5), we get

$$y_1(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt. \quad \dots (7)$$

Putting $n = 2$ in (3), the second-order approximation $y_2(x)$ is given by

$$y_2(x) = f(x) + \lambda \int_a^b K(x, t) y_1(t) dt.$$

or
$$y_2(x) = f(x) + \lambda \int_a^b K(x, z) y_1(z) dz. \quad \dots (8)$$

Replacing x by z in (7), we get
$$y_1(z) = f(z) + \lambda \int_a^b K(z, t) f(t) dt. \quad \dots (9)$$

Substituting the above value of $y_1(z)$ in (8), we get

$$y_2(x) = f(x) + \lambda \int_a^b K(x, z) \left[f(z) + \lambda \int_a^b K(z, t) f(t) dt \right] dz$$

or
$$y_2(x) = f(x) + \lambda \int_a^b K(x, z) f(z) dz + \lambda^2 \int_a^b K(x, z) \left[\int_a^b K(z, t) f(t) dt \right] dz \quad \dots (10)$$

or
$$y_2(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b f(t) \left[\int_a^b K(x, z) K(z, t) dz \right] dt$$

 [On changing the order of integration in third term on R.H.S of (10)]

or
$$y_2(x) = f(x) + \lambda \int_a^b K_1(x, t) f(t) dt + \lambda^2 \int_a^b K_2(x, t) f(t) dt, \text{ using (4A) and (4B)}$$

or
$$y_2(x) = f(x) + \sum_{m=1}^2 \lambda^m \int_a^b K_m(x, t) f(t) dt. \quad \dots (11)$$

Proceeding likewise, we easily obtain by Mathematical induction the n th approximate solution $y_n(x)$ of (1) as

$$y_n(x) = f(x) + \sum_{m=1}^n \lambda^m \int_a^b K_m(x, t) f(t) dt. \quad \dots (12)$$

Proceeding to the limit as $n \rightarrow \infty$, we obtain the so called *Neumann series*.

$$y(x) = \lim_{n \rightarrow \infty} y_n(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(x, t) f(t) dt \quad \dots (13)$$

We now determine the resolvent kernel (or reciprocal kernel) $R(x, t; \lambda)$ or $\Gamma(x, t; \lambda)$ in terms of the iterated kernels $K_m(x, t)$. For this purpose, by changing the order of integration and summation in the so called Neumann series (13), we obtain

$$y(x) = f(x) + \lambda \int_a^b \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) \right] f(t) dt. \quad \dots (14)$$

Comparing (14) with
$$y(x) = f(x) + \lambda \int_a^b R(x, t; \lambda) f(t) dt, \quad \dots (15)$$

here
$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t). \quad \dots (16)$$

Determination of the conditions of convergence of (13).

Consider the partial sum (12) and apply the Schwarz inequality [refer Art. 1.16, Chapter 1] to the general term of this series. This leads us to

$$\left| \int_a^b K_m(x, t) f(t) dt \right|^2 \leq \left(\int_a^b |K_m(x, t)|^2 dt \right) \int_a^b |f(t)|^2 dt. \quad \dots (17)$$

$$\text{Let } D = \text{norm of } f(t) = \left[\int_a^b |f(t)|^2 dt \right]^{1/2}. \quad \dots (18)$$

Further, let C_m^2 denote the upper bound of the integral $\int_a^b |K_m(x, t)|^2 dt$,

$$\text{so that } \int_a^b |K_m(x, t)|^2 dt \leq C_m^2 \quad \dots (19)$$

Using (18) and (19), (17) reduces to

$$\left| \int_a^b K_m(x, t) f(t) dt \right|^2 \leq C_m^2 D^2 \quad \dots (20)$$

Now, applying the Schwarz inequality to relation

$$K_m(x, t) = \int_a^b K_{m-1}(x, z) K(z, t) dz,$$

$$\text{we get } |K_m(x, t)|^2 \leq \left(\int_a^b |K_{m-1}(x, z)|^2 dz \right) \times \left(\int_a^b |K(z, t)|^2 dz \right),$$

which when integrated with respect to t , gives

$$\int_a^b |K_m(x, t)|^2 dt \leq B^2 C_{m-1}^2, \quad \dots (21)$$

$$\text{where } B^2 = \int_a^b \int_a^b |K(x, t)|^2 dx dt. \quad \dots (22)$$

The inequality (21) gives rise to the recurrence relation

$$C_m^2 \leq B^{2m-2} C_1^2. \quad \dots (23)$$

$$\text{Using (20) and (23), we get } \left| \int_a^b K_m(x, t) f(t) dt \right|^2 \leq C_1^2 D^2 B^{2m-2} \quad \dots (24)$$

showing that the general term of the partial sum (12) has a magnitude less than the quantity $DC_1 |\lambda|^m B^{m-1}$. Hence the infinite series (13) converges faster than the geometric series with common ratio $|\lambda| B$. It follows that, if the condition

$$|\lambda| B < 1 \quad \text{or} \quad |\lambda| < \frac{1}{\left[\int_a^b \int_a^b |K(x, t)|^2 dx dt \right]^{1/2}} \quad \dots (25)$$

is satisfied, then the series (13) will be uniformly convergent.

Uniqueness of solution for a given λ .

If possible, let (1) possess two solutions $y_1(x)$ and $y_2(x)$. Then we have

$$y_1(x) = f(x) + \lambda \int_a^b K(x, t) y_1(t) dt \quad \dots (26)$$

$$\text{and } y_2(x) = f(x) + \lambda \int_a^b K(x, t) y_2(t) dt \quad \dots (27)$$

$$\text{Let } y_1(x) - y_2(x) = \phi(x). \quad \dots (28)$$

Subtracting (27) from (26), we have

$$y_1(x) - y_2(x) = \lambda \int_a^b K(x, t) [y_1(t) - y_2(t)] dt$$

$$\text{or} \quad \phi(x) = \lambda \int_a^b K(x, t) \phi(t) dt, \quad \text{using (28)} \quad \dots (29)$$

which is homogeneous integral equation. Applying the Schwarz inequality to (29), we have

$$|\phi(x)|^2 \leq |\lambda|^2 \left(\int_a^b |K(x, t)|^2 dt \right) \times \left(\int_a^b |\phi(t)|^2 dt \right) \quad \dots (30)$$

Integrating with respect to x , (30) gives

$$\int_a^b |\phi(x)|^2 dx \leq |\lambda|^2 \left(\int_a^b \int_a^b |K(x, t)|^2 dx dt \right) \times \left(\int_a^b |\phi(x)|^2 dx \right)$$

$$\text{or} \quad \int_a^b |\phi(x)|^2 dx \leq |\lambda|^2 B^2 \int_a^b |\phi(x)|^2 dx, \quad \text{by (22)}$$

$$\text{or} \quad (1 - |\lambda|^2 B^2) \int_a^b |\phi(x)|^2 dx \leq 0, \quad \dots (31)$$

giving $\phi(x) = 0$, using (25)

$$\text{or} \quad y_1(x) - y_2(x) = 0 \quad \text{or} \quad y_1(x) = y_2(x),$$

showing that (1) has a unique solution

From the uniqueness of the solution of (1), we now proceed to show that the resolvent kernel $R(x, t; \lambda)$ is also unique. If possible, let equation (1) have, with $\lambda = \lambda_0$, two resolvent kernels $R_1(x, t; \lambda_0)$ and $R_2(x, t; \lambda_0)$. In view of the uniqueness of the solution (1), an arbitrary function $f(x)$ satisfies the identity

$$f(x) + \lambda_0 \int_a^b R_1(x, t; \lambda_0) f(t) dt \equiv f(x) + \lambda_0 \int_a^b R_2(x, t; \lambda_0) f(t) dt. \quad \dots (32)$$

Setting $F(x, t; \lambda_0) = R_1(x, t; \lambda_0) - R_2(x, t; \lambda_0)$, (32) reduces to

$$\int_a^b F(x, t; \lambda_0) f(t) dt \equiv 0, \quad \dots (33)$$

for an arbitrary function $f(t)$. Let us choose $f(t) = \overline{F(x, t; \lambda_0)}$ with fixed x . Here $\overline{F(x, t; \lambda_0)}$ denotes the complex conjugate of $F(x, t; \lambda_0)$. Then (33) reduces to

$$\int_a^b |F(x, t; \lambda_0)|^2 dt \equiv 0$$

$$\Rightarrow F(x, t; \lambda_0) = 0 \Rightarrow R_1(x, t; \lambda_0) - R_2(x, t; \lambda_0) = 0 \Rightarrow R_1(x, t; \lambda_0) = R_2(x, t; \lambda_0)$$

showing that the resolvent kernel is unique.

The above analysis can be summed up in the following basic theorem.

Theorem. To each $\mathcal{L}_2$ -kernel $K(x, t)$, there corresponds a unique resolvent kernel $R(x, t; \lambda)$ which is an analytic function of λ , regular at least inside the circle $|\lambda| < B^{-1}$, and represented by the power series

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t).$$

Furthermore, if $f(x)$ is also an $\mathcal{L}_2$ -function, then the unique $\mathcal{L}_2$ -solution of the Fredholm equation

$$y(x) = f(x) + \int_a^b K(x, t) y(t) dt$$

valid in the circle $|\lambda| < B^{-1}$ is given by the formula

$$y(x) = f(x) + \lambda \int_a^b R(x, t; \lambda) f(t) dt.$$

5.8 SOME IMPORTANT THEOREMS

Theorem I. Let $R(x, t; \lambda)$ be the resolvent (or reciprocal) kernel of a Fredholm integral equation.

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt,$$

then the resolvent kernel satisfies the integral equation

$$R(x, t; \lambda) = K(x, t) + \lambda \int_a^b K(x, z) R(z, t; \lambda) dz.$$

Proof. We know that $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t), \quad \dots (1)$$

where iterated kernels (or functions) are given by

$$K_1(x, t) = K(x, t) \quad \dots (2A)$$

and

$$K_m(x, t) = \int_a^b K(x, z) K_{m-1}(z, t) dz. \quad \dots (2B)$$

Now, from (1), we have

$$\begin{aligned} R(x, t; \lambda) &= K_1(x, t) + \sum_{m=2}^{\infty} \lambda^{m-1} K_m(x, t) \\ &= K(x, t) + \sum_{m=2}^{\infty} \lambda^{m-1} \int_a^b K(x, z) K_{m-1}(z, t) dz, \text{ using (2A) and (2B)} \\ &= K(x, t) + \sum_{n=1}^{\infty} \lambda^n \int_a^b K(x, z) K_n(z, t) dz \quad (\text{Setting } m-1 = n) \\ &= K(x, t) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K(x, z) K_m(z, t) dz \\ &= K(x, t) + \lambda \sum_{m=1}^{\infty} \lambda^{m-1} \int_a^b K(x, z) K_m(z, t) dz \\ &= K(x, t) + \lambda \int_a^b \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(z, t) \right] K(x, z) dz \\ &\quad [\text{on changing the order of summation and integration}] \\ &= K(x, t) + \lambda \int_a^b R(z, t; \lambda) K(x, z) dz, \text{ using (1)} \end{aligned}$$

$$\therefore R(x, t; \lambda) = K(x, t) + \lambda \int_a^b K(x, z) R(z, t; \lambda) dz.$$

Theorem II. The series for the resolvent kernel $R(x, t; \lambda)$.

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) \quad \dots (1)$$

is absolutely and uniformly convergent for all values of x and t in the circle $|\lambda| < B^{-1}$.

Proof. In addition to the assumptions of Art. 5.7, we shall use the following inequality

$$\int_a^b |K(x, t)|^2 dx < E^2, \quad E = \text{constant.} \quad \dots (2)$$

(2) follows from one of the conditions for the kernel $K(x, t)$ to be an $\mathcal{L}_2$ kernel. We have, the recurrence formula

$$K_m(x, t) = \int_a^b K_{m-1}(x, z) K(z, t) dz \quad \dots (3)$$

Applying the Schwarz inequality to (3), we have

$$|K_m(x, t)|^2 \leq \left(\int_a^b |K_{m-1}(x, z)|^2 dz \right) \times \left(\int_a^b |K(z, t)|^2 dz \right) \quad \dots (4)$$

Using inequality (23) of Art. 5.7, (4) reduces to $|K_m(x, t)| \leq C_1 E B^{m-1}$

so that

$$\lambda^{m-1} |K_m(x, t)| \leq C_1 E (\lambda^{m-1} B^{m-1}),$$

showing that the series (1) is dominated by the *geometric series* with the general term $C_1 E (\lambda^{m-1} B^{m-1})$ and hence the result

Theorem III. *The resolvent kernel satisfies the integro-differential equation.*

$$\partial R(x, t; \lambda) / \partial \lambda = \int_a^b R(x, z; \lambda) R(z, t; \lambda) dz$$

Proof. We have
$$\int_a^b R(x, z; \lambda) R(z, t; \lambda) dz = \int_a^b \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, z) \sum_{n=1}^{\infty} \lambda^{n-1} K_n(z, t) dz$$

In view of the result of theorem II, both the series on R.H.S. are absolutely and uniformly convergent. Hence we can multiply the series under the integral sign and integrate it term by term.

Thus, we obtain
$$\int_a^b R(x, z; \lambda) R(z, t; \lambda) dz = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda^{m+n-2} K_{m+n}(x, t) \quad \dots (1)$$

$$[\text{Using the result } \int_a^b K_m(x, z) K_n(z, t) dz = K_{m+n}(x, t)]$$

On setting $m + n = p$ and changing the order of summation on R.H.S. of (1), we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda^{m+n-2} K_{m+n}(x, t) &= \sum_{p=2}^{\infty} \sum_{n=1}^{p-1} \lambda^{p-2} K_p(x, t) \\ &= \sum_{p=2}^{\infty} (p-1) \lambda^{p-2} K_p(x, t) = \frac{\partial R(x, t; \lambda)}{\partial \lambda} \quad \dots (2) \end{aligned}$$

From (1) and (2), we get the required result.

5.9 SOLVED EXAMPLES BASED ON SOLUTION OF FREDHOLM INTEGRAL EQUATION OF SECOND KIND BY SUCCESSIVE APPROXIMATIONS OR ITERATIVE METHOD. (Refer Art. 5.7)

Type1: Determination of iterated kernels (or functions) for $y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt$.

Ex. 1. Find the iterated kernels (or functions) for the following kernels :

(i) $K(x, t) = \sin(x - 2t)$, $0 \leq x \leq 2\pi$, $0 \leq t \leq 2\pi$. (ii) $K(x, t) = e^x \cos t$; $a = 0$, $b = \pi$.

[Meerut 2008]

(iii) $K(x, t) = x + \sin t$; $a = -\pi$, $b = \pi$. (iv) $K(x, t) = x - t$; $a = 0$, $b = 1$.

[Kanpur 2007]

Sol. (i) Iterated kernel $K_n(x, t)$ are given by

$$K_1(x, t) = K(x, t) \quad \dots (1)$$

and

$$K_n(x, t) = \int_0^{2\pi} K(x, z)K_{n-1}(z, t) dz, (n = 2, 3, \dots) \quad \dots (2)$$

$$[\because 0 \leq x \leq 2\pi, 0 \leq t \leq 2\pi \Rightarrow a = 0 \text{ and } b = 2\pi]$$

From (1),

$$K_1(x, t) = K(x, t) = \sin(x - 2t). \quad \dots (3)$$

Putting $n = 2$ in (2), we have

$$\begin{aligned} K_2(x, t) &= \int_0^{2\pi} K(x, z)K_1(z, t) dz = \int_0^{2\pi} \sin(x - 2z)\sin(z - 2t) dz, \text{ using (3)} \\ &= \frac{1}{2} \int_0^{2\pi} [\cos(x + 2t - 3z) - \cos(x - 2t - z)] dz = \frac{1}{2} \left[-\frac{1}{3} \sin(x + 2t - 3z) + \sin(x - 2t - z) \right]_0^{2\pi} \\ &= 0, \text{ on simplification.} \end{aligned}$$

$\therefore$

$$K_2(x, t) = 0 \quad \dots (4)$$

Putting $n = 3$ in (2), we have

$$K_3(x, t) = \int_0^{2\pi} K(x, z)K_2(z, t) dz = 0. \quad [\because K_2(z, t) = 0, \text{ by (4)}]$$

Thus, $K_1(x, t) = \sin(x - 2t)$ and $K_n(x, t) = 0$ for $n = 2, 3, 4, \dots$

Part (ii) Iterated kernels $K_n(x, t)$ are given by

$$K_1(x, t) = K(x, t) \quad \dots (1)$$

and

$$K_n(x, t) = \int_0^{\pi} K(x, z)K_{n-1}(z, t) dz. \quad \dots (2)$$

From (1),

$$K_1(x, t) = K(x, t) = e^x \cos t. \quad \dots (3)$$

Putting $n = 2$ in (2), we have

$$\begin{aligned} K_2(x, t) &= \int_0^{\pi} K(x, z)K_1(z, t) dz = \int_0^{\pi} e^x \cos z e^z \cos t dz, \text{ using (3)} \\ &= e^x \cos t \int_0^{\pi} e^z \cos z dz = e^x \cos t \left[\frac{e^z}{1^2 + 1^2} (\cos z + \sin z) \right]_0^{\pi} \\ &\quad \left[\because \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \right] \\ &= e^x \cos t \{ -(1/2) \times e^{\pi} - (1/2) \}. \end{aligned} \quad \dots (4)$$

$\therefore$

$$K_2(x, t) = (-1)^1 \frac{1 + e^{\pi}}{2} e^x \cos t.$$

Next, putting $n = 3$ in (2), we have

$$\begin{aligned} K_3(x, t) &= \int_0^{\pi} K(x, z)K_2(z, t) dz = \int_0^{\pi} e^x \cos z \left\{ -\frac{1 + e^{\pi}}{2} e^z \cos t \right\} dz, \text{ using (3) and (4)} \\ &= -\frac{1 + e^{\pi}}{2} e^x \cos t \int_0^{\pi} e^z \cos z dz = -\frac{1 + e^{\pi}}{2} e^x \cos t \left(-\frac{1 + e^{\pi}}{2} \right), \text{ as before} \end{aligned}$$

$\therefore$

$$K_3(x, t) = (-1)^2 \left(\frac{1 + e^{\pi}}{2} \right)^2 e^x \cos t \quad \dots (5)$$

ans so on Noting (3) (4) and (5), we see that the iterated kernels are given by

$$K_n(x, t) = (-1)^{n-1} \left(\frac{1 + e^\pi}{2} \right)^{n-1} e^x \cos t, n = 1, 2, 3, \dots$$

Part (iii) Iterated kernels are given by

$$K_1(x, t) = K(x, t) \quad \dots (1)$$

and

$$K_n(x, t) = \int_{-\pi}^{\pi} K(x, z) K_{n-1}(z, t) dz. \quad \dots (2)$$

From (1),

$$K_1(x, t) = K(x, t) = x + \sin t. \quad \dots (3)$$

Putting $n = 2$ in (2), we have

$$\begin{aligned} K_2(x, t) &= \int_{-\pi}^{\pi} K(x, z) K_1(z, t) dz = \int_{-\pi}^{\pi} (x + \sin z) (z + \sin t) dz \\ &= x \int_{-\pi}^{\pi} z dz + \sin t \int_{-\pi}^{\pi} \sin z dz + x \sin t \int_{-\pi}^{\pi} dz + \int_{-\pi}^{\pi} z \sin z dz \\ &= x \left[\frac{z^2}{2} \right]_{-\pi}^{\pi} + \sin t [-\cos z]_{-\pi}^{\pi} + x \sin t [z]_{-\pi}^{\pi} + [-z \cos z]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos z dz \\ &\quad \text{[Integrating the last integral by method of integration by parts]} \\ &= 2\pi x \sin t + 2\pi, \text{ on simplification} \\ \therefore K_2(x, t) &= 2\pi (1 + x \sin t). \quad \dots (4) \end{aligned}$$

Next, putting $n = 3$ in (2), we have

$$\begin{aligned} K_3(x, t) &= \int_{-\pi}^{\pi} K(x, z) K_2(z, t) dz = \int_{-\pi}^{\pi} (x + \sin z) \{2\pi(1 + z \sin t)\} dz \\ &= 2\pi \int_{-\pi}^{\pi} (x + xz \sin t + \sin z + z \sin z \sin t) dz \\ &= 2\pi x [z]_{-\pi}^{\pi} + 2\pi x \sin t \left[\frac{z^2}{2} \right]_{-\pi}^{\pi} + 2\pi [-\cos z]_{-\pi}^{\pi} + 2\pi \sin t \left[[-z \cos z]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos z dz \right] \\ &= 4\pi^2 + 4\pi^2 \sin t. \\ \therefore K_3(x, t) &= 4\pi^2 (x + \sin t) = 4\pi^2 K_1(x, t), \text{ using (3)} \quad \dots (5) \end{aligned}$$

Now, putting $n = 4$ in (2), we have

$$\begin{aligned} K_4(x, t) &= \int_{-\pi}^{\pi} K(x, z) K_3(z, t) dz = \int_{-\pi}^{\pi} (x + \sin z) \{4\pi^2(z + \sin t)\} dz, \text{ using (3) and (5)} \\ &= 4\pi^2 \int_{-\pi}^{\pi} (x + \sin z) (z + \sin t) dz = 4\pi^2 \times 2\pi (1 + x \sin t), \text{ as before} \end{aligned}$$

$$\therefore K_4(x, t) = 4\pi^2 K_2(x, t), \text{ using (4)} \quad \dots (6)$$

Proceeding like wise, we have

$$K_5(x, t) = 4\pi^2 K_3(x, t) = 16\pi^4 K_1(x, t), \text{ by (5)} \quad \dots (7)$$

and

$$K_6(x, t) = 4\pi^2 K_4(x, t) = 16\pi^4 K_2(x, t), \text{ by (6)} \quad \dots (8)$$

Observing (3), (5), (7) etc. and also (4), (6), (8) etc. we can write the required iterated kernels $K_n(x, t)$ as follows :

If $n = 2m-1$, then $K_{2m-1}(x, t) = (2\pi)^{2m-2}(x + \sin t)$, $m = 1, 2, 3, \dots$... (9)

If $n = 2m$, then $K_{2m}(x, t) = (2\pi)^{2m-1}(1 + x \sin t)$, $m = 1, 2, 3, \dots$... (10)

Part (iv). Iterated kernels $K_n(x, t)$ are given by

$$K_1(x, t) = K(x, t) \quad \dots (1)$$

and

$$K_n(x, t) = \int_0^1 K(x, z) K_{n-1}(z, t) dz. \quad \dots (2)$$

From (1), $K_1(x, t) = K(x, t) = x - t$ (3)

Putting $n = 2$ in (2), we have

$$\begin{aligned} K_2(x, t) &= \int_0^1 K(x, z) K_1(z, t) dz = \int_0^1 (x - z)(z - t) dz, \text{ by (3)} \\ &= \int_0^1 [(x+t)z - z^2 - xt] dz = \left[(x+t)\frac{z^2}{2} - \frac{z^3}{3} - xt z \right]_0^1 \end{aligned}$$

$$\therefore K_2(x, t) = \frac{x+t}{2} - \frac{1}{3} - xt. \quad \dots (4)$$

Next, putting $n = 3$ in (2), we have

$$\begin{aligned} K_3(x, t) &= \int_0^1 K(x, z) K_2(z, t) dz = \int_0^1 (x - z) \left[\frac{z+t}{2} - \frac{1}{3} - zt \right] dt, \text{ using (3) and (4)} \\ &= \int_0^1 \left[z \left(\frac{x}{2} - xt - \frac{t}{2} + \frac{1}{3} \right) - z^2 \left(\frac{1}{2} - t \right) + x \left(\frac{t}{2} - \frac{1}{3} \right) \right] dz \\ &= \left[\frac{z^2}{2} \left(\frac{x}{2} - xt - \frac{t}{2} + \frac{1}{3} \right) - \frac{z^3}{2} \left(\frac{1}{2} - t \right) + xz \left(\frac{t}{2} - \frac{1}{3} \right) \right]_0^1 \\ &= \frac{1}{2} \left(\frac{x}{2} - xt - \frac{t}{2} + \frac{1}{3} \right) - \frac{1}{3} \left(\frac{1}{2} - t \right) + x \left(\frac{t}{2} - \frac{1}{3} \right) = -\frac{x}{12} + \frac{t}{12} = -\frac{x-t}{12} \end{aligned}$$

$$\therefore K_3(x, t) = -(1/12) \times (x - t) = -(1/12) \times K_1(x, t), \quad \dots (5)$$

Now, putting $n = 4$ in (2), we have

$$\begin{aligned} K_4(x, t) &= \int_0^1 K(x, z) K_3(z, t) dz = \int_0^1 (x - z) \left[-\frac{z-t}{12} \right] dz, \text{ by (3) and (5)} \\ &= -\frac{1}{12} \int_0^1 (x - z)(z - t) dz = -\frac{1}{12} \left(\frac{x+t}{2} - \frac{1}{3} - xt \right), \text{ as before} \end{aligned}$$

$$\therefore K_4(x, t) = -\frac{1}{12} \left(\frac{x+t}{2} - \frac{1}{3} - xt \right) = -\frac{1}{12} K_2(x, t) \quad \dots (6)$$

Similarly, we find

$$K_5(x, t) = -\frac{1}{12} K_3(x, t) = \left(-\frac{1}{12} \right)^2 K_1(x, t), \text{ by (5)} \quad \dots (7)$$

and

$$K_6(x, t) = -\frac{1}{12} K_4(x, t) = \left(-\frac{1}{12} \right)^2 K_2(x, t), \text{ by (6)} \quad \dots (8)$$

and so on.

Observing (3), (5), (7) etc. and also (4), (6), (8) etc., we can write the required iterated kernels $K_n(x, t)$ as follows:

$$\text{If } n = 2m - 1, \quad \text{then} \quad K_{2m-1}(x, t) = \frac{(-1)^{m-1}}{12^{m-1}}(x-t), m = 1, 2, 3, \dots \quad \dots (9)$$

$$\text{If } n = 2m, \quad \text{then} \quad K_{2m}(x, t) = \frac{(-1)^{m-1}}{12^{m-1}} \left(\frac{x+t}{2} - \frac{1}{3} - xt \right), m = 1, 2, \dots \quad \dots (10)$$

Type 2. Determination of the resolvent kernel or reciprocal kernel $R(x, t; \lambda)$ or $\Gamma(x, t; \lambda)$.

$$\text{If } K_n(x, t) \text{ be iterated kernels then} \quad R(x, t; \lambda) = \Gamma(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t).$$

Ex. 2. Determine the resolvent kernels for the Fredholm integral equation having kernels :

(i) $K(x, t) = e^{x+t}$; $a = 0, b = 1$. [Kanpur 2007, 08, 11]

(ii) $K(x, t) = (1+x)(1-t)$; $a = -1, b = 1$. [Kanpur 2006, 10, Meerut 2004, 2012]

Sol. (i) Iterated kernels $K_m(x, t)$ are given by

$$K_1(x, t) = K(x, t). \quad \dots (1)$$

and

$$K_m(x, t) = \int_0^1 K(x, z) K_{m-1}(z, t) dz \quad \dots (2)$$

From (1),

$$K_1(x, t) = K(x, t) = e^{x+t}. \quad \dots (3)$$

Putting $n = 2$ in (2), we have

$$\begin{aligned} K_2(x, t) &= \int_0^1 K(x, z) K_1(z, t) dz = \int_0^1 e^{x+z} e^{z+t} dz, \text{ using (3)} \\ &= e^{x+t} \int_0^1 e^{2z} dz = e^{x+t} \left[\frac{1}{2} e^{2z} \right]_0^1 = e^{x+t} \left(\frac{e^2 - 1}{2} \right), \end{aligned} \quad \dots (4)$$

Putting $n = 3$ in (2), we have

$$\begin{aligned} K_3(x, t) &= \int_0^1 K(x, z) K_2(z, t) dz = \int_0^1 e^{x+z} e^{z+t} \left(\frac{e^2 - 1}{2} \right) dz \\ &= e^{x+t} \left(\frac{e^2 - 1}{2} \right) \int_0^1 e^{2z} dz = e^{x+t} \left(\frac{e^2 - 1}{2} \right)^2, \text{ as before} \end{aligned} \quad \dots (5)$$

and so on. Observing (3), (4) and (5), we may write

$$K_m(x, t) = e^{x+t} \left(\frac{e^2 - 1}{2} \right)^{m-1}, m = 1, 2, 3, \dots \quad \dots (6)$$

Now, the required resolvent kernel is given by

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = \sum_{m=1}^{\infty} \lambda^{m-1} e^{x+t} \left(\frac{e^2 - 1}{2} \right)^{m-1} = e^{x+t} \sum_{m=1}^{\infty} \left\{ \frac{\lambda(e^2 - 1)}{2} \right\}^{m-1} \quad \dots (7)$$

$$\text{But} \quad \sum_{m=1}^{\infty} \left\{ \frac{\lambda(e^2 - 1)}{2} \right\}^{m-1} = 1 + \frac{\lambda(e^2 - 1)}{2} + \left\{ \frac{\lambda(e^2 - 1)}{2} \right\}^2 + \dots$$

which is an infinite geometric series with common ratio $\{\lambda(e^2 - 1)\}/2$.

$$\therefore \sum_{m=1}^{\infty} \left\{ \frac{\lambda(e^2-1)}{2} \right\}^{m-1} = \frac{1}{1 - \{\lambda(e^2-1)\}/2} = \frac{2}{2 - \lambda(e^2-1)},$$

provided $\left| \frac{\lambda(e^2-1)}{2} \right| < 1$ or $|\lambda| < \frac{2}{e^2-1}$... (9)

Using (8) and (9), (7) reduces to

$$R(x, t; \lambda) = \frac{2e^{x+t}}{2 - \lambda(e^2-1)}, \quad \text{provided } |\lambda| < \frac{2}{e^2-1} \quad \dots (10)$$

Part (ii) Iterated Kernels $K_m(x, t)$ are given by

$$K_1(x, t) = K(x, t) \quad \dots (1)$$

and

$$K_m(x, t) = \int_{-1}^1 K(x, z) K_{m-1}(z, t) dz. \quad \dots (2)$$

From (1),

$$K_1(x, t) = K(x, t) = (1+x)(1-t). \quad \dots (3)$$

Putting $n = 2$ in (2), we have

$$\begin{aligned} K_2(x, t) &= \int_{-1}^1 K(x, z) K_1(z, t) dz = \int_{-1}^1 (1+x)(1-z)(1+z)(1-t) dz, \text{ by (3)} \\ &= (1+x)(1-t) \int_{-1}^1 (1-z^2) dz = (1+x)(1-t) \left[z - \frac{1}{3}z^3 \right]_{-1}^1 \end{aligned}$$

$$\therefore K_2(x, t) = (2/3) \times (1+x)(1-t). \quad \dots (4)$$

Next, putting $n = 3$ in (3), we have

$$\begin{aligned} K_3(x, t) &= \int_{-1}^1 K(x, z) K_2(z, t) dz = \int_{-1}^1 (1+x)(1-z) \cdot \frac{2}{3} (1+z)(1-t) dz \\ &= \frac{2}{3} (1+x)(1-t) \int_{-1}^1 (1-z^2) dz = \left(\frac{2}{3} \right)^2 (1+x)(1-t), \text{ as before.} \end{aligned} \quad \dots (5)$$

and so on. Observing (3), (4) and (5), we may write

$$K_m(z, t) = \left(\frac{2}{3} \right)^{m-1} (1+x)(1-t). \quad \dots (6)$$

Now, the required resolvent kernel is given by

$$\begin{aligned} R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = \sum_{m=1}^{\infty} \lambda^{m-1} \left(\frac{2}{3} \right)^{m-1} (1+x)(1-t), \text{ by (6)} \\ &= (1+x)(1-t) \sum_{m=1}^{\infty} \left(\frac{2\lambda}{3} \right)^{m-1} \end{aligned} \quad \dots (7)$$

But

$$\sum_{m=1}^{\infty} \left(\frac{2\lambda}{3} \right)^{m-1} = 1 + \frac{2\lambda}{3} + \left(\frac{2\lambda}{3} \right)^2 + \left(\frac{2\lambda}{3} \right)^3 + \dots$$

which is an infinite geometric series with common ratio $(2\lambda/3)$.

$$\therefore \sum_{m=1}^{\infty} \left(\frac{2\lambda}{3} \right)^{m-1} = \frac{1}{1 - (2\lambda/3)} = \frac{3}{3 - 2\lambda}, \quad \dots (8)$$

provided $|2\lambda/3| < 1$ or $|\lambda| < 3/2$ (9)

Using (8) and (9), (7) reduces to

$$R(x, t; \lambda) = \frac{3(1+x)(1-t)}{3-2\lambda}, \quad \text{provided} \quad |\lambda| < \frac{3}{2} \quad \dots (10)$$

Type 3 : Solution of Fredholm integral equation with help of the resolvent kernel.

Working Rule: Let $y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt$... (1)

be given Fredholm integral equation. Let $K_m(x, t)$ be m th iterated kernel and let $R(x, t; \lambda)$ be the resolvent kernel of (1). Then we have

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) \quad \dots (2)$$

Suppose the sum of infinite series (2) exists and so $R(x, t, \lambda)$ can be obtained in the closed form. Then, the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_a^b R(x, t; \lambda) f(t) dt. \quad \dots (3)$$

Ex. 3. Solve $y(x) = x + \int_0^{1/2} y(t) dt$ [Kanpur 2005, Meerut 2006, 08]

Sol. Given $y(x) = x + \int_0^{1/2} y(t) dt$ (1)

Comparing (1) with $y(x) = f(x) + \lambda \int_0^{1/2} K(x, t) y(t) dt$,

we have $f(x) = x$, $\lambda = 1$, $K(x, t) = 1$ (2)

Let $K_m(x, t)$ be the m th iterated kernel. Then, we have

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

and

$$K_m(x, t) = \int_0^{1/2} K(x, z) K_{m-1}(z, t) dz. \quad \dots (4)$$

From (1),

$$K_1(x, t) = K(x, t) = 1, \text{ by (2)} \quad \dots (5)$$

Putting $m = 2$ in (4), we have

$$K_2(x, t) = \int_0^{1/2} K(x, z) K_1(z, t) dz = \int_0^{1/2} dz = [z]_0^{1/2} = \frac{1}{2}. \quad \dots (6)$$

Next, putting $m = 3$ in (4), we have

$$\begin{aligned} K_3(x, t) &= \int_0^{1/2} K(x, z) K_2(z, t) dz = \int_0^{1/2} \frac{1}{2} dz, \text{ by (5) and (6)} \\ &= (1/2)^2, \end{aligned} \quad \dots (7)$$

and so on. Observing (5), (6) and (7), we find

$$K_m(x, t) = (1/2)^{m-1} \quad \dots (8)$$

Now, the resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1}, \text{ using (2) and (8)} \quad \dots (9)$$

But $\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots$

which is an infinite geometric series with common ratio $1/2$.

$$\therefore \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1} = \frac{1}{1-(1/2)} = 2.$$

Substituting the above value in (9), we have $R(x, t; \lambda) = 2$... (10)

Finally, the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^{1/2} R(x, t; \lambda) f(t) dt \quad \text{or} \quad y(x) = x + \int_0^{1/2} (2t) dt, \text{ by (2) and (10)}$$

$$\text{or} \quad y(x) = x + 2 \left[t^2 / 2 \right]_0^{1/2} = x + (1/4)$$

$$\text{Ex. 4. Solve } y(x) = e^x - \frac{1}{2}e + \frac{1}{2} + \frac{1}{2} \int_0^1 y(t) dt,$$

$$\text{Sol. Given} \quad y(x) = e^x - \frac{1}{2}e + \frac{1}{2} + \frac{1}{2} \int_0^1 y(t) dt, \quad \dots (1)$$

$$\text{Comparing (1) with} \quad y(x) = f(x) + \lambda \int_0^1 K(x, t) y(t) dt,$$

$$\text{we have} \quad f(x) = e^x - \frac{1}{2}e + \frac{1}{2}, \quad \lambda = \frac{1}{2}, \quad K(x, t) = 1. \quad \dots (2)$$

Let $K_m(x, t)$ by the m th iterated kernel. Then, we have

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

$$\text{and} \quad K_m(x, t) = \int_0^1 K(x, z) K_{m-1}(z, t) dz. \quad \dots (4)$$

$$\text{From (1),} \quad K_1(x, t) = K(x, t) = 1, \text{ by (2)} \quad \dots (5)$$

Putting $m = 2$ in (4) and using (5), we have

$$K_2(x, t) = \int_0^1 K(x, z) K_1(z, t) dz = \int_0^1 dz = 1. \quad \dots (6)$$

Next, putting $m = 3$ in (4), and using (5) and (6), we have

$$K_3(x, t) = \int_0^1 K(x, z) K_2(z, t) dz = \int_0^1 dz = 1 \quad \dots (7)$$

and so on. Observing (5), (6) and (7), we find

$$K_m(x, t) = 1 \quad \text{for} \quad m = 1, 2, 3, \dots \quad \dots (8)$$

Now, the resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1}, \text{ using (2) and (8)} \quad \dots (9)$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots = \frac{1}{1-(1/2)}$$

$$\therefore \quad R(x, t; \lambda) = 2. \quad \dots (10)$$

Finally, the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t) dt$$

or
$$y(x) = e^x - \frac{1}{2}e + \frac{1}{2} + \frac{1}{2} \int_0^1 2 \left(e^t - \frac{1}{2}e + \frac{1}{2} \right) dt, \text{ using (2) and (10)}$$

or
$$y(x) = e^x - \frac{1}{2}e + \frac{1}{2} + \left[e^t - \frac{et}{2} + \frac{1}{2}t \right]_0^1$$

or
$$y(x) = e^x - \frac{1}{2}e + \frac{1}{2} + \left[e - \frac{1}{2}e + \frac{1}{2} - 1 \right] = e^x, \quad \text{i.e.,} \quad y(x) = e^x$$

Ex. 5. Solve the following integral equations by the method of successive approximations :

(i) $y(x) = \frac{5x}{6} + \frac{1}{2} \int_0^1 xt y(t) dt.$ [Meerut 2000, 01, 02]

(ii) $y(x) = x + \lambda \int_0^1 xt y(t) dt.$

(iii) $y(x) = \sin x - \frac{x}{4} + \frac{1}{4} \int_0^{\pi/2} xt y(t) dt.$ [Kanpur 2005]

Sol. (i) Given $y(x) = \frac{5x}{6} + \frac{1}{2} \int_0^1 xt y(t) dt.$... (1)

Comparing (1) with
$$y(x) = f(x) + \lambda \int_0^1 K(x, t) y(t) dt$$

we have $f(x) = \frac{5x}{6}, \quad \lambda = \frac{1}{2}, \quad K(x, t) = xt$... (2)

Let $K_m(x, t)$ by the m th iterated kernel. Then

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

and

$$K_m(x, t) = \int_0^1 K(x, z) K_{m-1}(z, t) dz. \quad \dots (4)$$

From (1),

$$K_1(x, t) = K(x, t) = xt \quad \dots (5)$$

Putting $m = 2$ in (4), we have

$$K_2(x, t) = \int_0^1 K(x, z) K_1(z, t) dz = \int_0^1 (xz)(zt) dz = xt \int_0^1 z^2 dz, \text{ by (5)}$$

$$\therefore K_2(x, t) = (1/3) \times xt \quad \dots (6)$$

Next, putting $m = 3$ in (4), we have

$$K_3(x, t) = \int_0^1 K(x, z) K_2(z, t) dz = \int_0^1 (xz) \left(\frac{1}{3} zt \right) dz, \text{ using (5) and (6)}$$

$$= \frac{1}{3} xt \int_0^1 z^2 dz = \left(\frac{1}{3} \right)^2 xt, \quad \dots (7)$$

and so on. Observing (5), (6) and (7), we find

$$K_m(x, t) = (1/3)^{m-1} \times xt. \quad \dots (8)$$

Now, the resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = \sum_{m=1}^{\infty} \left(\frac{1}{3} \right)^{m-1} \left(\frac{1}{3} \right)^{m-1} xt, \text{ by (2) and (8)}$$

$$= xt \sum_{m=1}^{\infty} \left(\frac{1}{6}\right)^{m-1} = xt \left[1 + \frac{1}{6} + \left(\frac{1}{6}\right)^2 + \dots \right] = \frac{xt}{1-(1/6)}$$

$$R(x, t; \lambda) = (6/5) \times xt. \quad \dots (9)$$

Finally, the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t) dt \quad \text{or} \quad y(x) = \frac{5x}{6} + \frac{1}{2} \int_0^1 \left(\frac{6xt}{5} \times \frac{5t}{6} \right) dt, \text{ by (2) and (9)}$$

$$\text{or} \quad y(x) = \frac{5x}{6} + \frac{1}{2} x \int_0^1 t^2 dt = \frac{5x}{6} + \frac{x}{2} \left[\frac{t^3}{3} \right]_0^1 \quad \text{or} \quad y(x) = \frac{5x}{6} + \frac{x}{6} = x.$$

(ii) Proceed like part (i). Required solution is $y(x) = (3x)/(3 - \lambda)$, where $|\lambda| < 3$.

(iii) Proceed like part (i). Required solution is $y(x) = \sin x$.

Ex. 6. Using iterative method, solve $y(x) = f(x) + \lambda \int_0^1 e^{x-t} y(t) dt$.

$$\text{Sol. Given} \quad y(x) = f(x) + \lambda \int_0^1 e^{x-t} y(t) dt. \quad \dots (1)$$

$$\text{Comparing (1) with} \quad y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt,$$

$$\text{we have} \quad f(x) = f(x), \quad \lambda = \lambda, \quad K(x, t) = e^{x-t} \quad \dots (2)$$

Let $K_m(x, t)$ be the m th iterated kernel. Then

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

and

$$K_m(x, t) = \int_0^1 K(x, z) K_{m-1}(z, t) dz. \quad \dots (4)$$

$$\text{From (1),} \quad K_1(x, t) = K(x, t) = e^{x-t} \quad \dots (5)$$

Putting $m = 2$ in (4), and using (5), we have

$$K_2(x, t) = \int_0^1 K(x, z) K_1(z, t) dz = \int_0^1 e^{x-z} e^{z-t} dz = e^{x-t} \int_0^1 dz = e^{x-t}. \quad \dots (6)$$

Next, putting $m = 3$ in (4), we have

$$\begin{aligned} K_3(x, t) &= \int_0^1 K(x, z) K_2(z, t) dz = \int_0^1 e^{x-z} e^{z-t} dz, \text{ by (5) and (6)} \\ &= e^{x-t}, \text{ as before.} \end{aligned} \quad \dots (7)$$

and so on. Observing (5), (6) and (7), we find

$$K_m(x, t) = e^{x-t} \text{ for } m = 1, 2, 3, \dots \quad \dots (8)$$

Now, the resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = \sum_{m=1}^{\infty} \lambda^{m-1} e^{x-t} = e^{x-t} \sum_{m=1}^{\infty} \lambda^{m-1} = e^{x-t} (1 + \lambda + \lambda^2 + \dots)$$

$$\therefore R(x, t; \lambda) = e^{x-t} / (1 - \lambda), \quad \text{provided} \quad |\lambda| < 1. \quad \dots (9)$$

Finally, the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t) dt \quad \text{or} \quad y(x) = f(x) + \lambda \int_0^1 e^{x-t} \frac{1}{1 - \lambda} f(t) dt$$

or
$$y(x) = f(x) + \frac{\lambda}{1-\lambda} \int_0^1 e^{x-t} f(t) dt, \text{ where } |\lambda| < 1.$$

Ex. 7. Solve by the method of successive approximation :

$$y(x) = \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2} + \frac{1}{2} \int_0^1 t y(t) dt. \quad [\text{Meerut 2010, G.N.D.U., Amritsar 2004, Kanpur 2005}]$$

Sol. Given
$$y(x) = \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2} + \frac{1}{2} \int_0^1 t y(t) dt. \quad \dots (1)$$

Comparing (1) with
$$y(x) = f(x) + \lambda \int_0^1 K(x, t) y(t) dt,$$

we have
$$f(x) = \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2}, \quad \lambda = \frac{1}{2}, \quad K(x, t) = t. \quad \dots (2)$$

Let $K_m(x, t)$ by the m th iterated kernel. Then, we have

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

and
$$K_m(x, t) = \int_0^1 K(x, z) K_{m-1}(z, t) dz. \quad \dots (4)$$

From (1),
$$K_1(x, t) = K(x, t) = t. \quad \dots (5)$$

Putting $m = 2$ in (4), we have

$$K_2(x, t) = \int_0^1 K(x, z) K_1(z, t) dz = \int_0^1 zt dz = t \left[\frac{z^2}{2} \right]_0^1 = \frac{1}{2}t. \quad \dots (6)$$

Putting $m = 3$ in (4), and using (2) and (6), we have

$$K_3(x, t) = \int_0^1 K(x, z) K_2(z, t) dz = \int_0^1 z \left(\frac{1}{2}t \right) dz = \left(\frac{1}{2} \right)^2 t \quad \dots (7)$$

and so on. Observing (5), (6) and (7), we find

$$K_m(x, t) = (1/2)^{m-1} t. \quad \dots (8)$$

Now, the resolvent kernel $R(x, t; \lambda)$ is given by

$$\begin{aligned} R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = \sum_{m=1}^{\infty} \left(\frac{1}{2} \right)^{m-1} \cdot \left(\frac{1}{2} \right)^{m-1} t, \text{ by (2) and (8)} \\ &= t \sum_{m=1}^{\infty} \left(\frac{1}{4} \right)^{m-1} = t \left[1 + \frac{1}{4} + \left(\frac{1}{4} \right)^2 + \dots \right] = t \frac{1}{1 - (1/4)} = \frac{4t}{3}. \quad \dots (9) \end{aligned}$$

Finally, the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t) dt$$

or
$$y(x) = \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2} + \frac{1}{2} \int_0^1 \frac{4t}{3} \left(\frac{3}{2}e^t - \frac{1}{2}te^t - \frac{1}{2} \right) dt, \text{ by (2) and (9)}$$

$$= \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2} + \int_0^1 te^t dt - \frac{1}{3} \int_0^1 t^2 e^t dt - \frac{1}{3} \int_0^1 t dt$$

$$\begin{aligned}
&= \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2} + \int_0^1 te^t dt - \frac{1}{3} \left\{ \left[t^2 e^t \right]_0^1 - \int_0^1 2t e^t dt \right\} - \frac{1}{3} \left[\frac{t^2}{2} \right]_0^1 \\
&= \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2} + \int_0^1 te^t dt - \frac{e}{3} + \frac{2}{3} \int_0^1 t e^t dt - \frac{1}{6} \\
&= \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{2}{3} - \frac{e}{3} + \frac{5}{3} \int_0^1 t e^t dt \\
&= \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{2}{3} - \frac{e}{3} + \frac{5}{3} \left\{ \left[t e^t \right]_0^1 - \int_0^1 e^t dt \right\} \\
&= \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{2}{3} - \frac{e}{3} + \frac{5}{3} \left\{ e - \left[e^t \right]_0^1 \right\} \\
&= \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{2}{3} - \frac{e}{3} + \frac{5}{3} \{ e - (e-1) \} \\
\therefore y(x) &= (3/2) \times e^x - (1/2) \times xe^x - (1/3) \times e + 1
\end{aligned}$$

Ex. 8. By iterative method, solve $y(x) = 1 + \lambda \int_0^\pi \sin(x+t)y(t) dt$.

Sol. Given $y(x) = 1 + \lambda \int_0^\pi \sin(x+t)y(t) dt$ (1)

Comparing (1) with $y(x) = f(x) + \lambda \int_0^\pi K(x,t)y(t) dt$,

we have $f(x) = 1$, $\lambda = \lambda$, $K(x, t) = \sin(x+t)$ (2)

Let $K_m(x, t)$ be the m th iterated kernel. Then, we have

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

and

$$K_m(x, t) = \int_0^\pi K(x, z)K_{m-1}(z, t) dz. \quad \dots (4)$$

From (1), $K_1(x, t) = K(x, t) = \sin(x+t)$... (5)

Putting $m = 2$ in (4), we have

$$\begin{aligned}
K_2(x, t) &= \int_0^\pi K(x, z)K_1(z, t) dz = \int_0^\pi \sin(x+z)\sin(z+t) dz \\
&= \frac{1}{2} \int_0^\pi [\cos(x-t) - \cos(2z+x+t)] dz = \frac{1}{2} \left[z \cos(x-t) - \frac{1}{2} \sin(2z+x+t) \right]_0^\pi \\
&= \frac{1}{2} [\pi \cos(x-t) - \frac{1}{2} \sin(x+t) + \frac{1}{2} \sin(x+t)] = \frac{\pi}{2} \cos(x-t). \quad \dots (6)
\end{aligned}$$

Next, putting $m = 3$ in (4), we have

$$\begin{aligned}
K_3(x, t) &= \int_0^\pi K(x, z) K_2(z, t) dz = \int_0^\pi \frac{\pi}{2} \sin(x+z) \cos(z-t) dz, \text{ by (5) and (6)} \\
&= \frac{\pi}{4} \int_0^\pi [\sin(2z+x-t) + \sin(x+t)] dz
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{4} \left[-\frac{1}{2} \cos(2z + x - t) + z \sin(x + t) \right]_0^\pi \\
&= \frac{\pi}{4} \left[-\frac{1}{2} \cos(x - t) + \frac{1}{2} \cos(x - t) + \pi \sin(x + t) \right] = \left(\frac{\pi}{2} \right)^2 \sin(x + t) \quad \dots (7)
\end{aligned}$$

Now, putting $m = 4$ in (4), we have

$$K_4(x, t) = \int_0^\pi K(x, z) K_3(z, t) dz = \int_0^\pi \left(\frac{\pi}{2} \right)^2 \sin(x + z) \cdot \sin(z + t) dz, \text{ by (5) and (7)}$$

$$= \left(\frac{\pi}{2} \right)^2 \int_0^\pi \sin(x + z) \sin(z + t) dz = \left(\frac{\pi}{2} \right)^2 \left(\frac{\pi}{2} \right) \cos(x - t), \text{ as before}$$

$$\therefore K_4(x, t) = (\pi/2)^3 \times \cos(x - t). \quad \dots (8)$$

Next, putting $m = 5$ in (4), we have

$$K_5(x, t) = \int_0^\pi K(x, z) K_4(z, t) dz = \int_0^\pi \left(\frac{\pi}{2} \right)^3 \sin(x + z) \cdot \cos(z - t) dz, \text{ by (5) and (8)}$$

$$= \left(\frac{\pi}{2} \right)^3 \int_0^\pi \sin(x + z) \cos(z - t) dz = \left(\frac{\pi}{2} \right)^3 \left(\frac{\pi}{2} \right) \sin(x + t), \text{ as before}$$

$$\therefore K_5(x, t) = (\pi/2)^4 \times \sin(x + t), \quad \dots (9)$$

and so on. Taking advantage of symmetry and noting that all odd iterated kernels involve $\sin(x + t)$ whereas all even iterated kernels involve $\cos(x - t)$, we may write

$$K_6(x, t) = (\pi/2)^5 \times \cos(x - t), K_7(x, t) = (\pi/2)^6 \times \sin(x + t) \text{ etc.} \quad \dots (10)$$

Now, the resolvent kernel $R(x, t; \lambda)$ is given by

$$\begin{aligned}
R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = K_1(x, t) + \lambda K_2(x, t) + \lambda^2 K_3(x, t) + \lambda^3 K_4(x, t) + \dots \\
&= K_1(x, t) + \lambda^2 K_3(x, t) + \lambda^4 K_5(x, t) + \dots \\
&\quad + \lambda [K_2(x, t) + \lambda^2 K_4(x, t) + \lambda^4 K_6(x, t) + \dots] \\
&= \sin(x + t) \left\{ 1 + \left(\frac{\lambda \pi}{2} \right)^2 + \left(\frac{\lambda \pi}{2} \right)^4 + \dots \right\} + \frac{\lambda \pi}{2} \cos(x - t) \left\{ 1 + \left(\frac{\lambda \pi}{2} \right)^2 + \left(\frac{\lambda \pi}{2} \right)^4 + \dots \right\} \\
&\quad \text{[using (5), (6), (8), (9) and (10) etc]} \\
&= \left\{ \sin(x + t) + \frac{\lambda \pi}{2} \cos(x - t) \right\} \left\{ 1 + \left(\frac{\lambda \pi}{2} \right)^2 + \left(\frac{\lambda \pi}{2} \right)^4 + \dots \right\} \\
&= \left\{ \sin(x + t) + \frac{\lambda \pi}{2} \cos(x - t) \right\} \cdot \frac{1}{1 - (\lambda \pi / 2)^2}, \text{ provided } \left| \frac{\lambda \pi}{2} \right| < \lambda \text{ or } |\lambda| < \frac{2}{\pi}
\end{aligned}$$

$$\text{Thus, } R(x, t; \lambda) = 2 \{ \sin(x + t) + \lambda \pi \cos(x - t) \} / (4 - \lambda^2 \pi^2) \quad \dots (11)$$

Finally, the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^\pi R(x, t; \lambda) f(t) dt$$

$$\begin{aligned}
&= 1 + \frac{2\lambda}{4 - \lambda^2 \pi^2} \int_0^\pi \{2 \sin(x+t) + \lambda \pi \cos(x-t)\} dt, \text{ by (2) and (11)} \\
&= 1 + \frac{2\lambda}{4 - \lambda^2 \pi^2} [-2 \cos(x+t) - \lambda \pi \sin(x-t)]_0^\pi \\
&= 1 + \frac{2\lambda}{4 - \lambda^2 \pi^2} \{-2 \cos(\pi+x) - \lambda \pi \sin(x-\pi) + 2 \cos x + \lambda \pi \sin x\}
\end{aligned}$$

$$\therefore y(x) = 1 + \frac{4\lambda}{4 - \lambda^2 \pi^2} (2 \cos + \lambda \pi \sin x), \text{ where } |\lambda| < \frac{2}{\pi}.$$

Ex. 9. Consider $y(x) = 1 + \lambda \int_0^1 (1 - 3xt) y(t) dt.$

Evaluate the resolvent kernel. For what values of λ the solution does not exist. Obtain solution of the above integral equation.

Sol. Given $y(x) = 1 + \lambda \int_0^1 (1 - 3xt) y(t) dt. \quad \dots (1)$

Comparing (1) with $y(x) = f(x) + \lambda \int_0^1 K(x, t) y(t) dt$

we have $f(x) = 1, \quad \lambda = 1, \quad K(x, t) = 1 - 3xt. \quad \dots (2)$

Let $K_m(x, t)$ be the m th iterated kernel. Then, we have

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

and

$$K_m(x, t) = \int_0^1 K(x, z) K_{m-1}(z, t) dz. \quad \dots (4)$$

From (1), $K_1(x, t) = K(x, t) = 1 - 3xt. \quad \dots (5)$

Putting $m = 2$ in (4), we have

$$\begin{aligned}
K_2(x, t) &= \int_0^1 K(x, z) K_1(z, t) dz = \int_0^1 (1 - 3xz) (1 - 3zt) dz \\
&= \int_0^1 \{1 - 3z(x+t) + 9xz^2\} dz = \left[z - \frac{3z^2}{2}(x+t) + 3xz^3 \right]_0^1 \\
&= 1 - (3/2) \times (x+t) + 3xt. \quad \dots (6)
\end{aligned}$$

Next, putting $m = 3$ in (4), we have

$$\begin{aligned}
K_3(x, t) &= \int_0^1 K(x, z) K_2(z, t) dz = \int_0^1 (1 - 3xz) \left\{ 1 - \frac{3}{2}(z+t) + 3zt \right\} dz \\
&= \int_0^1 (1 - 3xz) \left\{ \left(1 - \frac{3}{2}t \right) - 3z \left(\frac{1}{2} - t \right) \right\} dz \\
&= \int_0^1 \left\{ \left(1 - \frac{3}{2}t \right) - 3z \left(\frac{1}{2} - t + x - \frac{3}{2}xt \right) + 9xz^2 \left(\frac{1}{2} - t \right) \right\} dz \\
&= \left[\left(1 - \frac{3}{2}t \right) z - \frac{3}{2}z^2 \left(\frac{1}{2} - t + x - \frac{3}{2}xt \right) + 3xz^3 \left(\frac{1}{2} - t \right) \right]_0^1
\end{aligned}$$

$$= 1 - \frac{3}{2}t - \frac{3}{2}\left(\frac{1}{2} - t + x - \frac{3}{2}xt\right) + 3x\left(\frac{1}{2} - t\right) = \frac{1}{4}(1 - 3xt)$$

$$\therefore K_3(x, t) = (1/4) \times K_1(x, t), \text{ using (5)} \quad \dots (7)$$

Now, putting $m = 4$ in (4), we have

$$K_4(x, t) = \int_0^1 K(x, z)K_3(z, t) dz = \int_0^1 (1 - 3xz) \cdot \frac{1}{4}(1 - 3zt) dz, \text{ by (5) and (7)}$$

$$= \frac{1}{4} \int_0^1 (1 - 3xz)(1 - 3zt) dz = \frac{1}{4} \left\{ 1 - \frac{3}{2}(x+t) + 3xt \right\}, \text{ as before}$$

$$\therefore K_4(x, t) = \frac{1}{4} \left\{ 1 - \frac{3}{2}(x+t) + 3xt \right\} = \frac{1}{4} K_2(x, t). \text{ by (6)} \quad \dots (8)$$

Next, putting $m = 5$ in (4), we have

$$K_5(x, t) = \int_0^1 K(x, z)K_4(z, t) dz = \int_0^1 (1 - 3xz) \cdot \frac{1}{4} \left\{ 1 - \frac{3}{2}(z+t) + 3zt \right\} dz$$

$$= \frac{1}{4} \int_0^1 (1 - 3xz) \left\{ 1 - \frac{3}{2}(z+t) + 3zt \right\} dz = \frac{1}{4} \times \frac{1}{4}(1 - 3xt), \text{ as before}$$

$$\therefore K_5(x, t) = (1/4)^2 \times (1 - 3xt) = (1/4)^2 \times K_1(x, t), \text{ by (5)} \quad \dots (9)$$

and so on. On observing (5), (6), (7), (8) and (9), we find that all odd iterated kernels involve $K_1(x, t)$ and all even iterated kernels involve $K_2(x, t)$. Hence, by symmetry, we may write

$$K_6(x, t) = (1/4)^2 \times K_2(x, t), \quad K_7(x, t) = (1/4)^3 \times K_1(x, t), \quad \text{and so on.} \quad \dots (10)$$

Now, the resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = K_1(x, t) + \lambda K_2(x, t) + \lambda^2 K_3(x, t) + \lambda^3 K_4(x, t) \dots$$

$$= K_1(x, t) + \lambda^2 K_3(x, t) + \lambda^4 K_5(x, t) + \dots + \lambda [K_2(x, t) + \lambda^2 K_4(x, t) + \lambda^4 K_6(x, t) + \dots]$$

$$= K_1(x, t) + \frac{\lambda^2}{4} K_1(x, t) + \frac{\lambda^4}{4^2} K_1(x, t) + \dots + \lambda \left[K_2(x, t) + \frac{\lambda^2}{4} K_2(x, t) + \frac{\lambda^4}{4^2} K_2(x, t) + \dots \right]$$

[using (7), (8), (9), (10)]

$$= K_1(x, t) \left\{ 1 + (\lambda^2/4) + (\lambda^2/4)^2 + \dots \right\} + \lambda K_2(x, t) \left\{ 1 + (\lambda^2/4) + (\lambda^2/4)^2 + \dots \right\}$$

$$= \{K_1(x, t) + \lambda K_2(x, t)\} \left\{ 1 + (\lambda^2/4) + (\lambda^2/4)^2 + \dots \right\}$$

$$= \{K_1(x, t) + \lambda K_2(x, t)\} \frac{1}{1 - (\lambda^2/4)}, \quad \text{provided } \lambda^2 < 4 \quad \text{or} \quad |\lambda| < 2$$

$$= \frac{4}{4 - \lambda^2} \left[1 - 3xt + \lambda \left\{ 1 - \frac{3}{2}(x+t) + 3xt \right\} \right]$$

$$\therefore R(x, t; \lambda) = \frac{4}{4 - \lambda^2} \left[1 + \lambda - \frac{3}{2}x\lambda - 3t \left(x + \frac{1}{2}\lambda - x\lambda \right) \right] \quad \dots (11)$$

Finally, the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t) dt$$

or

$$y(x) = 1 + \frac{4\lambda}{4 - \lambda^2} \int_0^1 \left[1 + \lambda - \frac{3\lambda}{2}x - 3t \left(x + \frac{1}{2}\lambda - x\lambda \right) \right] dt, \text{ by (2) and (11)}$$

$$\begin{aligned}
&= 1 + \frac{4\lambda}{4-\lambda^2} \left[\left(1 + \lambda - \frac{3\lambda}{2}x \right) t - \frac{3}{2}t^2 \left(x + \frac{1}{2}\lambda - x\lambda \right) \right]_0^1 \\
&= 1 + \frac{4\lambda}{4-\lambda^2} \left[1 + \lambda - \frac{3\lambda x}{2} - \frac{3}{2} \left(x + \frac{\lambda}{2} - x\lambda \right) \right] = 1 + \frac{4\lambda}{4-\lambda^2} \left[1 + \lambda - \frac{3}{2}x - \frac{3\lambda}{4} \right] \\
&= 1 + \frac{4\lambda}{4-\lambda^2} \times \frac{4+\lambda-6x}{4} = \frac{4-\lambda^2 + (4+\lambda-6x)}{4-\lambda^2} \\
\therefore y(x) &= \frac{4+2\lambda(2-3x)}{4-\lambda^2}, \quad \text{provided} \quad |\lambda| < 2.
\end{aligned}$$

Since solution exists only when $|\lambda| < 2$, it follows that (1) will not possess solution if $|\lambda| > 2$.

Type 4 : Solution of Fredholm integral equation when the resolvent kernel cannot be obtained in closed form i.e., the sum of infinite series occurring in the formula of the resolvent kernel cannot be determined. In such integral equation, we use the method of successive approximations to find solutions upto third order.

Working Rule : Let the given Fredholm integral equation of the second kind be

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt \quad \dots (1)$$

As zero-order approximation, we take $y_0(x) = f(x)$... (2)

If n th order approximation be $y_n(x)$, then

$$y_n(x) = f(x) + \lambda \int_a^b K(x, t) y_{n-1}(x). \quad \dots (3)$$

With help of (2) and (3), we easily obtain $y_1(x)$, $y_2(x)$ and $y_3(x)$.

Remark. Sometimes the zero-order approximation is mentioned in the problem. In that case, we modify equation (2) according to data of the problem.

Ex. 10. (a) Solve the following integral equation $y(x) = 1 + \lambda \int_0^1 (x+t) y(t) dt$,

by the method of successive approximation to third order.

(b) Solve the integral equation $y(x) = 1 + \lambda \int_0^1 (x+t) y(t) dt$ by the method of successive approximation upto second order for $y_0(x) = 1$ (Kanpur 2009)

Sol. (a). Given $y(x) = 1 + \lambda \int_0^1 (x+t) y(t) dt$, ... (1)

Let $y_0(x)$ denote the zero-order approximation, Then, we may take

$$y_0(x) = 1. \quad \dots (2)$$

If $y_n(x)$ denotes the n th order approximation, then we know that

$$y_n(x) = 1 + \lambda \int_0^1 (x+t) y_{n-1}(t) dt. \quad \dots (3)$$

Putting $n = 1$ in (3), $y_1(x) = 1 + \lambda \int_0^1 (x+t) y_0(t) dt = 1 + \lambda \int_0^1 (x+t) dt$, by (2)

or
$$y_1(x) = 1 + \lambda \left[xt + \frac{1}{2}t^2 \right]_0^1 = 1 + \lambda \left(x + \frac{1}{2} \right). \quad \dots (4)$$

Next, putting $n = 2$ in (3), we have

$$\begin{aligned}
 y_2(x) &= 1 + \lambda \int_0^1 (x+t) y_1(t) dt = 1 + \lambda \int_0^1 (x+t) \left\{ 1 + \lambda \left(t + \frac{1}{2} \right) \right\} dt, \text{ by (4)} \\
 &= 1 + \lambda \int_0^1 (x+t) \left\{ \left(1 + \frac{\lambda}{2} \right) + \lambda t \right\} dt = 1 + \lambda \int_0^1 \left[x \left(1 + \frac{\lambda}{2} \right) + t \left(1 + \frac{\lambda}{2} + \lambda x \right) + \lambda t^2 \right] dt, \\
 &= 1 + \lambda \left[x \left(1 + \frac{\lambda}{2} \right) t + \frac{t^2}{2} \left(1 + \frac{\lambda}{2} + \lambda x \right) + \frac{\lambda t^3}{3} \right]_0^1 \\
 &= 1 + \lambda \left[x \left(1 + \frac{\lambda}{2} \right) + \frac{1}{2} \left(1 + \frac{\lambda}{2} + \lambda x \right) + \frac{\lambda}{3} \right] = 1 + \lambda \left(x + \frac{1}{2} \right) + \lambda^2 \left(x + \frac{7}{12} \right). \quad \dots (5)
 \end{aligned}$$

Finally, putting $n = 3$ in (3), we have

$$\begin{aligned}
 y_3 &= 1 + \lambda \int_0^1 (x+t) y_2(t) dt = 1 + \lambda \int_0^1 (x+t) \left\{ 1 + \lambda \left(t + \frac{1}{2} \right) + \lambda^2 \left(t + \frac{7}{12} \right) \right\} dt \\
 &= 1 + \lambda \int_0^1 (x+t) \left\{ \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} \right) + \lambda t(1+\lambda) \right\} dt \\
 &= 1 + \lambda \int_0^1 \left\{ x \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} \right) + t \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} + x\lambda + x\lambda^2 \right) + \lambda t^2(1+\lambda) \right\} dt \\
 &= 1 + \lambda \left[x \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} \right) t + \frac{t^2}{2} \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} + x\lambda + x\lambda^2 \right) + \frac{1}{3} \lambda t^3(1+\lambda) \right]_0^1 \\
 &= 1 + \lambda x \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} \right) + \frac{\lambda}{2} \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} + x\lambda + x\lambda^2 \right) + \frac{1}{3} \lambda^2(1+\lambda). \\
 \therefore \quad y_3(x) &= 1 + \lambda \left(x + \frac{1}{2} \right) + \lambda^2 \left(x + \frac{7}{12} \right) + \lambda^3 \left(\frac{13}{12} x + \frac{5}{8} \right)
 \end{aligned}$$

(b) Do as in part (a). We are given $y_0(x) = 1$. The required solution to second order is given by (5) of part (a).

Ex. 11. Solve the inhomogeneous Fredholm integral equation of the second kind

$y(x) = 2x + \lambda \int_0^1 (x+t) y(t) dt$, by the method of successive approximations to the third order by taking $y_0(x) = 1$.

Sol. Given
$$y(x) = 2x + \lambda \int_0^1 (x+t) y(t) dt, \quad \dots (1)$$

Let $y_0(x)$ denote the zero-order approximation, then given that $y_0(x) = 1. \quad \dots (2)$

If $y_n(x)$ denotes the n th order approximation, then we know that

$$y_n(x) = 2x + \lambda \int_0^1 (x+t) y_{n-1}(t) dt. \quad \dots (3)$$

Putting $n = 1$ in (3), $y_1(x) = 2x + \lambda \int_0^1 (x+1) y_0(t) dt = 2x + \lambda \int_0^1 (x+1) dt$, using (2)

or
$$y_1(x) = 2x + \lambda \left[tx + \frac{t^2}{2} \right]_0^1 = 2x + \lambda \left(x + \frac{1}{2} \right). \quad \dots (4)$$

Next, putting $n = 2$ in (3), we have

$$\begin{aligned} y_2(x) &= 2x + \lambda \int_0^1 (x+t) y_1(t) dt = 2x + \lambda \int_0^1 (x+t) \left\{ 2t + \lambda \left(t + \frac{1}{2} \right) \right\} dt, \text{ by (4)} \\ &= 2x + \lambda \int_0^1 (x+t) \left\{ \frac{\lambda}{2} + t(2+\lambda) \right\} dt = 2x + \lambda \int_0^1 \left\{ \frac{x\lambda}{2} + t \left(\frac{\lambda}{2} + 2x + x\lambda \right) + t^2(2+\lambda) \right\} dt \\ &= 2x + \lambda \left[\frac{x\lambda t}{2} + \frac{t^2}{2} \left(\frac{\lambda}{2} + 2x + x\lambda \right) + \frac{t^3}{3} (2+\lambda) \right]_0^1 \\ &= 2x + \lambda \left[\frac{x\lambda}{2} + \frac{1}{2} \left(\frac{\lambda}{2} + 2x + x\lambda \right) + \frac{1}{3} (2+\lambda) \right] = 2x + \lambda \left(x + \frac{2}{3} \right) + \lambda^2 \left(x + \frac{7}{12} \right) \quad \dots (5) \end{aligned}$$

Finally, putting $n = 3$ in (3), we have

$$\begin{aligned} y_3(x) &= 2x + \lambda \int_0^1 (x+t) y_2(t) dt = 2x + \lambda \int_0^1 (x+t) \left\{ 2t + \lambda \left(t + \frac{2}{3} \right) + \lambda^2 \left(t + \frac{7}{12} \right) \right\} dt \\ &= 2x + \lambda \int_0^1 (x+t) \left\{ \frac{2\lambda}{3} + \frac{7\lambda^2}{12} + t(2+\lambda+\lambda^2) \right\} dt \\ &= 2x + \lambda \int_0^1 \left\{ x \left(\frac{2\lambda}{3} + \frac{7\lambda^2}{12} \right) + t \left(\frac{2\lambda}{3} + \frac{7\lambda^2}{12} + 2x + x\lambda + x\lambda^2 \right) + t^2(2+\lambda+\lambda^2) \right\} dt \\ &= 2x + \lambda \left[xt \left(\frac{2\lambda}{3} + \frac{7\lambda^2}{12} \right) + \frac{t^2}{2} \left(\frac{2\lambda}{3} + \frac{7\lambda^2}{12} + 2x + x\lambda + x\lambda^2 \right) + \frac{t^3}{3} (2+\lambda+\lambda^2) \right]_0^1 \\ &= 2x + \lambda \left[x \left(\frac{2\lambda}{3} + \frac{7\lambda^2}{12} \right) + \frac{1}{2} \left(\frac{2\lambda}{3} + \frac{7\lambda^2}{12} + 2x + x\lambda + x\lambda^2 \right) + \frac{1}{3} (2+\lambda+\lambda^2) \right] \end{aligned}$$

or
$$y(x) = 2x + \lambda \left(x + \frac{2}{3} \right) + \lambda^2 \left(\frac{7}{6}x + \frac{2}{3} \right) + \lambda^3 \left(\frac{13}{12}x + \frac{5}{8} \right)$$

5.10. RECIPROCAL FUNCTIONS.

Let $K_n(x, t)$, $n = 1, 2, 3, \dots$ be iterated kernels (or functions) given by

$$K_1(x, t) = K(x, t) \quad \dots (1)$$

and

$$K_n(x, t) = \int_a^b K(x, z) K_{n-1}(z, t) dz. \quad \dots (2)$$

Let $-k(x, t) = K_1(x, t) + K_2(x, t) + \dots + K_n(x, t) + \dots \quad \dots (3)$

Let $K(x, t)$ be real and continuous in a rectangle R , for which $a \leq x \leq b$, $a \leq t \leq b$. Let $K(x, t) \neq 0$ and let M be the maximum value of $|K(x, t)|$ in R , that is, $|K(x, t)| \leq M$ in R . Then, if $M(b-a) < 1$, we easily find that the infinite series (3) for $k(x, t)$ is absolutely and uniformly convergent. Hence, $k(x, t)$ is real and continuous in R .

We know that
$$K_{p+q}(x, t) = \int_a^b K_p(x, z) K_q(z, t) dz. \quad \dots (4)$$

Now, re-writing (3), we have

$$-k(x, t) - K_1(x, t) = K_2(x, t) + K_3(x, t) + \dots + K_n(x, t) + \dots \quad \dots (5)$$

or
$$-k(x, t) - K(x, t) = K_2(x, t) + K_3(x, t) + \dots + K_n(x, t) + \dots \quad \dots (6)$$

Using (4), (6) may be written as

$$\begin{aligned} -k(x, t) - K(x, t) &= \int_a^b K_1(x, z) K_1(z, t) dz + \int_a^b K_1(x, z) K_2(z, t) dz + \dots \\ &= \int_a^b K_1(x, z) [K_1(z, t) + K_2(z, t) + K_3(z, t) + \dots] dz \end{aligned}$$

Thus,
$$-k(x, t) - K(x, t) = -\int_a^b K(x, z) k(z, t) dz, \text{ using (1) and (3)} \quad \dots (7)$$

Again, using (4), (6) may be written in another form as follows :

$$\begin{aligned} -k(x, t) - K(x, t) &= \int_a^b K_1(x, z) K_1(z, t) dz + \int_a^b K_2(x, z) K_1(z, t) dz + \dots \\ &= \int_a^b [K_1(x, z) + K_2(x, z) + K_3(x, z) + \dots] K_1(z, t) dz \\ &= -\int_a^b k(x, z) K_1(z, t) dz, \text{ using (1) and (3)} \quad \dots (8) \end{aligned}$$

From (7) and (8), we have

$$k(x, t) + K(x, t) = \int_a^b K(x, z) k(z, t) dz \quad \dots (9A)$$

and

$$k(x, t) + K(x, t) = \int_a^b k(x, z) K(z, t) dz. \quad \dots (9B)$$

Two functions $K(x, t)$ and $k(x, t)$ are known as *reciprocal* if they are both real and continuous in R and if they satisfy the condition (9A) or (9B).

Theorem I. If $K(x, t)$ is real and continuous in R , there exists a reciprocal functions $k(x, t)$ given by

$$-k(x, t) = K_1(x, t) + K_2(x, t) + K_3(x, t) + \dots$$

where $K_1(x, t), K_2(x, t), \dots$ are iterated functions (or kernels), provided that $M(b-a) < 1$, where M is the maximum value of $|K(x, t)|$ in rectangle R , for which $a \leq x \leq b$ and $a \leq t \leq b$.

Proof. Left as an exercise.

Theorem II. Volterra's solution of Fredholm Integral equation of the second kind

Given.
$$y(x) = f(x) + \int_a^b K(x, t) y(t) dt. \quad \dots (1)$$

If (i) $K(x, t)$ is real and continuous in rectangle R , for which $a \leq x \leq b$ and $a \leq t \leq b$. $K(x, t) \neq 0$.

(ii) $f(x)$ is real and continuous in I and $f(x) \neq 0$.

(iii) If a function $k(x, t)$ reciprocal to $K(x, t)$ exists, then the integral equation (1) has a unique continuous solution in I given by

$$y(x) = f(x) - \int_a^b k(x, t) f(t) dt. \quad \dots (2)$$

Proof. Re-writing (1), we have
$$y(x) = f(x) + \int_a^b K(x, z) y(z) dz. \quad \dots (3)$$

Replaing x by t in (3), we have
$$y(t) = f(t) + \int_a^b K(t, z) y(z) dz. \quad \dots (4)$$

Multiplying both sides of (4) by $k(x, t)$ and then integrating both sides w.r.t. ' t ' from a to b , we have

$$\int_a^b k(x, t) y(t) dt = \int_a^b k(x, t) f(t) dt + \int_a^b k(x, t) \left\{ \int_a^b K(t, z) y(z) dz \right\} dt$$

or
$$\int_a^b k(x, t) y(t) dt = \int_a^b k(x, t) f(t) dt + \int_a^b y(z) \left\{ \int_a^b k(x, t) K(t, z) dt \right\} dz \quad \dots (5)$$

[changing the order of integration]

Since $k(x, t)$ and $K(x, t)$ are reciprocal functions, we have by definition

$$\int_a^b k(x, t) K(t, z) dt = k(x, z) + K(x, z). \quad \dots (6)$$

Using (6), (5) becomes

$$\int_a^b k(x, t) y(t) dt = \int_a^b k(x, t) f(t) dt + \int_a^b y(z) [k(x, z) + K(x, z)] dz$$

or
$$\int_a^b k(x, z) y(z) dz = \int_a^b k(x, t) f(t) dt + \int_a^b k(x, z) y(z) dz + \int_a^b K(x, z) y(z) dz$$

or
$$0 = \int_a^b k(x, t) f(t) dt + \int_a^b K(x, t) y(t) dt \quad \dots (7)$$

From (1)
$$\int_a^b K(x, t) y(t) dt = y(x) - f(x) \quad \dots (8)$$

Using (8), (7) becomes
$$0 = \int_a^b k(x, t) f(t) dt + y(x) - f(x)$$

or
$$y(x) = f(x) - \int_a^b k(x, t) f(t) dt,$$

showing that if (1) has a continous solution, then it is given by (2) and it is unique solution.

We now show that the expression of $y(x)$ given by (2) is, indeed, a solution of (1). Re-writing (2), we have

$$f(x) = y(x) + \int_a^b k(x, t) f(t) dt \quad \dots (9)$$

Clearly, (9) is an integral equation for the determination of $f(x)$. The function reciprocal to $k(x, t)$ is $K(x, t)$. Hence, making use of the method of getting solution (2) from (1), we find that if (9) has a conitnuous solution, it is unique and is given by

$$f(x) = y(x) - \int_a^b K(x, t) y(t) dt \quad \text{or} \quad y(x) = f(x) + \int_a^b K(x, t) y(t) dt,$$

which is the equation (1) from which we started. Hence we find that (1) is satisfied by the value of $y(x)$ given by (2).

ILLUSTRATIVE SOLVED EXAMPLES

Ex. 1. Solve $y(x) = f(x) + \frac{1}{2} \int_0^1 e^{x-t} y(t) dt$.

Sol. Given
$$y(x) = f(x) + \frac{1}{2} \int_0^1 e^{x-t} y(t) dt. \quad \dots (1)$$

Comparing (1) with
$$y(x) = f(x) + \int_0^1 K(x, t) y(t) dt,$$

we have
$$K(x, t) = (1/2) \times e^{x-t}. \quad \dots (2)$$

Let $k(x, t)$ be the reciprocal function of $K(x, t)$. Then if $K_1(x, t)$, $K_2(x, t)$, ... be iterated functions, then

$$\therefore -k(x, t) = K_1(x, t) + K_2(x, t) + K_3(x, t) + \dots \quad \dots (3)$$

Now, the solution of (1) is given by

$$y(x) = f(x) - \int_0^1 k(x, t) f(t) dt. \quad \dots (4)$$

We know that iterated functions are given by

$$K_1(x, t) = K(x, t) \quad \dots (5)$$

and
$$K_n(x, t) = \int_0^1 K(x, z) K_{n-1}(z, t) dz, (n = 2, 3, \dots) \quad \dots (6)$$

From (5) and (2),
$$K_1(x, t) = K(x, t) = (1/2) \times e^{x-t}. \quad \dots (7)$$

Putting $n = 2$ in (6), we have

$$K_2(x, t) = \int_0^1 K(x, z) K_1(z, t) dz = \int_0^1 \frac{1}{2} e^{x-z} \cdot \frac{1}{2} e^{z-t} dz = \frac{1}{2^2} e^{x-t} \int_0^1 dz = \frac{1}{2^2} e^{x-t}. \quad \dots (8)$$

Next, putting $n = 3$ in (6), we have

$$K_3(x, t) = \int_0^1 K(x, z) K_2(z, t) dz = \int_0^1 \frac{1}{2} e^{x-z} \cdot \frac{1}{2^2} e^{z-t} dz = \frac{1}{2^3} e^{x-t} \int_0^1 dz = \frac{1}{2^3} e^{x-t}, \quad \dots (9)$$

and so on, Observing (7), (8) and (9), we may write

$$K_n(x, t) = (1/2^n) \times e^{x-t}, \quad n = 1, 2, 3 \dots \quad \dots (10)$$

Substituting the above values of $K_1(x, t)$, $K_2(x, t)$, $K_3(x, t)$, ... etc. in (3), we have

$$\begin{aligned} -k(x, t) &= \frac{1}{2} e^{x-t} + \frac{1}{2^2} e^{x-t} + \frac{1}{2^3} e^{x-t} + \dots = \frac{1}{2} e^{x-t} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots \right) = \frac{1}{2} e^{x-t} \frac{1}{1 - (1/2)} \\ \therefore -k(x, t) &= e^{x-t} \quad \text{or} \quad k(x, t) = -e^{x-t}. \quad \dots (11) \end{aligned}$$

Substituting the above value of $k(x, t)$ in (4), we have

$$y(x) = f(x) + \int_0^1 e^{x-t} f(t) dt,$$

which is the required solution.

Ex. 2. Solve the following integral equations.

(i) $y(x) = x + \int_0^1 y(t) dt$

Ans. $y(x) = x$

$$(ii) \quad y(x) = \frac{5x}{6} + \frac{1}{2} \int_0^1 xt \, y(t) \, dt \quad \text{Ans. } y(x) = x$$

$$(iii) \quad y(x) = \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2} + \frac{1}{2} \int_0^1 t \, y(t) \, dt. \quad \text{Ans. } y(x) = \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{3}e + 1$$

$$(iv) \quad y(x) = \sin x - \frac{x}{4} + \frac{1}{4} \int_0^{\pi/2} xt \, y(t) \, dt. \quad \text{Ans. } y(x) = \sin x$$

Sol. Solve as in the above solved Ex. 1.

EXERCISE 5A

1. Find the iterated kernels $K_n(x, t)$ of the following kernels for specified a and b .

$$(i) \quad K(x, t) = x - t; \quad a = -1, \quad b = 1.$$

$$(ii) \quad K(x, t) = \sin(x - t); \quad a = 0, \quad b = \pi/2 \quad \text{for } n = 2, 3$$

$$(iii) \quad K(x, t) = (x - t)^2; \quad a = -1, \quad b = 1 \quad \text{for } n = 2, 3. \quad (\text{Kanpur 2008, 09})$$

$$(iv) \quad K(x, t) = xe^t; \quad a = 0, \quad b = 1.$$

$$(v) \quad K(x, t) = e^{[x-t]}; \quad a = 0, \quad b = 1 \quad \text{for } n = 2$$

$$(vi) \quad K(x, t) = e^{|x|+t}. \quad a = -1, \quad b = 1 \quad \text{for } n = 2$$

$$\text{Ans. (i)} \quad K_{2m-1}(x, t) = \left(-\frac{4}{3}\right)^{m-1} (x - t), \quad K_{2m}(x, t) = 2(-1)^m \left(\frac{4}{3}\right)^{m-1} \left(xt + \frac{1}{3}\right), \quad m = 1, 2, 3, \dots$$

$$(ii) \quad K_2(x, t) = \frac{1}{2} \sin(x + t) - \frac{1}{4} \pi \cos(x - t), \quad K_3(x, t) = \frac{1}{16} (4 - \pi^2) \sin(x - t).$$

$$(iii) \quad K_2(x, t) = (2/3) \times (x + t)^2 + 2x^2t^2 + (4/3) \times xt + (2/5).$$

$$K_3(x, t) = (56/45) \times (x^2 + t^2) + (8/3) \times x^2t^2 - (32/9) \times xt + (8/15).$$

$$(iv) \quad K_n(x, t) = xe^t, \quad n = 1, 2, 3, \dots$$

$$(v) \quad K_2(x, t) = \begin{cases} \frac{e^{x+t} + e^{2-x-t}}{2} + (t - x - 1)e^{t-x}, & 0 \leq x \leq t \\ \frac{e^{x+t} + e^{2-x-t}}{2} + (x - t - 1)e^{x-t}, & t \leq x \leq 1. \end{cases}$$

$$(vi) \quad K_2(x, t) = \begin{cases} (1/2) \times (e^2 + 1)e^{t-x}, & -1 \leq x \leq 0 \\ (1/2) \times (e^2 + 1)e^{t+x}, & 0 \leq x \leq 1. \end{cases}$$

2. (a) Construct the resolvent kernels for the following kernels for specified a and b .

$$(i) \quad K(x, t) = \sin x \cos t; \quad a = 0, \quad b = \pi/2. \quad [\text{Kanpur 2006}] \quad (ii) \quad K(x, t) = xe^t; \quad a = -1, \quad b = 1.$$

$$(iii) \quad K(x, t) = x^2t^2; \quad a = -1, \quad b = 1.$$

$$(iv) \quad K(x, t) = xt; \quad a = -1, \quad b = 1.$$

$$(v) \quad K(x, t) = \sin x \cos t + \cos 2x \sin 2t; \quad a = 0, \quad b = 2\pi.$$

$$(vi) \quad K(x, t) = 1 + (2x - 1)(2t - 1); \quad a = 0, \quad b = 1.$$

$$(vii) \quad K(x, t) = xt + x^2t^2; \quad a = -1, \quad b = 1$$

$$\text{Ans. (i)} \quad R(x, t; \lambda) = \frac{2 \sin x \cos t}{2 - \lambda}; \quad |\lambda| < 2.$$

$$(ii) \quad R(x, t; \lambda) = \frac{xe^{t+1}}{e - 2\lambda}; \quad |\lambda| < \frac{e}{2}.$$

$$(iii) \quad R(x, t; \lambda) = (5x^2t^2)/(5 - 2\lambda); \quad |\lambda| < 5/2.$$

$$(iv) \quad R(x, t; \lambda) = (3xt)/(3 - 2\lambda); \quad |\lambda| < 3/2. \quad (v) \quad R(x, t; \lambda) = \sin x \cos t + \cos 2x \sin 2t.$$

$$(vi) R(x, t; \lambda) = \frac{1}{1-\lambda} + \frac{3(2x-1)(2t-1)}{3-\lambda}; |\lambda| < 1 \quad (vii) R(x, t; \lambda) = \frac{3xt}{3-2\lambda} + \frac{5x^2t^2}{5-2\lambda}; |\lambda| < 3/2$$

2. (b) Find the resolvent kernel associated with the following kernels :

- (i) $|x-t|$, in the interval $(0, 1)$ (ii) $e^{-|x-t|}$, in the interval $(0, 1)$
 (iii) $\cos(x+t)$, in the interval $(0, 2\pi)$.

3. Solve $y(x) = \frac{5x}{6} - \frac{1}{9} + \frac{1}{3} \int_0^1 (x+t) y(t) dt$, by the method of successive approximations.

[Ans. $y(x) = x$.]

4. Consider the integral equation $y(x) = 1 + \lambda \int_0^1 xt y(t) dt$.

- (i) Make use of the relation $|\lambda| < B^{-1}$ to show that the iterative procedure is valid for $|\lambda| < 3$.

- (ii) Show that the iterative procedure leads formally to the solution

$$y(x) = 1 + x[(\lambda/2) + (\lambda^2/6) + (\lambda^3/18) + \dots].$$

- (iii) Use the method of chapter 4 to obtain the exact solution

$$y(x) = 1 + [3\lambda x/2(x-\lambda)], \quad \lambda \neq 3.$$

5. Explain the method of solving a Fredholm integral equation by the method of successive approximation and hence solve the integral equation

$$y(x) = x + \int_0^{1/2} y(t) dt. \quad [\text{Ans. } y(x) = x + \text{constant}]$$

6. Solve the integral equation by the method of iterated kernel

$$y(x) = \frac{5x}{6} + \frac{1}{2} \int_0^1 xt y(t) dt. \quad [\text{Ans. } y(x) = x.]$$

7. Find the iterated kernel of the kernel $K(x, t) = x + \sin t$, in the interval $(-\pi, \pi)$.

[Hint. Here $a = -\pi$, $b = \pi$. Refer solved Ex. 1. part (iii) of Art. 5.9]

8. Explain the method of successive substitution for solving Fredholm integral equation of second kind and show that the solution is unique. How does this method differ from the method of successive approximation?

9. Define iterated kernels. Prove that the n th iterated kernel $K_n(x, t)$ satisfies the relation

$$K_n(x, t) = \int_a^b K_m(x, z) K_{n-m}(z, t) dz, \quad m < n.$$

10. Define resolvent kernel and find the resolvent kernel of the kernel $K(x, t) = 1 - 3xt$ in $(0, 1)$.

11. Explain the iterative method (method of successive approximation) of solving the Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt$$

and obtain the sufficient condition for the convergence of the iterative procedure.

12. What do you understand by iterated function, reciprocal function and resolvent kernel. If $k(x, t)$ and $K(x, t)$ and reciprocal functions, then prove that

$$K(x, t) + k(x, t) = \int_a^b K(x, z) k(z, t) dz.$$

13. Solve the integral equation $y(x) = 1 + \lambda \int_0^1 (x+t) y(t) dt$, by the method of successive approximations and show that the estimate afforded by the relation $|\lambda| < B^{-1}$ is conservative in this case.

14. Obtain the radius of convergence of the Neumann series when the function $f(x)$ and the kernel $K(x, t)$ are continuous in the interval (a, b) .

5.11. SOLUTION OF VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND BY SUCCESSIVE APPROXIMATIONS. ITERATIVE METHOD. NEUMANN SERIES.

[Meerut 2000, 01, 03, 07]

Consider Volterra integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt. \quad \dots (1)$$

As a zero-order approximation to the required solution $y(x)$, let us take

$$y_0(x) = f(x). \quad \dots (2)$$

Further, if $y_n(x)$ and $y_{n-1}(x)$ are the n th order and $(n-1)$ th order approximations respectively, then these are connected by

$$y_n(x) = f(x) + \lambda \int_a^x K(x, t) y_{n-1}(t) dt. \quad \dots (3)$$

We know that the iterated kernels (or iterated functions) $K_n(x, t)$, $(n = 1, 2, 3, \dots)$ are defined by

$$K_1(x, t) = K(x, t) \quad \dots (4A)$$

and
$$K_n(x, t) = \int_t^x K(x, z) K_{n-1}(z, t) dz. \quad \dots (4B)$$

Putting $n = 1$ in (3), the first-order approximation $y_1(x)$ is given by

$$y_1(x) = f(x) + \lambda \int_a^x K(x, t) y_0(t) dt. \quad \dots (5)$$

But, from (2),
$$y_0(t) = f(t) \quad \dots (6)$$

Substituting the above value of $y_0(t)$ in (5), we get

$$y_1(x) = f(x) + \lambda \int_a^x K(x, t) f(t) dt. \quad \dots (7)$$

Putting $n = 2$ in (3), the second-order approximation $y_2(x)$ is given by

$$y_2(x) = f(x) + \lambda \int_a^x K(x, t) y_1(t) dt$$

or
$$y_2(x) = f(x) + \lambda \int_a^x K(x, z) y_1(z) dz. \quad \dots (8)$$

Replacing x by z in (7), we have

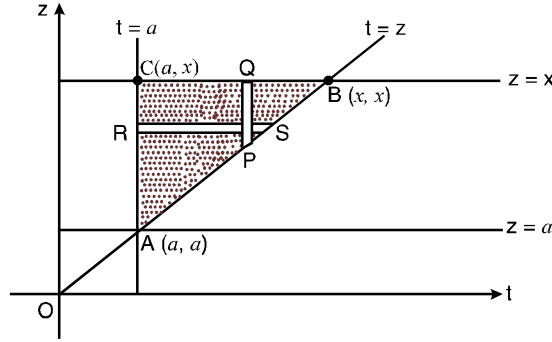
$$y_1(z) = f(z) + \lambda \int_a^z K(z, t) f(t) dt. \quad \dots (9)$$

Substituting the above value of $y_1(z)$ in (8), we get

$$y_2(x) = f(x) + \lambda \int_a^x K(x, z) [f(z) + \lambda \int_a^z K(z, t) f(t) dt] dz$$

or
$$y_2(x) = f(x) + \lambda \int_a^x K(x, z) f(z) dz + \lambda^2 \int_{z=a}^x K(x, z) \left[\int_{t=a}^z K(z, t) f(t) dt \right] dz. \quad \dots (10)$$

Now, consider the double integral on the R.H.S. of (10). The limits of integration are given by $t = a$, $t = z$, $z = a$, $z = x$. Clearly the region of integration is the triangle ABC as shown in the following figure. In double integral, clearly strips have been taken parallel to t -axis (see strip RS)



When we wish to change the order of integration in the above mentioned double integral, we shall take strips parallel to z -axis (see strip PQ). Then for the same region (triangle ABC), we see that the limits of z are $z = t$ to $z = x$ and limits for t are $t = a$ to $t = x$.

$$\therefore \int_{z=a}^x K(x, z) \left[\int_{t=a}^z K(z, t) f(t) dt \right] dz = \int_{t=a}^x f(t) \left[\int_{z=t}^x K(x, z) K(z, t) dz \right] dt.$$

Using the above equivalent value of the double integral in (10), we obtain

$$y_2(x) = f(x) + \lambda \int_a^x K(x, z) f(z) dz + \lambda^2 \int_{t=a}^x f(t) \left[\int_{z=t}^x K(x, z) K(z, t) dz \right] dt$$

or
$$y_2(x) = f(x) + \lambda \int_a^x K(x, t) f(t) dt + \lambda^2 \int_a^x f(t) K_2(x, t) dt$$

$$\left[\begin{array}{l} \text{For } n = 2 \text{ in (4B), } K_2(x, t) = \int_t^x K(x, z) K_1(z, t) dz \\ \text{or } K_2(x, t) = \int_t^x K(x, z) K(z, t) dz, \text{ by (4A)} \end{array} \right]$$

or
$$y_2(x) = f(x) + \lambda \int_a^x K_1(x, t) f(t) dt + \lambda^2 \int_a^x K_2(x, t) f(t) dt, \text{ using (4A).}$$

or
$$y_2(x) = f(x) + \sum_{m=1}^2 \lambda^m \int_a^x K_m(x, t) f(t) dt. \quad \dots (11)$$

Proceeding likewise, we easily obtain by Mathematical induction the n th approximate solution $y_n(x)$ of (1) given by

$$y_n(x) = f(x) + \sum_{m=1}^n \lambda^m \int_a^x K_m(x, t) f(t) dt. \quad \dots (12)$$

Proceeding to the limit as $n \rightarrow \infty$, we obtain the so called Neumann series

$$y(x) = \lim_{n \rightarrow \infty} y_n(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^x K_m(x, t) f(t) dt. \quad \dots (13)$$

We now determine the resolvent kernel (or reciprocal kernel) $R(x, t; \lambda)$ or $\Gamma(x, t; \lambda)$ in terms of the iterated kernels $K_m(x, t)$. For this purpose, by changing the order of integration and summation (13), we obtain

$$y(x) = f(x) + \lambda \int_a^x \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) \right] f(t) dt. \quad \dots (14)$$

$$\text{Comparing (14) with} \quad y(x) = f(x) + \lambda \int_a^x R(x, t; \lambda) f(t) dt, \quad \dots (15)$$

$$\text{here} \quad R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t). \quad \dots (16)$$

The series (16) converges absolutely and uniformly when $K(x, t)$ is continuous in R .

5.12. AN IMPORTANT THEOREM.

Let $R(x, t; \lambda)$ be the resolvent (or reciprocal) kernel of a Volterra integral equation.

$$y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt,$$

then the resolvent kernel satisfies the integral equation

$$R(x, t; \lambda) = K(x, t) + \lambda \int_t^x K(x, z) R(z, t; \lambda) dz. \quad (\text{Meerut 2010; Kanpur 2007})$$

Proof. We know that $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t), \quad \dots (1)$$

where iterated kernels (or functions) are given by

$$K_1(x, t) = K(x, t) \quad \dots (2A)$$

and

$$K_m(x, t) = \int_t^x K(x, z) K_{m-1}(z, t) dz \quad \dots (2B)$$

Now, from (1), we have

$$\begin{aligned} R(x, t; \lambda) &= K_1(x, t) + \sum_{m=2}^{\infty} \lambda^{m-1} K_m(x, t) \\ &= K(x, t) + \sum_{m=2}^{\infty} \lambda^{m-1} \int_t^x K(x, z) K_{m-1}(z, t) dz, \text{ using (2A) and (2B)} \\ &= K(x, t) + \sum_{n=1}^{\infty} \lambda^n \int_t^x K(x, z) K_n(z, t) dz, \text{ setting } m-1 = n \\ &= K(x, t) + \sum_{m=1}^{\infty} \lambda^m \int_t^x K(x, z) K_m(z, t) dz, \\ &= K(x, t) + \lambda \sum_{m=1}^{\infty} \lambda^{m-1} \int_t^x K(x, z) K_m(z, t) dz \\ &= K(x, t) + \lambda \int_t^x \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(z, t) \right] K(x, z) dz \\ &\quad \text{[on changing the order of summation and integration]} \\ &= K(x, t) + \lambda \int_t^x R(z, t; \lambda) K(x, z) dz, \text{ using (1)} \\ \therefore \quad R(x, t; \lambda) &= K(x, t) + \lambda \int_t^x K(x, z) R(z, t; \lambda) dz. \end{aligned}$$

5.13. SOLVED EXAMPLES BASED ON SOLUTION OF VOLTERRA INTEGRAL EQUATION OF SECOND KIND BY SUCCESSIVE APPROXIMATIONS (OR ITERATIVE METHOD).

Type 1. Determination of resolvent kernel or reciprocal kernel for Volterra integral

equation
$$y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt.$$

Ex. 1. Find the resolvent kernel of the Volterra integral equation with the kernel $K(x, t) = 1$.
[Kanpur 2005, 2006]

Sol. Iterated kernels $K_n(x, t)$ are given by

$$K_1(x, t) = K(x, t) \quad \dots (1)$$

and
$$K_n(x, t) = \int_t^x K(x, z) K_{n-1}(z, t) dz, \quad n = 1, 2, 3, \dots \quad \dots (2)$$

Given
$$K(x, t) = 1. \quad \dots (3)$$

$\therefore$ From (1) and (3),
$$K_1(x, t) = K(x, t) = 1. \quad \dots (4)$$

Putting $n = 2$ in (2), and using (4), we have

$$K_2(x, t) = \int_t^x K(x, z) K_1(z, t) dz = \int_t^x dz = [z]_t^x = x - t. \quad \dots (5)$$

Next, putting $n = 3$ in (2), we have

$$\begin{aligned} K_3(x, t) &= \int_t^x K(x, z) K_2(z, t) dz = \int_t^x 1 \cdot (z - t) dz, \text{ by (4) and (5)} \\ &= \left[\frac{(z - t)^2}{2} \right]_t^x = \frac{(x - t)^2}{2!} \end{aligned} \quad \dots (6)$$

Now, putting $n = 4$ in (2), we have

$$\begin{aligned} K_4(x, t) &= \int_t^x K(x, z) K_3(z, t) dz = \int_t^x 1 \cdot \frac{(z - t)^2}{2!} dz, \text{ by (4) and (6)} \\ &= \frac{1}{2!} \left[\frac{(z - t)^3}{3} \right]_t^x = \frac{(x - t)^3}{3!} \end{aligned} \quad \dots (7)$$

and so on. Observing (4), (5), (6) and (7) etc, we find by mathematical induction, that

$$K_n(x, t) = \frac{(x - t)^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \dots \quad \dots (8)$$

Now, by the definition of the resolvent kernel, we have

$$\begin{aligned} R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = K_1(x, t) + \lambda K_2(x, t) + \lambda^2 K_3(x, t) + \dots \\ &= 1 + \frac{\lambda(x - t)}{1!} + \frac{[\lambda(x - t)]^2}{2!} + \frac{[\lambda(x - t)]^3}{3!} + \dots, \text{ by (8)} \\ &= e^{\lambda(x - t)}. \end{aligned}$$

Ex. 2. Find the resolvent kernel of the Volterra integral equation with the kernel

$$K(x, t) = e^{x-t}.$$

Sol. Iterated kernels $K_n(x, t)$ are given by

$$K_1(x, t) = K(x, t) \quad \dots (1)$$

and
$$K_n(x, t) = \int_t^x K(x, z) K_{n-1}(z, t) dz, \quad n = 2, 3, \dots \quad \dots (2)$$

Given
$$K(x, t) = e^{x-t}. \quad \dots (3)$$

$\therefore$ From (1) and (3),
$$K_1(x, t) = K(x, t) = e^{x-t}. \quad \dots (4)$$

Putting $n = 2$ in (2), we have

$$\begin{aligned} K_2(x, t) &= \int_t^x K(x, z) K_1(z, t) dz \\ &= \int_t^x e^{x-z} e^{z-t} dz = e^{x-t} \int_t^x dz = e^{x-t} (x-t). \end{aligned} \quad \dots (5)$$

Next, putting $n = 3$ in (2), and using (5) we have

$$\begin{aligned} K_3(x, t) &= \int_t^x K(x, z) K_2(z, t) dz = \int_t^x e^{x-z} (z-t) e^{z-t} dz = e^{x-t} \int_t^x (z-t) dz \\ &= e^{x-t} \left[\frac{(z-t)^2}{2} \right]_t^x = e^{x-t} \frac{(x-t)^2}{2!} \end{aligned} \quad \dots (6)$$

Now, putting $n = 4$ in (2) and using (6), we have

$$\begin{aligned} K_4(x, t) &= \int_t^x k(x, z) k_3(z, t) dz = \int_t^x e^{x-z} e^{z-t} \frac{(z-t)^2}{2!} dz = \frac{e^{x-t}}{2!} \int_t^x (z-t)^2 dz \\ &= \frac{e^{x-t}}{2!} \left[\frac{(z-t)^3}{3} \right]_t^x = e^{x-t} \frac{(x-t)^3}{3!}, \end{aligned} \quad \dots (7)$$

and so on. Observing (4), (5), (6) and (7) etc. we find by methematical induction, that

$$K_n(x, t) = e^{x-t} \frac{(x-t)^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \dots \quad \dots (8)$$

Now, by the definition of the resolvent kernel, we have

$$\begin{aligned} R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = K_1(x, t) + \lambda K_2(x, t) + \lambda^2 K_3(x, t) + \dots \\ &= e^{x-t} + e^{x-t} \frac{\lambda(x-t)}{1!} + e^{x-t} \frac{[\lambda(x-t)]^2}{2!} + \dots, \quad \text{by (8)} \\ &= e^{x-t} \left[1 + \frac{\lambda(x-t)}{1!} + \frac{[\lambda(x-t)]^2}{2!} + \dots \right] \\ &= e^{x-t} e^{\lambda(x-t)} = e^{(x-t) + \lambda(x-t)} = e^{(x-t)(1+\lambda)}. \end{aligned}$$

Ex. 3. Find the resolvent kernel of the Volterra integral equation with the kernel $K(x, t) = (2 + \cos x) / (2 + \cos t)$. (Meerut 2010, II)

Sol. Iterated kernels $K_n(x, t)$ are given by

$$K_1(x, t) = K(x, t) \quad \dots (1)$$

and
$$K_n(x, t) = \int_t^x K(x, z) K_{n-1}(z, t) dz, \quad n = 2, 3, \dots \quad \dots (2)$$

Given
$$K(x, t) = (2 + \cos x) / (2 + \cos t) \quad \dots (3)$$

$$\therefore \text{ From (1) and (3), } K_1(x, t) = K(x, t) = (2 + \cos x)/(2 + \cos t). \quad \dots(4)$$

Putting $n = 2$ in (2) and using (4), we have

$$\begin{aligned} K_2(x, t) &= \int_t^x K(x, z) K_1(z, t) dz \\ &= \int_t^x \frac{2 + \cos x}{2 + \cos z} \cdot \frac{2 + \cos z}{2 + \cos t} dz = \frac{2 + \cos x}{2 + \cos t} \int_t^x dz = \frac{2 + \cos x}{2 + \cos t} (x - t). \end{aligned} \quad \dots (5)$$

Next, putting $n = 3$ in (3), we have

$$\begin{aligned} K_3(x, t) &= \int_t^x K(x, z) K_2(z, t) dz = \int_t^x \frac{2 + \cos x}{2 + \cos z} \cdot \frac{2 + \cos z}{2 + \cos t} (z - t) dz, \text{ by (3) and (5)} \\ &= \frac{2 + \cos x}{2 + \cos t} \int_t^x (z - t) dz = \frac{2 + \cos x}{2 + \cos t} \left[\frac{(z - t)^2}{2} \right]_t^x = \frac{2 + \cos x}{2 + \cos t} \cdot \frac{(x - t)^2}{2!}. \end{aligned} \quad \dots (6)$$

Now, putting $n = 4$ in (2), we have

$$\begin{aligned} K_4(x, t) &= \int_t^x K(x, z) K_3(z, t) dz = \int_t^x \frac{2 + \cos x}{2 + \cos z} \cdot \frac{2 + \cos z}{2 + \cos t} \cdot \frac{(z - t)^2}{2!} dz, \text{ by (3) and (6)} \\ &= \frac{2 + \cos x}{2 + \cos t} \cdot \frac{1}{2!} \left[\frac{(z - t)^3}{3} \right]_t^x = \frac{2 + \cos x}{2 + \cos t} \cdot \frac{(x - t)^3}{3!}, \end{aligned} \quad \dots (7)$$

and so on. Observing (4), (5), (6) and (7) etc. we find by mathematical intuition that

$$K_n(x, t) = \frac{2 + \cos x}{2 + \cos t} \cdot \frac{(x - t)^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \quad \dots (8)$$

Now, by the definition of the resolvent kernel, we get

$$\begin{aligned} R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = K_1(x, t) + \lambda K_2(x, t) + \lambda^2 K_3(x, t) + \dots \\ &= \frac{2 + \cos x}{2 + \cos t} + \frac{2 + \cos x}{2 + \cos t} \cdot \frac{\lambda(x - t)}{1!} + \frac{2 + \cos x}{2 + \cos t} \cdot \frac{\{\lambda(x - t)\}^2}{2!} + \dots, \quad \text{by (8)} \\ &= \frac{2 + \cos x}{2 + \cos t} \left[1 + \frac{\lambda(x - t)}{1!} + \frac{\{\lambda(x - t)\}^2}{2!} + \frac{\{\lambda(x - t)\}^3}{3!} + \dots \right] = \frac{2 + \cos x}{2 + \cos t} e^{\lambda(x - t)}. \end{aligned}$$

EXERCISE 5B

Find the resolvent kernels for Volterra integral equations with the following kernels.

1. $K(x, t) = e^{x^2 - t^2}$. Ans. $e^{x^2 - t^2} e^{\lambda(x - t)}$.
2. $K(x, t) = \frac{1 + x^2}{1 + t^2}$. Ans. $\frac{1 + x^2}{1 + t^2} e^{\lambda(x - t)}$.
3. $K(x, t) = a^{x - t}$ ($a > 0$). Ans. $a^{x - t} e^{\lambda(x - t)}$.
4. $K(x, t) = \frac{\cosh x}{\cosh t}$. Ans. $\frac{\cosh x}{\cosh t} e^{\lambda(x - t)}$.
5. $K(x, t) = x - t$. Ans. $(1/\sqrt{\lambda}) \times \sinh \{(x - t)\sqrt{\lambda}\}, (\lambda > 0)$.

Type 2. Solution of Volterra integral equation with help of the resolvent kernel.

Working Rule: Let
$$y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt \quad \dots (1)$$

be given Volterra integral equation. Let $K_m(x, t)$ be the m th iterated kernel and let $R(x, t; \lambda)$ be the resolvent kernel of (1). Then, we have

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t). \quad \dots (2)$$

Suppose the sum of infinite series (2) exists and so $R(x, t; \lambda)$ can be obtained in the closed form. Then the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_a^x R(x, t; \lambda) f(t) dt. \quad \dots (3)$$

Ex. 4. With the aid of the resolvent kernel, find the solution of the integral equation

$$y(x) = e^{x^2} + \int_0^x e^{x^2-t^2} y(t) dt. \quad [\text{Meerut 2004, II}]$$

Sol. Given
$$y(x) = e^{x^2} + \int_0^x e^{x^2-t^2} y(t) dt. \quad \dots (1)$$

Comparing (1) with
$$y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt,$$

we have $f(x) = e^{x^2}, \quad \lambda = 1, \quad K(x, t) = e^{x^2-t^2}. \quad \dots (2)$

Let $K_m(x, t)$ be the m th iterated kernel. then, we have

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

and

$$K_m(x, t) = \int_t^x K(x, z) K_{m-1}(z, t) dz. \quad \dots (4)$$

From (2) and (3), $K_1(x, t) = K(x, t) = e^{x^2-t^2}. \quad \dots (5)$

Putting $m = 2$ in (4), we have

$$\begin{aligned} K_2(x, t) &= \int_t^x K(x, z) K_1(z, t) dz = \int_t^x e^{x^2-z^2} e^{z^2-t^2} dz, \text{ by (5)} \\ &= e^{x^2-t^2} \int_t^x dz = e^{x^2-t^2} (x-t). \end{aligned} \quad \dots (6)$$

Next, putting $m = 3$ in (4), we have

$$\begin{aligned} K_3(x, t) &= \int_t^x K(x, z) K_2(z, t) dz = \int_t^x e^{x^2-z^2} \cdot e^{z^2-t^2} (z-t) dz, \text{ by (5) and (6)} \\ &= e^{x^2-t^2} \int_t^x (z-t) dz = e^{x^2-t^2} \left[\frac{(z-t)^2}{2} \right]_t^x = e^{x^2-t^2} \frac{(x-t)^2}{2!}. \end{aligned} \quad \dots (7)$$

Now, putting $m = 4$ in (4), we have

$$\begin{aligned} K_4(x, t) &= \int_t^x K(x, z) K_3(z, t) dz \\ &= \int_t^x e^{x^2-z^2} e^{z^2-t^2} \frac{(z-t)^2}{2!} dz = \frac{e^{x^2-t^2}}{2!} \left[\frac{(z-t)^3}{3} \right]_t^x = e^{x^2-t^2} \frac{(x-t)^3}{3!}, \dots (8) \end{aligned}$$

and so on. Observing (5), (6), (7) and (8) etc., by mathematical induction, we have

$$K_m(x, t) = e^{x^2-t^2} \frac{(x-t)^{m-1}}{(m-1)!}, \quad m = 1, 2, 3, \dots \quad \dots (9)$$

Now, by the definition of the resolvent kernel, we have

$$\begin{aligned} R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = K_1(x, t) + \lambda K_2(x, t) + \lambda^2 K_3(x, t) + \dots \\ &= e^{x^2-t^2} + e^{x^2-t^2} \frac{(x-t)}{1!} + e^{x^2-t^2} \frac{(x-t)^2}{2!} + \dots, \text{ using (2) and (9)} \\ &= e^{x^2-t^2} \left[1 + \frac{(x-t)}{1!} + \frac{(x-t)^2}{2!} + \dots \right] = e^{x^2-t^2} e^{x-t} \quad \dots (10) \end{aligned}$$

Finally, the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt$$

or

$$y(x) = e^{x^2} + \int_0^x e^{x^2-t^2} e^{x-t} e^{t^2} dt, \text{ using (2) and (10)}$$

or

$$y(x) = e^{x^2} + e^{x^2+x} \int_0^x e^{-t} dt = e^{x^2} + e^{x^2+x} [-e^{-t}]_0^x$$

or

$$y(x) = e^{x^2} + e^{x^2+x} [-e^{-x} + 1] = e^{x^2} - e^{x^2} + e^{x^2+x} \quad \text{or} \quad y(x) = e^{x^2+x}.$$

Ex. 5. Solve the following integral equation by successive approximation

$$y(x) = f(x) + \lambda \int_0^x e^{x-t} y(t) dt. \text{ and find the resolvent kernel.}$$

Sol. Given

$$y(x) = f(x) + \lambda \int_0^x e^{x-t} y(t) dt \quad \dots (1)$$

Comparing (1) with

$$y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt,$$

here

$$K(x, t) = e^{x-t}. \quad \dots (2)$$

Proceed as in solved Ex. 2. and show that

$$R(x, t; \lambda) = e^{(x-t)(1+\lambda)} \quad \dots (3)$$

Now, the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt$$

or

$$y(x) = f(x) + \lambda \int_0^x e^{(x-t)(1+\lambda)} f(t) dt, \text{ by (3).}$$

Ex. 6. By means of resolvent kernel, find the solution of

$$y(x) = e^x \sin x + \int_0^x \frac{2 + \cos x}{2 + \cos t} y(t) dt. \quad [Meerut 2009]$$

Sol. Given

$$y(x) = e^x \sin x + \int_0^x \frac{2 + \cos x}{2 + \cos t} y(t) dt. \quad \dots (1)$$

Comparing (1) with

$$y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt,$$

we have $f(x) = e^x \sin x$, $\lambda = 1$, $K(x, t) = \frac{2 + \cos x}{2 + \cos t}$... (2)

Proceed as in solved Ex. 3. and show that

$$R(x, t; \lambda) = \frac{2 + \cos x}{2 + \cos t} e^{x-t} \quad [\text{Note that here } \lambda = 1, \text{ by (2)}] \quad \dots (3)$$

The required solution is given by

$$\begin{aligned} y(x) &= f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt \\ &= e^x \sin x + \int_0^x \frac{2 + \cos x}{2 + \cos t} e^{x-t} e^t \sin t dt, \text{ using (2) and (3)} \\ &= e^x \sin x - (2 + \cos x) e^x \int_0^x \frac{-\sin t}{2 + \cos t} dt \\ &= e^x \sin x - e^x (2 + \cos x) [\log (2 + \cos t)]_0^x \end{aligned}$$

$$\therefore y(x) = e^x \sin x - e^x (2 + \cos x) \log \frac{2 + \cos x}{3}$$

Ex. 7. Solve $y(x) = \sin x + 2 \int_0^x e^{x-t} y(t) dt$, [G.N.D.U. Amritsar 2004, Meerut 2000, 02, 03, 12; Kanpur 2009]

Sol. Given $y(x) = \sin x + 2 \int_0^x e^{x-t} y(t) dt$ (1)

Comparing (1) with

$$y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt,$$

we have $f(x) = \sin x$, $\lambda = 2$, $K(x, t) = e^{x-t}$ (2)

Proceed as in Ex. 2. and show that

$$R(x, t; \lambda) = e^{(x-t)(1+\lambda)} = e^{3(x-t)}. \quad [\because \lambda = 2, \text{ by (2)}]$$

Now, the required solution of (1) is

$$\begin{aligned} y(x) &= f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt \\ &= \sin x + 2 \int_0^x e^{3(x-t)} \sin t dt = \sin x + 2e^{3x} \int_0^x e^{-3t} \sin t dt \\ &= \sin x + 2e^{3x} \left[\frac{e^{-3t}}{10} (-3 \sin t - \cos t) \right]_0^x \quad \text{as } \int e^{ax} \sin bx dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2} \end{aligned}$$

or $y(x) = \sin x + \frac{e^{3x}}{5} [e^{-3x} (-3 \sin x - \cos x) + 1]$ or $y(x) = \frac{1}{5} e^{3x} + \frac{2}{5} \sin x - \frac{1}{5} \cos x$

Ex. 8. With help of the resolvent kernel, find the solution of the integral equation

$$y(x) = 1 + x^2 + \int_0^x \frac{1+x^2}{1+t^2} y(t) dt. \quad [\text{Amritsar 2004, Meerut 2000, 08}]$$

Sol. Given
$$y(x) = 1 + x^2 + \int_0^x \frac{1+x^2}{1+t^2} y(t) dt. \quad \dots (1)$$

Comparing (1) with
$$y(x) = f(x) + \int_0^x K(x, t) y(t) dt$$

we have $f(x) = 1 + x^2, \quad \lambda = 1, \quad K(x, t) = (1 + x^2)/(1 + t^2). \quad \dots (2)$

Let $K_m(x, t)$ be the m th iterated kernel. Then, we have

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

and

$$K_m(x, t) = \int_t^x K(x, z) K_{m-1}(z, t) dz. \quad \dots (4)$$

From (2) and (3), $K_1(x, t) = K(x, t) = (1 + x^2) / (1 + t^2) \quad \dots (5)$

Putting $m = 2$ in (4), we have

$$\begin{aligned} K_2(x, t) &= \int_t^x K(x, z) K_1(z, t) dz = \int_t^x \frac{1+x^2}{1+z^2} \cdot \frac{1+z^2}{1+t^2} dz, \text{ by (5)} \\ &= \frac{1+x^2}{1+t^2} \int_t^x dz = \frac{1+x^2}{1+t^2} (x-t). \end{aligned} \quad \dots (6)$$

Next, putting $m = 3$ in (4), we have

$$\begin{aligned} K_3(x, t) &= \int_t^x K(x, z) K_2(z, t) dz = \int_t^x \frac{1+x^2}{1+z^2} \cdot \frac{1+z^2}{1+t^2} \cdot (z-t) dz, \text{ by (5) and (6)} \\ &= \frac{1+x^2}{1+t^2} \int_t^x (x-t) dz = \frac{1+x^2}{1+t^2} \left[\frac{(z-t)^2}{2} \right]_t^x = \frac{1+x^2}{1+t^2} \cdot \frac{(x-t)^2}{2!}. \end{aligned} \quad \dots (7)$$

Now, putting $m = 4$ in (4), we have

$$\begin{aligned} K_4(x, t) &= \int_t^x K(x, z) K_3(z, t) dz = \int_t^x \frac{1+x^2}{1+z^2} \cdot \frac{1+z^2}{1+t^2} \cdot \frac{(z-t)^2}{2!} dz, \text{ by (5) and (7)} \\ &= \frac{1+x^2}{1+t^2} \frac{1}{2!} \int_t^x (z-t)^2 dz = \frac{1+x^2}{1+t^2} \frac{1}{2!} \left[\frac{(z-t)^3}{3} \right]_t^x = \frac{1+x^2}{1+t^2} \frac{(x-t)^3}{3!}, \end{aligned} \quad \dots (8)$$

and so on. Observing (5) (6), (7) and (8) etc. by mathematical induction, it follows that

$$K_m(x, t) = \frac{1+x^2}{1+t^2} \frac{(x-t)^{m-1}}{(m-1)!}, \quad m = 1, 2, 3, \dots \quad \dots (9)$$

Now, by the definition of the resolvent kernel $R(x, t; \lambda)$, we have

$$\begin{aligned} R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = \sum_{m=1}^{\infty} K_m(x, t), \quad [\because \lambda = 1] \\ &= K_1(x, t) + K_2(x, t) + K_3(x, t) + \dots \\ &= \frac{1+x^2}{1+t^2} + \frac{1+x^2}{1+t^2} \frac{(x-t)}{1!} + \frac{1+x^2}{1+t^2} \frac{(x-t)^2}{2!} + \dots \\ &= \frac{1+x^2}{1+t^2} \left[1 + \frac{(x-t)}{1!} + \frac{(x-t)^2}{2!} + \dots \right] = \frac{1+x^2}{1+t^2} e^{x-t}. \end{aligned} \quad \dots (10)$$

Finally, the required solution of (1) is given by

$$\begin{aligned}
 y(x) &= f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt \\
 &= 1 + x^2 + \int_0^x \frac{1+x^2}{1+t^2} e^{x-t} (1+t^2) dt, \text{ using (2) and (10)} \\
 &= 1 + x^2 + (1+x^2) e^x \int_0^x e^{-t} dt = 1 + x^2 + e^x (1+x^2) \left[-e^{-t} \right]_0^x \\
 &= 1 + x^2 + e^x (1+x^2) [-e^{-x} + 1] = 1 + x^2 - (1+x^2) + e^x (1+x^2) \\
 \therefore y(x) &= e^x (1+x^2).
 \end{aligned}$$

Ex. 9. Solve $y(x) = 1 + \int_0^x y(t) dt$. [Kanpur 2007, 08]

Sol. Given $y(x) = 1 + \int_0^x y(t) dt$(1)

Comparing (1) with $y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt$,

we have $f(x) = 1$, $\lambda = 1$, $K(x, t) = 1$(2)

Proceeding as in Ex. 1, we have

$$R(x, t; \lambda) = e^{\lambda(x-t)} = e^{x-t} \quad [\because \lambda = 1, \text{ by (2)}] \quad \dots (3)$$

Now, the required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt$$

or

$$\begin{aligned}
 y(x) &= 1 + \int_0^x e^{x-t} dt, \text{ using (1)} \\
 &= 1 + e^x \int_0^x e^{-t} dt = 1 + e^x \left[-e^{-t} \right]_0^x = 1 + e^x [-e^{-x} + 1] = 1 - 1 + e^x. \\
 \therefore y(x) &= e^x.
 \end{aligned}$$

Ex. 10. Solve $y(x) = x + \int_0^x (t-x)y(t) dt$.

Sol. Given $y(x) = x + \int_0^x (t-x)y(t) dt$(1)

Comparing (1) with $y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt$,

we have $f(x) = x$, $\lambda = 1$, $K(x, t) = t - x$(2)

Let $K_m(x, t)$ be the m th iterated kernel. Then

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

and

$$K_m(x, t) = \int_t^x K(x, z) K_{m-1}(z, t) dz, \quad m = 2, 3, \dots \quad \dots (4)$$

From (2) and (3), $K_1(x, t) = K(x, t) = t - x$... (5)

Putting $m = 2$ in (4) and using (5), we get

$$\begin{aligned} K_2(x, t) &= \int_t^x K(x, z) K_1(z, t) dz = \int_t^x (z - x)(t - z) dz \\ &= \left[(t - z) \frac{(z - x)^2}{2} \right]_t^x - \int_t^x (-1) \frac{(z - x)^2}{2} dz, \text{ integrating by parts} \\ &= \frac{1}{2} \int_t^x (z - x)^2 dz = \frac{1}{2} \left[\frac{(z - x)^3}{3} \right]_t^x \\ \therefore K_2(x, t) &= - (t - x)^3 / 3! \end{aligned} \quad \dots (6)$$

Next, putting $m = 3$ in (4), we have

$$\begin{aligned} K_3(x, t) &= \int_t^x K(x, z) K_2(z, t) dz = \int_t^x (z - x) \left\{ -\frac{(t - z)^3}{3!} \right\} dz, \text{ by (5) and (6)} \\ &= -\frac{1}{3!} \int_t^x (z - x)(t - z)^3 dz = -\frac{1}{3!} \left\{ \left[(z - x) \frac{(t - z)^4}{(-4)} \right]_t^x + \int_t^x 1 \cdot \frac{(t - z)^4}{(-4)} dz \right\} \\ &= -\frac{1}{4 \cdot 3!} \int_t^x (t - z)^4 dz = -\frac{1}{4 \cdot 3!} \left[\frac{(t - z)^5}{(-5)} \right]_t^x \\ \therefore K_3(x, t) &= (t - x)^5 / 5!, \end{aligned} \quad \dots (7)$$

and so on. Observing (5), (6) and (7) etc. by mathematical induction, we have

$$K_m(x, t) = (-1)^{m-1} \frac{(t - x)^{2m-1}}{(2m-1)!}, \quad m = 1, 2, 3, \dots \quad \dots (8)$$

Now, by the definition of the resolvent kernel

$$\begin{aligned} R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = \sum_{m=1}^{\infty} K_m(x, t) \quad [\because \lambda = 1, \text{ by (2)}] \\ &= K_1(x, t) + K_2(x, t) + K_3(x, t) + \dots \\ &= \frac{(t - x)}{1!} - \frac{(t - x)^3}{3!} + \frac{(t - x)^5}{5!} - \dots = \sin(t - x) \end{aligned} \quad \dots (9)$$

Finally, the required solution of (1) is given by

$$\begin{aligned} y(x) &= f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt = x + \int_0^x t \sin(t - x) dt, \text{ by (2) and (9)} \\ &= x + \left[t \{-\cos(t - x)\} \right]_0^x - \int_0^x 1 \cdot \{-\cos(t - x)\} dt = x - x + [\sin(t - x)]_0^x \\ \therefore y(x) &= \sin x. \end{aligned}$$

Ex. 11. Solve $y(x) = 1 + \int_0^x (t - x) y(t) dt.$ [Meerut 2008, 10]

Sol. Comparing (1) with $y(x) = 1 + \int_0^x (t - x) y(t) dt,$... (1)

here $f(x) = 1,$ $\lambda = 1,$ $K(x, t) = t - x.$... (2)

Proceed as in solved Ex. 10 to show that $R(x, t; \lambda) = \sin(t - x)$... (3)

Now the required solution is given by

$$y(x) = f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt \quad \text{or} \quad y(x) = 1 + \int_0^x \sin(t - x) dt, \text{ by (2) and (3)}$$

$$\text{or} \quad y(x) = 1 + [-\cos(t - x)]_0^x = 1 - 1 + \cos x. \quad \text{or} \quad y = \cos x$$

Ex. 12. Solve $y(x) = \cos x - x - 2 + \int_0^x (t - x) y(t) dt$.

$$\text{Sol. Given} \quad y(x) = \cos x - x - 2 + \int_0^x (t - x) y(t) dt. \quad \dots (1)$$

$$\text{Comparing (1) with} \quad y(x) = f(x) + \lambda \int_0^x (t - x) y(t) dt,$$

$$\text{here} \quad f(x) = \cos x - x - 2, \quad \lambda = 1, \quad K(x, t) = t - x, \quad \dots (2)$$

$$\text{Proceeding as in solved Ex. 10, we have} \quad R(x, t; \lambda) = \sin(t - x) \quad \dots (3)$$

Now, the required solution of (1) is given by

$$\begin{aligned} y(x) &= f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt \\ &= \cos x - x - 2 + \int_0^x \sin(t - x) (\cos t - t - 2) dt, \text{ using (2)} \\ &= \cos x - x - 2 + \int_0^x \sin(t - x) \cos t dt - \int_0^x t \sin(t - x) dt - 2 \int_0^x \sin(t - x) dt \\ &= \cos x - x - 2 + \frac{1}{2} \int_0^x [\sin(2t - x) - \sin x] dt - \int_0^x t \sin(t - x) dt - 2 \int_0^x \sin(t - x) dt \\ &= \cos x - x - 2 + \frac{1}{2} \left[-\frac{\cos(2t - x)}{2} - t \sin x \right]_0^x \\ &\quad - \left\{ [-t \cos(t - x)]_0^x - \int_0^x 1 \cdot \{-\cos(t - x)\} dt \right\} - 2[-\cos(t - x)]_0^x \\ &\quad \text{[Integrating by parts the 2nd integral]} \\ &= \cos x - x - 2 + \frac{1}{2} \left[-\frac{\cos x}{2} - x \sin x + \frac{\cos x}{2} \right] - \left\{ -x + [\sin(t - x)]_0^x \right\} + 2(1 - \cos x) \\ &= \cos x - x - 2 - (1/2) \times x \sin x + x + \sin(-x) + 2 - 2 \cos x \\ \therefore \quad y(x) &= -\cos x - \sin x - (x/2) \times \sin x. \end{aligned}$$

EXERCISE 5C

Solve the following integral equations :

$$1. \quad y(x) = e^x + \int_0^x e^{x-t} y(t) dt. \quad \text{Ans. } y(x) = e^{2x}.$$

$$2. \quad y(x) = x \cdot 3^x - \int_0^x 3^{x-t} y(t) dt. \quad \text{Ans. } y(x) = 3^x (1 - e^{-x})$$

$$3. \quad y(x) = 1 - 2x - \int_0^x e^{x^2-t^2} y(t) dt. \quad \text{Ans. } y(x) = e^{x^2-x} - 2x.$$

$$4. \quad y(x) = e^{x^2+2x} + 2 \int_0^x e^{x^2-t^2} y(x) dt. \quad \text{Ans. } y(x) = e^{x^2+2x} (1 + 2x).$$

$$5. \quad y(x) = \frac{1}{1+x^2} + \int_0^x \sin(x-t) y(t) dt. \quad \text{Ans. } y(x) = \frac{1}{1+x^2} + x \tan^{-1} x - \frac{1}{2} \log(1+x^2)$$

$$6. \quad y(x) = x e^{x^2/2} + \int_0^x e^{-(x-t)} y(t) dt. \quad \text{Ans. } y(x) = e^{x^2/2} (x+1) - 1.$$

$$7. \quad y(x) = e^{-x} + \int_0^x e^{-(x-t)} \sin(x-t) y(t) dt. \quad \text{Ans. } y(x) = e^{-x} [1 + (x^2/2)].$$

$$8. \quad y(x) = e^x - \int_0^x e^{x-t} y(t) dt. \quad \text{Ans. } y(x) = 1.$$

Type 3. Solution of Volterra integral equation when the sum of the infinite series occurring in the formula for resolvent kernel cannot be obtained in closed form. In such problems we use the following formula for solution, which is also known as Neumann series :

$$y(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^x K_m(x, t) f(t) dt, \quad \dots (i)$$

where $K_m(x, t)$ is the m th iterated kernel. Here (i) is solution of given Volterra integral equation

$$y(x) = f(x) + \lambda \int_a^x K(x, t) y(t) dt. \quad \dots (ii)$$

Ex. 13. Find the Neumann series for the solution of the integral equation

$$y(x) = 1 + x + \lambda \int_0^x (x-t) y(t) dt.$$

$$\text{Sol. Given} \quad y(x) = 1 + x + \lambda \int_0^x (x-t) y(t) dt. \quad \dots (1)$$

$$\text{Comparint (1) with} \quad y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt.$$

$$\text{here} \quad f(x) = 1 + x, \quad \lambda = \lambda, \quad K(x, t) = x - t. \quad \dots (2)$$

Let $K_m(x, t)$ be the m th iterated kernel. Then

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

$$\text{and} \quad K_m(x, t) = \int_t^x K(x, z) K_{m-1}(z, t) dz. \quad \dots (4)$$

$$\text{From (2) and (3),} \quad K_1(x, t) = K(x, t) = x - t. \quad \dots (5)$$

Putting $m = 2$ in (4) and using (5), we have

$$K_2(x, t) = \int_t^x K(x, z) K_1(z, t) dz = \int_t^x (x-z)(z-t) dz, \text{ using (5)}$$

$$= \left[(x-z) \frac{(z-t)^2}{2} \right]_t^x - \int_t^x \frac{(-1)(z-t)^2}{2} dz.$$

$$\therefore K_2(x, t) = \frac{1}{2} \int_t^x (z-t)^2 dz = \frac{1}{2} \left[\frac{(z-t)^3}{3} \right]_t^x = \frac{(x-t)^3}{3!} \quad \dots (6)$$

Putting $m = 3$ in (4), we have

$$\begin{aligned}
 K_3(x, t) &= \int_t^x K(x, z) K_2(z, t) dz = \int_t^x (x-z) \frac{(z-t)^3}{3!} dz, \text{ using (6)} \\
 &= \left[(x-z) \frac{(z-t)^4}{4 \cdot 3!} \right]_t^x - \int_t^x (-1) \frac{(z-t)^4}{4 \cdot 3!} dz \\
 &= \frac{1}{4!} \int_t^x (z-t)^4 dz = \frac{1}{4!} \left[\frac{(z-t)^5}{5} \right]_t^x = \frac{(x-t)^5}{5!}, \quad \dots (7)
 \end{aligned}$$

and so on.

Now the Neumann series for the solution of (1) is given by

$$y(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_0^x K_m(x, t) f(t) dt, \text{ i.e.,}$$

$$y(x) = 1 + x + \lambda \int_0^x K_1(x, t) (1+t) dt + \lambda^2 \int_0^x K_2(x, t) (1+t) dt + \lambda^3 \int_0^x K_3(x, t) (1+t) dt + \dots, \quad [\text{using (2)}]$$

$$\begin{aligned}
 &= 1 + x + \lambda \int_0^x (x-t) (1+t) dt + \lambda^2 \int_0^x \frac{(x-t)^3}{3!} (1+t) dt + \lambda^3 \int_0^x \frac{(x-t)^5}{5!} (1+t) dt + \dots \\
 &= 1 + x + \lambda \left\{ \left[(1+t) \frac{(x-t)^2}{(-2)} \right]_0^x - \int_0^x \frac{(x-t)^2}{(-2)} dt \right\} + \frac{\lambda^2}{3!} \left\{ \left[(1+t) \frac{(x-t)^4}{(-4)} \right]_0^x \right. \\
 &\quad \left. - \int_0^x \frac{(x-t)^4}{(-4)} dt \right\} + \frac{\lambda^3}{5!} \left\{ \left[(1+t) \frac{(x-t)^6}{(-6)} \right]_0^x - \int_0^x \frac{(x-t)^6}{(-6)} dt \right\} + \dots \\
 &= 1 + x + \lambda \left\{ \frac{x^2}{2} + \frac{1}{2} \left[\frac{(x-t)^3}{(-3)} \right]_0^x \right\} + \frac{\lambda^2}{3!} \left\{ \frac{x^4}{4} + \frac{1}{4} \left[\frac{(x-t)^5}{(-5)} \right]_0^x \right\} + \frac{\lambda^3}{5!} \left\{ \frac{x^6}{6} + \frac{1}{6} \left[\frac{(x-t)^7}{(-7)} \right]_0^x \right\} + \dots \\
 &= 1 + x + \lambda \left(\frac{x^2}{2} + \frac{x^3}{6} \right) + \frac{\lambda^2}{6} \left(\frac{x^4}{4} + \frac{x^5}{20} \right) + \frac{\lambda^3}{120} \left(\frac{x^6}{6} + \frac{x^7}{42} \right) + \dots
 \end{aligned}$$

or

$$y(x) = 1 + x + \lambda \left(\frac{x^2}{2!} + \frac{x^3}{3!} \right) + \lambda^2 \left(\frac{x^4}{4!} + \frac{x^5}{5!} \right) + \lambda^3 \left(\frac{x^6}{6!} + \frac{x^7}{7!} \right) + \dots \quad \dots (8)$$

Remark 1. In particular case, if $\lambda = 1$, then (8) reduces to

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = e^x. \quad \dots (9)$$

Remark 2. In above solved example, the resolvent kernel is given by

$$\begin{aligned}
 R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = K_1(x, t) + \lambda K_2(x, t) + \lambda^2 K_3(x, t) + \dots \\
 &= (x-t) + \lambda \frac{(x-t)^3}{3!} + \lambda^2 \frac{(x-t)^5}{5!} + \dots,
 \end{aligned}$$

whose sum cannot be obtained in closed form. Hence solution cannot be obtained by the usual formula

$$y(x) = f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt.$$

We, therefore, find solution by using Neumann series. Due to the same reason, in the following example we have to use Neumann series.

Ex. 14. Solve the Volterra integral equation $y(x) = 1 + \int_0^x xt y(t) dt.$

Sol. Given $y(x) = 1 + \int_0^x xt y(t) dt.$... (1)

Comparing (1) with $y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt,$

here $f(x) = 1,$ $\lambda = 1,$ $K(x, t) = xt.$... (2)

Let $K_m(x, t)$ be the m th iterated kernel. Then

$$K_1(x, t) = K(x, t) \quad \dots (3)$$

and

$$K_m(x, t) = \int_t^x K(x, z) K_m(z, t) dz. \quad \dots (4)$$

From (2) and (3),

$$K_1(x, t) = K(x, t) = xt. \quad \dots (5)$$

Putting $m = 2$ in (4), we have

$$\begin{aligned} K_2(x, t) &= \int_t^x K(x, z) K_1(z, t) dz = \int_t^x (xz) (zt) dz = xt \int_t^x \left[\frac{z^3}{3} \right]_t^x, \text{ by (5)} \\ &= (xt/3) \times (x^3 - t^3) = (1/3) \times (x^4 t - xt^4). \end{aligned} \quad \dots (6)$$

Next, putting $m = 3$ in (4) we have

$$\begin{aligned} K_3(x, t) &= \int_t^x K(x, z) K_2(z, t) dz = \int_t^x (xz) \cdot \frac{zt}{3} (z^3 - t^3) dz, \text{ by (5) and (6)} \\ &= \frac{xt}{3} \int_t^x (z^5 - z^2 t^3) dz = \frac{xt}{3} \left[\frac{z^6}{6} - \frac{z^3 t^3}{3} \right]_t^x = \frac{xt}{3} \left(\frac{x^6}{6} - \frac{x^3 t^3}{3} - \frac{t^6}{6} + \frac{t^6}{3} \right) \\ &= (1/18) \times (x^7 t - 2x^4 t^4 + xt^7) \end{aligned} \quad \dots (7)$$

Next, putting $m = 4$ in (4), we have

$$\begin{aligned} K_4(x, t) &= \int_t^x K(x, z) K_3(z, t) dz = \int_t^x (xz) \times \frac{1}{18} (z^7 t - 2z^4 t^4 + zt^7) dz \\ &= \frac{x}{18} \int_t^x (z^8 t - 2z^5 t^4 + z^2 t^7) dz = \frac{x}{18} \left[\frac{z^9 t}{9} - \frac{z^6 t^4}{3} + \frac{z^3 t^7}{3} \right]_t^x \\ &= \frac{x}{18} \left(\frac{x^9 t}{9} - \frac{x^6 t^4}{3} + \frac{x^3 t^7}{3} - \frac{t^{10}}{9} + \frac{t^{10}}{3} - \frac{t^{10}}{3} \right) = \frac{1}{162} (x^{10} t - 3x^7 t^4 + 3x^4 t^7 - xt^{10}), \quad \dots (8) \end{aligned}$$

and so on.

Now the solution (1) is given by the Neumann series

$$y(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_0^x K_m(x, t) f(t) dt = 1 + \sum_{m=1}^{\infty} \lambda^m \int_0^x K_m(x, t) dt, \text{ using (2)}$$

$$\begin{aligned}
&= 1 + \int_0^x K_1(x, t) dt + \int_0^x K_2(x, t) dt + \int_0^x K_3(x, t) dt + \dots \\
&= 1 + \int_0^x xt dt + \frac{1}{3} \int_0^x (x^4 t - xt^4) dt + \frac{1}{18} \int_0^x (x^7 t - 2x^4 t^4 + xt^7) dt \\
&\quad + \frac{1}{162} \int_0^x (x^{10} t - 3x^7 t^4 + 3x^4 t^7 - xt^{10}) dt + \dots \\
&= 1 + x \left[\frac{t^2}{2} \right]_0^x + \frac{1}{3} \left[\frac{x^4 t^2}{2} - \frac{xt^5}{5} \right]_0^x + \frac{1}{18} \left[\frac{x^7 t^2}{2} - \frac{2x^4 t^5}{5} + \frac{xt^8}{8} \right]_0^x \\
&\quad + \frac{1}{162} \left[\frac{x^{10} t^2}{2} - \frac{3x^7 t^5}{5} + \frac{3x^4 t^8}{8} - \frac{xt^{11}}{11} \right]_0^x \\
&= 1 + \frac{x^3}{2} + \frac{1}{3} \left(\frac{x^6}{2} - \frac{x^6}{5} \right) + \frac{1}{18} \left(\frac{x^9}{2} - \frac{2x^9}{5} + \frac{x^9}{8} \right) + \frac{1}{162} \left(\frac{x^{12}}{2} - \frac{3x^{12}}{5} + \frac{3x^{12}}{8} - \frac{x^{12}}{11} \right) + \dots
\end{aligned}$$

or

$$y(x) = 1 + \frac{x^3}{2} + \frac{x^6}{2.5} + \frac{x^9}{2.5.8} + \frac{x^{12}}{2.5.8.11} + \dots$$

Type 4. The method of successive approximations for solving Volterra integral equation of the second kind :

$$y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt. \quad \dots (1)$$

Working Rule: Let $f(x)$ be continuous in $[0, a]$ and $K(x, t)$ be continuous for $0 \leq x \leq a, 0 \leq t \leq x$.

We start with some function $y_0(x)$ continuous in $[0, a]$. Replacing $y(t)$ on R.H.S of (1) by $y_0(x)$, we obtain

$$y_1(x) = f(x) + \lambda \int_0^x K(x, t) y_0(t) dt. \quad \dots (2)$$

$y_1(x)$ given by (2) is itself continuous in $[0, a]$. Proceeding likewise we arrive at a sequence of functions $y_0(x), y_1(x), \dots, y_n(x), \dots$, where

$$y_n(x) = f(x) + \lambda \int_0^x K(x, t) y_{n-1}(t) dt. \quad \dots (3)$$

In view of continuity of $f(x)$ and $K(x, t)$, the sequence $\{y_n(x)\}$ converges, as $n \rightarrow \infty$ to obtain the solution $y(x)$ of given integral equation (1).

Remark. As a particular case, when $y_0(x) = f(x)$, we obtain the so called Neumann series (refer Art. 5.11).

Ex. 15. Using the method of successive approximations, solve the integral equation

$$y(x) = 1 + \int_0^x y(t) dt, \text{ taking } y_0(x) = 0. \quad [\text{Kanpur 2007}]$$

Sol. Given $y(x) = 1 + \int_0^x y(t) dt, \quad \dots (1)$

and $y_0(x) = 0. \quad \dots (2)$

Comparing (1) with

$$y(x) = f(x) + \lambda \int_0^x K(x, y) y(t) dt,$$

here $f(x) = 1,$

$\lambda = 1,$

$K(x, t) = 1.$

... (3)

The n th order approximation is given by

$$y_n(x) = f(x) + \lambda \int_0^x K(x, t) y_{n-1}(t) dt$$

or

$$y_n(x) = 1 + \int_0^x y_{n-1}(t) dt, \text{ using (3)} \quad \dots (4)$$

Putting $n = 1$ in (4) and using (5), we have

$$y_1(x) = 1 + \int_0^x y_0(t) dt = 1 + \int_0^x (0) dt = 1. \quad \dots (5)$$

Next, putting $n = 2$ in (4) and using (5), we have

$$y_2(x) = 1 + \int_0^x y_1(t) dt = 1 + \int_0^x dt = 1 + [t]_0^x = 1 + x. \quad \dots (6)$$

Now, putting $n = 3$ in (4) and using (6), we have

$$y_3(x) = 1 + \int_0^x y_2(t) dt = 1 + \int_0^x (1 + t) dt = 1 + \left[t + \frac{t^2}{2} \right]_0^x = 1 + x + \frac{x^2}{2!} \quad \dots (7)$$

Next, putting $n = 4$ in (4) and using (7), we have

$$y_4(x) = 1 + \int_0^x y_3(t) dt = 1 + \int_0^x \left(1 + t + \frac{t^2}{2!} \right) dt = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \quad \dots (8)$$

and so on, Observing (5), (6), (7), (8) etc, we find

$$y_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{n-1}}{(n-1)!} \quad \dots (9)$$

Making $n \rightarrow \infty$, we find the required solution is given by

$$y(x) = \lim_{n \rightarrow \infty} y_n(x)$$

or

$$y(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \text{ad inf,}$$

or

$$y(x) = e^x.$$

Ex. 16. Using the method of successive approximations, solve the integral equation

$$y(x) = 1 + x - \int_0^x y(t) dt, \text{ taking } y_0(x) = 1.$$

[Kanpur 2010, 11; Meerut 2011; 2012]

Sol. Given

$$y(x) = 1 + x - \int_0^x y(t) dt, \quad \dots (1)$$

and

$$y_0(x) = 1. \quad \dots (2)$$

Comparing (1) with

$$y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt,$$

here

$f(x) = 1 + x,$

$\lambda = -1,$

$K(x, t) = 1.$

... (3)

The n th order approximation is given by

$$y_n(x) = f(x) + \lambda \int_0^x K(x, t) y_{n-1}(t) dt$$

or
$$y_n(x) = 1 + x - \int_0^x y_{n-1}(t) dt, \text{ by (3)} \quad \dots (4)$$

Putting $n = 1$ in (4), we have

$$y_1(x) = 1 + x - \int_0^x y_0(t) dt = 1 + x - \int_0^x dt, \text{ by (2)}$$

$\therefore y_1(x) = 1 + x - [t]_0^x = 1 + x - x = 1. \quad \dots (5)$

Next, putting $n = 2$ in (4) and using (5), we have

$$y_2(x) = 1 + x - \int_0^x y_1(t) dt = 1 + x - \int_0^x dt = 1 \quad \dots (6)$$

and so on. Observing (5) and (6), we find that

$$y_n(x) = 1, \text{ for } n = 1, 2, 3, \dots \quad \dots (7)$$

Hence, the required solution of (1) is given by $y(x) = \lim_{n \rightarrow \infty} y_n(x) = 1$

Ex.17. Using the method of successive approximations, solve the integral equation

$$y(x) = x - \int_0^x (x-t)y(t)dt, \quad y_0(x) = 0. \quad [\text{Kanpur 2006}]$$

Sol. Given
$$y(x) = x - \int_0^x (x-t)y(t) dt \quad \dots (1)$$

and
$$y_0(x) = 0 \quad \dots (2)$$

Comparing (1) with
$$y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt,$$

here $f(x) = x, \quad \lambda = -1, \quad \text{and} \quad K(x, t) = x - t. \quad \dots (3)$

The n th approximation $y_n(x)$ is given by

$$y_n(x) = f(x) + \lambda \int_0^x K(x, t) y_{n-1}(t) dt \quad \text{or} \quad y_n(x) = x - \int_0^x (x-t) y_{n-1}(t) dt \text{ by (3)} \quad \dots (4)$$

Putting $n = 1$ in (4) and using (2), we have

$$y_1(x) = x - \int_0^x (x-t)y_0(t) dt = x - \int_0^x (x-t) \times (0) dt = x \quad \dots (5)$$

Next putting $n = 2$ in (4), we have

$$y_2(x) = x - \int_0^x (x-t) y_1(t) dt = x - \int_0^x (x-t) t dt, \text{ by (5)}$$

or
$$y_2(x) = x - \left[\frac{xt^2}{2} - \frac{t^3}{3} \right]_0^x = x - \frac{x^3}{2} + \frac{x^3}{3} = x - \frac{x^3}{3!} \quad \dots (6)$$

Now, putting $n = 3$ in (4), we have

$$y_3(x) = x - \int_0^x (x-t) y_2(t) dt = x - \int_0^x (x-t) \left(t - \frac{t^3}{6} \right) dt, \text{ by (6)}$$

$$= x - \int_0^x \left(xt - \frac{xt^3}{6} - t^2 + \frac{t^4}{6} \right) dt = x - \left[\frac{xt^2}{2} - \frac{xt^4}{24} - \frac{t^3}{3} + \frac{t^5}{30} \right]_0^x$$

or
$$y_3(x) = x - \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^3}{3} - \frac{x^5}{30} = x - \frac{x^3}{6} + \frac{x^5}{120} = x - \frac{x^3}{3!} + \frac{x^5}{5!}, \quad \dots (7)$$

and so on. Observing (5), (6) and (7) we find that

$$y_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}.$$

The required solution $y(x)$ of (1) is given by $y(x) = \lim_{n \rightarrow \infty} y_n(x)$, i.e.,

$$y(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \dots \text{ad inf} \quad \text{or} \quad y(x) = \sin x.$$

Ex. 18. Using the method of successive approximations, solve the integral equation

$$y(x) = 1 + \int_0^x (x-t) y(t) dt, \text{ taking } y_0(x) = 1.$$

Sol. Given

$$y(x) = 1 + \int_0^x (x-t) y(t) dt \quad \dots (1)$$

and

$$y_0(x) = 0. \quad \dots (2)$$

Comparing (1) with

$$y(x) = f(x) + \lambda \int_0^x K(x,t) y(t) dt$$

here

$$f(x) = 1,$$

$$\lambda = 1,$$

$$K(x, t) = x - t.$$

$$\dots (3)$$

The n th order approximation is given by

$$y_n(x) = f(x) + \lambda \int_0^x K(x,t) y_{n-1}(t) dt$$

or

$$y_n(x) = 1 + \int_0^x (x-t) y_{n-1}(t) dt, \text{ by (3)} \quad \dots (4)$$

Putting $n = 1$ in (4), we have

$$y_1(x) = 1 + \int_0^x (x-t) y_0(t) dt = 1, \text{ by (2)} \quad \dots (5)$$

Next, putting $n = 2$ in (4), we have

$$\begin{aligned} y_2(x) &= 1 + \int_0^x (x-t) y_1(t) dt = 1 + \int_0^x (x-t) dt, \text{ by (5)} \\ &= 1 + \left[xt - \frac{t^2}{2} \right]_0^x = 1 + x^2 - \frac{x^2}{2} = 1 + \frac{x^2}{2} = 1 + \frac{x^2}{2!}. \end{aligned} \quad \dots (6)$$

Now, putting $n = 3$ in (4) and using (6), we have

$$\begin{aligned} y_3(x) &= 1 + \int_0^x (x-t) y_2(t) dt = 1 + \int_0^x (x-t) \left(1 + \frac{1}{2} t^2 \right) dt \\ &= 1 + \int_0^x \left(x + \frac{1}{2} xt^2 - t - \frac{1}{2} t^3 \right) dt = 1 + \left[xt + \frac{xt^3}{6} - \frac{t^2}{2} - \frac{t^4}{8} \right]_0^x \end{aligned}$$

or
$$y_3(x) = 1 + x^2 + \frac{x^4}{6} - \frac{x^2}{2} - \frac{x^4}{8} = 1 + \frac{x^2}{2} + \frac{x^4}{24} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}, \quad \dots (7)$$

and so on. Observing (5), (6) and (7), we have

$$y_n(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n-2}}{(2n-2)!}.$$

Now, the required solution $y(x)$ of (1) is given by
$$y(x) = \lim_{n \rightarrow \infty} y_n(x)$$

or
$$y(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n-2}}{(2n-2)!} + \dots \text{ad inf} \quad \text{or} \quad y(x) = \cosh x$$

Ex. 19. The integral equation $y(x) = x - \int_0^x (x-t)y(t) dt$ is solved by the method of successive approximations. Starting with initial approximation $y_0(x) = x$, the second approximation is given by

(a) $y_2(x) = x + x^3/3! + x^5/5!$ (b) $y_2(x) = x + x^3/3!$
 (c) $y_2(x) = x - x^3/3!$ (d) $y_2(x) = x - x^3/3! + x^5/5!$ [GATE 2005]

Sol. Given
$$y(x) = x - \int_0^x (x-t) y(t) dt \quad \dots (1)$$

and
$$y_0(x) = x \quad \dots (2)$$

Comparing (1) with $y(x) = f(x) + \lambda \int_0^x K(x,t) y(t) dt$, we get

$f(x) = x, \quad \lambda = -1 \quad \text{and} \quad K(x, t) = x - t \quad \dots (3)$

The n th order approximation is given by

$$y_n(x) = f(x) + \lambda \int_0^x K(x,t) y_{n-1}(t) dt$$

i.e.,
$$y_n(x) = x - \int_0^x (x-t) y_{n-1}(t) dt, \text{ using (3)} \quad \dots (4)$$

Putting $n = 1$ in (4), we have

$$y_1(x) = x - \int_0^x (x-t) y_0(t) dt = x - \int_0^x \{(x-t) \times t\} dt, \text{ by (2)}$$

or
$$y_1(x) = x - \left[xt^2/2 - t^3/3 \right]_0^x \quad \text{or} \quad y_1(x) = x - x^3/6 \quad \dots (5)$$

Putting $n = 2$ in (4), we have

$$y_2(x) = x - \int_0^x (x-t) y_1(t) dx = x - \int_0^x (x-t) (t - t^3/6) dt, \text{ by (5)}$$

or
$$y_2(x) = x - \int_0^x (xt - t^2 - xt^3/6 + t^4/6) dt = x - \left[\frac{xt^2}{2} - \frac{t^3}{3} - \frac{xt^4}{24} + \frac{t^5}{30} \right]_0^x$$

or
$$y_2(x) = x - (x^3/2 - x^3/3 - x^5/24 + x^5/30) = x - x^3/6 + x^5/120$$

Thus,
$$y_2(x) = x - x^3/3! + x^5/5!$$

EXERCISE 5 D

1. Using the method of successive approximations, solve the following integral equation with given value of $y_0(x)$ of zero-order approximation :

$$(i) \quad y(x) = 1 - \int_0^x (x-t) y(t) dt, \quad y_0(x) = 0. \quad \text{Ans. } y(x) = \cos x.$$

$$(ii) \quad y(x) = x + 1 - \int_0^x y(t) dt, \quad y_0(x) = x + 1. \quad \text{Ans. } y(x) = 1.$$

$$(iii) \quad y(x) = \frac{1}{2}x^2 + x - \int_0^x y(t) dt, \quad y_0(x) = 1. \quad \text{Ans. } y(x) = x.$$

$$(iv) \quad y(x) = \frac{1}{2}x^2 + x - \int_0^x y(t) dt, \quad y_0(x) = x. \quad \text{Ans. } y(x) = x.$$

$$(v) \quad y(x) = \frac{1}{2}x^2 + x - \int_0^x y(t) dt, \quad y_0(x) = \frac{1}{2}x^2 + x. \quad \text{Ans. } y(x) = x.$$

$$(vi) \quad y(x) = 1 + x + \int_0^x (x-t) y(t) dt, \quad y_0(x) = 1. \quad [\text{Meerut 2007}] \quad \text{Ans. } y(x) = e^x.$$

$$(vii) \quad y(x) = 2x + 2 - \int_0^x y(t) dt, \quad y_0(x) = 1. \quad \text{Ans. } y(x) = 2.$$

$$(viii) \quad y(x) = 2x + 2 - \int_0^x y(t) dt, \quad y_0(x) = 2. \quad \text{Ans. } y(x) = 2.$$

$$(ix) \quad y(x) = 2x^2 + 2 - \int_0^x x y(t) dt, \quad y_0(x) = 2. \quad \text{Ans. } y(x) = 2.$$

$$(x) \quad y(x) = 2x^2 + 2 - \int_0^x x y(t) dt, \quad y_0(x) = 2x. \quad \text{Ans. } y(x) = 2.$$

$$(xi) \quad y(x) = \frac{1}{2}x^3 - 2x - \int_0^x y(t) dt, \quad y_0(t) = x^2. \quad \text{Ans. } y(x) = x^2 - 2x.$$

2. Find an approximate solution of the integral equation $y(x) = \sinh x + \int_0^x e^{t-x} y(t) dt$, by the method of iteration.

3. Prove that the resolvent kernel for a Volterra integral equation of the second kind is an entire function of λ for any given (x, t) .

5.14 SOLUTION OF VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND

$$y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt, \quad \dots (1)$$

when its kernel $K(x, t)$ is of some particular forms.

Particular Form I* Suppose that the kernel $K(x, t)$ is a polynomial of degree $(n-1)$ in t , which can always be represented in the form

$$K(x, t) = a_0(x) + a_1(x)(x-t) + a_2(x) \frac{(x-t)^2}{2!} + \dots + a_{n-1}(x) \frac{(x-t)^{n-1}}{(n-1)!} \quad \dots (2)$$

* For proof, refer "A course in mathematical analysis", Volume III, Part II, by Edouard Goursat, Art. 94, page 7.

Then the resolvent kernel $R(x, t; \lambda)$ of (1) is given by

$$R(x, t; \lambda) = \frac{1}{\lambda} \frac{d^n g(x, t; \lambda)}{dx^n}, \quad \dots (3)$$

where $g(x, t; \lambda)$ is a solution of the differential equation

$$\frac{d^n g}{dx^n} - \lambda \left[a_0(x) \frac{d^{n-1} g}{dx^{n-1}} + a_1(x) \frac{d^{n-2} g}{dx^{n-2}} + \dots + a_{n-1}(x) g \right] = 0, \quad \dots (4)$$

satisfying the conditions

$$\left. \begin{aligned} g = \frac{dg}{dx} = \frac{d^2 g}{dx^2} = \dots = \frac{d^{n-2} g}{dx^{n-2}} = 0 \text{ when } x = t, \\ \frac{d^{n-1} g}{dx^{n-1}} = 1 \text{ when } x = t \end{aligned} \right\} \quad \dots (5)$$

and

The required solution of (1) is given by

$$y(x) = f(x) + \lambda \int_a^x R(x, t; \lambda) f(t) dt. \quad \dots (6)$$

Particular Form II. Suppose that the kernel $K(x, t)$ is a polynomial of degree $(n-1)$ in t , which can always be represented in the form

$$K(x, t) = b_0(t) + b_1(t)(t-x) + b_2(t) \frac{(t-x)^2}{2!} + \dots + b_{n-1}(t) \frac{(t-x)^{n-1}}{(n-1)!} \quad \dots (2)'$$

Then the resolvent kernel $R(x, t; \lambda)$ of (1) is given by

$$R(x, t; \lambda) = -\frac{1}{\lambda} \frac{d^n h(x, t; \lambda)}{dt^n}, \quad \dots (3)'$$

where $h(x, t; \lambda)$ is a solution of the differential equation

$$\frac{d^n h}{dt^n} + \lambda \left[b_0(t) \frac{d^{n-1} h}{dt^{n-1}} + b_1(t) \frac{d^{n-2} h}{dt^{n-2}} + \dots + b_{n-1}(t) h \right] = 0 \quad \dots (4)'$$

satisfying the conditions

$$\left. \begin{aligned} h = \frac{dh}{dt} = \frac{d^2 h}{dt^2} = \dots = \frac{d^{n-2} h}{dt^{n-2}} = 0 \text{ when } t = x, \\ \frac{d^{n-1} h}{dt^{n-1}} = 1 \text{ when } t = x. \end{aligned} \right\} \quad \dots (5)'$$

and

The required solution of (1) is again given by (6).

$$\text{Ex. 1. Solve } y(x) = 29 + 6x + \int_0^x [5 - 6(x-t)]y(t) dt. \quad [\text{Meerut 2006, 10}]$$

$$\text{Sol. Given } y(x) = 29 + 6x + \int_0^x [5 - 6(x-t)]y(t) dt \quad \dots (1)$$

$$\text{Comparing (1) with } y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt,$$

$$\text{here } f(x) = 29 + 6x \quad \text{and} \quad \lambda = 1. \quad \dots (2)$$

$$\text{and } K(x, t) = 5 - 6(x-t). \quad \dots (3)$$

Let $K(x, t) = a_0(x) + a_1(x)(x - t)$ (4)

Comparing (3) and (4), $a_0(x) = 5$ and $a_1(x) = -6$ (5)

Then the resolvent kernel $R(x, t; \lambda)$ of (1) is given by

$$R(x, t; \lambda) = \frac{d^2 g(x, t; \lambda)}{dx^2} \quad \text{or} \quad R(x, t; 1) = \frac{d^2 g(x, t; 1)}{dx^2} = \frac{d^2 g}{dx^2}, \text{ as } \lambda = 1. \quad \dots (6)$$

where $g(x, t; 1)$ satisfies the differential equation

$$\frac{d^2 g}{dx^2} - \lambda \left[a_0(x) \frac{dg}{dx} + a_1(x)g \right] = 0, \text{ where } \lambda = 1$$

or $\frac{d^2 g}{dx^2} - \left(5 \frac{dg}{dx} - 6g \right) = 0$ or $(D^2 - 5D + 6)g = 0$, when $D \equiv d/dx$... (7)

satisfying the conditions $g = 0$, when $x = t$ (8A)

and $dg/dx = 1$, when $x = t$ (8B)

Now, the auxiliary equation of (7) is $D^2 - 5D + 6 = 0$ giving $D = 3, 2$.

Hence the general solution of (7) is given by

$$g = Ae^{3x} + Be^{2x}. \quad \dots (9)$$

From (9) $dg/dx = 3Ae^{3x} + 2Be^{2x}$ (10)

Putting $x = t$ in (9) and (10) and using (8A) and (8B), we obtain

$$0 = Ae^{3t} + Be^{2t} \quad \dots (11)$$

and $1 = 3Ae^{3t} + 2Be^{2t}$ (12)

Solving (11) and (12), we have $A = e^{-3t}$ and $B = -e^{-2t}$ (13)

Substituting these values in (9), we have

$$g = g(x, t; 1) = e^{3(x-t)} - e^{2(x-t)} \quad \dots (14)$$

Differentiating both sides of (14) w.r.t. 'x', we have

$$dg/dx = 3e^{3(x-t)} - 2e^{2(x-t)} \quad \dots (15)$$

Differentiations both sides of (15) again w.r.t. 'x', we have

$$d^2 g/dx^2 = 9e^{3(x-t)} - 4e^{2(x-t)}. \quad \dots (16)$$

Using the above value of $d^2 g/dx^2$ in (6), we have

$$R(x, t; 1) = 9e^{3(x-t)} - 4e^{2(x-t)}. \quad \dots (17)$$

Hence the required solution of (1) is

$$y(x) = f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt = 29 + 6x + \int_0^x R(x, t; 1) (29 + 6t) dt, \text{ by (2)}$$

$$= 29 + 6x + \int_0^x [9e^{3(x-t)} - 4e^{2(x-t)}] (29 + 6t) dt, \text{ using (17)}$$

$$= 29 + 6x + \left[(29 + 6t) \{-3e^{3(x-t)} + 2e^{2(x-t)}\} \right]_0^x - \int_0^x 6\{-3e^{3(x-t)} + 2e^{2(x-t)}\} dt$$

[on integration by parts]

$$\begin{aligned}
&= 29 + 6x + (29 + 6x)(-3 + 2) - 29(-3e^{3x} + 2e^{2x}) - 6 \left[e^{3(x-t)} - e^{2(x-t)} \right]_0^x \\
&= 29 + 6x - 29x - 6x - 29(-3e^{3x} + 2e^{2x}) - 6[0 - (e^{3x} - e^{2x})] \\
\therefore y(x) &= 93e^{3x} - 64e^{2x}.
\end{aligned}$$

Ex. 2. Solve $y(x) = 1 - 2x - 4x^2 + \int_0^x [3 + 6(x-t) - 4(x-t)^2] y(t) dt$.

Sol. Given $y(x) = 1 - 2x - 4x^2 + \int_0^x [3 + 6(x-t) - 4(x-t)^2] y(t) dt$ (1)

Comparing (1) with $y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt$,

here $f(x) = 1 - 2x - 4x^2$ and $\lambda = 1$, ... (2)

and $K(x, t) = 3 + 6(x-t) - 4(x-t)^2$ (3)

Let $K(x, t) = a_0(x) + a_1(x)(x-t) + a_2(x) \frac{(x-t)^2}{2!}$ (4)

Comparing (3) and (4), $a_0(x) = 3$, $a_1(x) = 6$, $a_2(x) = -8$ (5)

Then the resolvent kernel $R(x, t; \lambda)$ of (1) is given by

$$R(x, t; \lambda) = \frac{d^3 g(x, t; \lambda)}{dx^3} \quad \text{or} \quad R(x, t; 1) = \frac{d^3 g(x, t; 1)}{dx^3} = \frac{d^3 g}{dx^3}, \text{ as } \lambda = 1. \quad \dots (6)$$

where $g(x, t; 1)$ satisfies the equation

$$\frac{d^3 g}{dx^3} - \lambda \left[a_0(x) \frac{d^2 g}{dx^2} + a_1(x) \frac{dg}{dx} + a_2(x) g \right] = 0 \quad \text{or} \quad \frac{d^3 g}{dx^3} - \left(3 \frac{d^2 g}{dx^2} + 6 \frac{dg}{dx} - 8g \right) = 0$$

or $(D^3 - 3D^2 - 6D + 8)g = 0$ $D \equiv d/dx$... (7)

satisfying the conditions $g = 0, dg/dx = 0$ when $x = t$ (8A)

and $d^2 g/dx^2 = 1$ when $x = t$ (8B)

Now, the auxiliary equation of (7) is

$$D^3 - 3D^2 - 6D + 8 = 0, \text{ i.e., } (D-1)(D+2)(D-4) = 0 \quad \text{giving} \quad D = 1, -2, 4.$$

Hence the general solution of (7) is

$$g = Ae^x + Be^{-2x} + Ce^{4x}. \quad \dots (9)$$

From (9) $dg/dx = Ae^x - 2Be^{-2x} + 4Ce^{4x}$... (10)

and $d^2 g/dx^2 = Ae^x + 4Be^{-2x} + 16Ce^{4x}$ (11)

Putting $x = t$ in (9), (10) and (11) and using (8A) and (8B), we have

$$\left. \begin{aligned} 0 &= Ae^t + Be^{-2t} + Ce^{4t}, \\ 0 &= Ae^t - 2Be^{-2t} + 4Ce^{4t}, \\ 1 &= Ae^t + 4Be^{-2t} + 16Ce^{4t}. \end{aligned} \right\} \quad \dots (12)$$

and

Solving (12) for A, B, C, we have

$$A = -(1/9) \times e^{-t}, \quad B = (1/18) \times e^{2t}, \quad C = (1/18) \times e^{-4t}. \quad \dots (13)$$

Substituting these values in (9), we have

$$g = g(x, t; 1) = -(1/9) \times e^{x-t} + (1/18) \times e^{-2x+2t} + (1/18) \times e^{4x-4t}. \quad \dots (14)$$

Differentiating both sides of (14) w.r.t. 'x', we get

$$\frac{dg}{dx} = -\frac{1}{9}e^{x-t} - \frac{1}{9}e^{-2x+2t} + \frac{1}{9}e^{4x-4t}. \quad \dots (15)$$

Again, differentiating both sides of (15) w.r.t. 'x', we get

$$\frac{d^2g}{dx^2} = -\frac{1}{9}e^{x-t} + \frac{2}{9}e^{-2x+2t} + \frac{8}{9}e^{4x-4t}. \quad \dots (16)$$

Finally, differentiating both sides of (16) w.r.t. 'x', we get

$$\frac{d^3g}{dx^3} = -\frac{1}{9}e^{x-t} - \frac{4}{9}e^{-2x+2t} + \frac{32}{9}e^{4x-4t}. \quad \dots (17)$$

Using the above value in (6), we have

$$R(x, t; \lambda) = -(1/9) \times e^{x-t} - (4/9) \times e^{-2(x-t)} + (32/9) \times e^{4(x-t)}. \quad \dots (18)$$

Hence the required solution of (1) is

$$y(x) = f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt$$

or

$$y(x) = 1 - 2x - 4x^2 + \int_0^x R(x, t; 1) (1 - 2t - 4t^2) dt, \text{ by (2)}$$

or

$$\begin{aligned} y(x) &= 1 - 2x - 4x^2 + \int_0^x (1 - 2t - 4t^2) \left[-\frac{1}{9}e^{x-t} - \frac{4}{9}e^{-2(x-t)} + \frac{32}{9}e^{4(x-t)} \right] dt, \text{ by (18)} \\ &= 1 - 2x - 4x^2 + \left[(1 - 2t - 4t^2) \left\{ \frac{1}{9}e^{x-t} - \frac{2}{9}e^{-2(x-t)} - \frac{8}{9}e^{4(x-t)} \right\} \right]_0^x \\ &\quad - \int_0^x (-2 - 8t) \left\{ \frac{1}{9}e^{x-t} - \frac{2}{9}e^{-2(x-t)} - \frac{8}{9}e^{4(x-t)} \right\} dt, \text{ integrating by parts} \\ &= 1 - 2x - 4x^2 + (1 - 2x - 4x^2) \left(\frac{1}{9} - \frac{2}{9} - \frac{8}{9} \right) - \left(\frac{1}{9}e^x - \frac{2}{9}e^{-2x} - \frac{8}{9}e^{4x} \right) \\ &\quad + \int_0^x (2 + 8t) \left\{ \frac{1}{9}e^{x-t} - \frac{2}{9}e^{-2(x-t)} - \frac{8}{9}e^{4(x-t)} \right\} dt \\ &= -\frac{1}{9}e^x + \frac{2}{9}e^{-2x} + \frac{8}{9}e^{4x} + \left[(2 + 8t) \left\{ -\frac{1}{9}e^{x-t} - \frac{1}{9}e^{-2(x-t)} + \frac{2}{9}e^{4(x-t)} \right\} \right]_0^x \\ &\quad - \int_0^x 8 \left\{ -\frac{1}{9}e^{x-t} - \frac{1}{9}e^{-2(x-t)} + \frac{2}{9}e^{4(x-t)} \right\} dt \\ &= -\frac{1}{9}e^x + \frac{2}{9}e^{-2x} + \frac{8}{9}e^{4x} + (2 + 8x) \left(-\frac{1}{9} - \frac{1}{9} + \frac{2}{9} \right) \\ &\quad - 2 \left(-\frac{1}{9}e^x - \frac{1}{9}e^{-2x} + \frac{2}{9}e^{4x} \right) - 8 \left[\frac{1}{9}e^{x-t} - \frac{1}{18}e^{-2(x-t)} - \frac{1}{18}e^{4(x-t)} \right]_0^x \end{aligned}$$

$$= \frac{1}{9}e^x + \frac{4}{9}e^{-2x} + \frac{4}{9}e^{4x} - 8 \left(\frac{1}{9} - \frac{1}{18} - \frac{1}{18} - \frac{1}{9}e^x + \frac{1}{18}e^{-2x} + \frac{1}{18}e^{4x} \right)$$

$$\therefore y(x) = e^x.$$

Ex. 3. Solve $y(x) = \cos x - x - 2 + \int_0^x (t-x) y(t) dt$.

Sol. Given $y(x) = \cos x - x - 2 + \int_0^x (t-x) y(t) dt$ (1)

Comparing (1) with $y(x) = f(x) + \lambda \int_0^x K(x, t) y(t) dt$,

here $f(x) = \cos x - x - 2$. $\lambda = 1$, ... (2)

and $K(x, t) = (t-x)$... (3)

Let $K(x, t) = b_0(t) + b_1(t)(t-x)$ (4)

Comparing (3) and (4), $b_0(t) = 0$, $b_1(t) = 1$... (5)

Then the resolvent kernel $R(x, t; \lambda)$ of (1) is given by

$$R(x, t; \lambda) = -\frac{1}{\lambda} \frac{d^2 h(x, t; \lambda)}{dt^2} \quad \text{or} \quad R(x, t; 1) = -\frac{d^2 h(x, t; 1)}{dt^2} = -\frac{d^2 h}{dt^2}, \text{ as } \lambda = 1 \quad \dots (6)$$

where h or $h(x, t, 1)$ satisfies the differential equation

$$\text{or} \quad \frac{d^2 h}{dt^2} + \lambda \left[b_0(t) \frac{dh}{dt} + b_1(t) h \right] = 0 \quad \text{or} \quad \frac{d^2 h}{dt^2} + (0 + h) = 0, \text{ using (2) and (5)}$$

$$\text{or} \quad (D^2 + 1) h = 0, \quad \text{where} \quad D \equiv d/dt, \quad \dots (7)$$

satisfying the conditions.

$$h = 0 \text{ at } t = x, \quad \dots (8A)$$

$$\text{and} \quad dh/dt = 1 \text{ at } t = x. \quad \dots (8B)$$

Now, the auxiliary equation of (7) is $D^2 + 1 = 0$ so that $D = 0 \pm i$.

Hence the general solution of (7) is

$$h = A \cos t + B \sin t. \quad \dots (9)$$

$$\text{From (9)} \quad dh/dt = -A \sin t + B \cos t. \quad \dots (10)$$

Putting $t = x$ in (9) and (10) and using (8A) and (8B), we have

$$0 = A \cos x + B \sin x \quad \dots (11)$$

$$\text{and} \quad 1 = -A \sin x + B \cos x. \quad \dots (12)$$

Solving (11) and (12) for A and B , we get

$$A = -\sin x \quad \text{and} \quad B = \cos x. \quad \dots (13)$$

Substituting these values in (9), we get

$$h = -\sin x \cos t + \cos x \sin t \quad \text{or} \quad h = \sin(t-x) \quad \dots (14)$$

$$\text{Differentiating (14) w.r.t. 't', we get} \quad dh/dt = \cos(t-x) \quad \dots (15)$$

$$\text{Again, differentiating (15) w.r.t. 't', we get} \quad d^2 h/dt^2 = -\sin(t-x). \quad \dots (16)$$

Using this value in (6), we have

$$R(x, t, 1) = -\{-\sin(t-x)\} = \sin(t-x). \quad \dots (17)$$

Hence the required solution is given by

$$y(x) = f(x) + \lambda \int_0^x R(x, t; \lambda) f(t) dt$$

or
$$y(x) = \cos x - x - 2 + \int_0^x R(x, t; 1) (\cos t - t - 2) dt, \text{ using (2)}$$

$$= \cos x - x - 2 + \int_0^x \sin(t - x) (\cos t - t - 2) dt, \text{ using (17)}$$

$$= \cos x - x - 2 + \int_0^x \sin(t - x) \cos t dt + \int_0^x (-t - 2) \sin(t - x) dt$$

$$= \cos x - x - 2 + \frac{1}{2} \int_0^x [\sin(2t - x) - \sin x] dt + [(-t - 2) \{-\cos(t - x)\}]_0^x$$

$$- \int_0^x (-1) \{-\cos(t - x)\} dt \quad [\text{In last term, integrating by parts}]$$

$$= \cos x - x - 2 + \frac{1}{2} \left[-\frac{\cos(2t - x)}{2} - t \sin x \right]_0^x + (x + 2) - 2 \cos x - [\sin(t - x)]_0^x$$

$$= -\cos x + \frac{1}{2} \left[-\frac{\cos x}{2} - x \sin x + \frac{\cos x}{2} \right] + \sin(-x)$$

or
$$y(x) = -\cos x - (x/2) \times \sin x - \sin x.$$

EXERCISES 5E

Find the resolvent kernels of Volterra integral equations with the following kernels (taking $\lambda = 1$).

1. $K(x, t) = 2 - (x - t).$ Ans. $e^{x-t}(x - t + 2).$

2. $K(x, t) = -2 + 3(x - t).$ Ans. $(1/4) \times e^{x-t} - (9/4) \times e^{-3(x-t)}.$

3. $K(x, t) = 2x$ Ans. $2x e^{x^2 - t^2}.$

4. $K(x, t) = -\frac{4x - 2}{2x + 1} + \frac{8(x - t)}{2x + 1}.$ Ans. $\frac{4t^2 + 1}{2(2t + 1)^2} \left[\frac{8}{4t^2 + 1} - 4e^{-2(x-t)} \right]$

5.15. SOLUTION OF VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND BY REDUCING TO DIFFERENTIAL EQUATION.

The whole procedure will be clear by following example.

Ex. 1. Solve $y(x) = 1 - 2x - 4x^2 + \int_0^x [3 + 6(x - t) - 4(x - t)^2] y(t) dt.$ [Meerut 2012]

Sol. Given
$$y(x) = 1 - 2x - 4x^2 + \int_0^x [3 + 6(x - t) - 4(x - t)^2] y(t) dt. \quad \dots (1)$$

Differentiating both sides of (1) w.r.t. x and using Leibnitz's rule of differentiation under the sign of integration (refer Art. 1.13), we have

$$y'(x) = -2 - 8x + \int_0^x \{6 - 8(x - t)\} y(t) dt + 3y(x). \quad \dots (2)$$

Again, differentiating both sides of (2) w.r.t. ' x ', and using Leibnitz's rule as before, we get

$$y''(x) = -8 - 8 \int_0^x y(t) dt + 6y(x) + 3y'(x). \quad \dots (3)$$

Finally, differentiating both sides of (3) w.r.t. ' x ', and using Leibnitz's rule as before, we get

$$y'''(x) = -8y(x) + 6y'(x) + 3y''(x). \quad \dots (4)$$

Putting $x = 0$ in (1), we have $y(0) = 1. \quad \dots (5)$

Next, putting $x = 0$ in (2), we have

$$y'(0) = -2 + 3y(0) = -2 + 3 = 1, \text{ using (5)} \quad \dots (6)$$

Now, putting $x = 0$ in (3), we have

$$y''(0) = -8 + 6y(0) + 3y'(0) = -8 + 6 + 3 = 1, \text{ using (5) and (6)} \quad \dots (7)$$

Re-writing (4), we have

$$\frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 8y = 0 \quad \text{or} \quad (D^3 - 3D^2 - 6D + 8) = 0. \quad \dots (8)$$

We shall solve the reduced differential equation by using the prescribed initial conditions (5), (6) and (7). The auxiliary equation of (8) is

$$D^3 - 3D^2 - 6D + 8 = 0 \quad \text{giving} \quad D = 1, 4, -2.$$

Hence the general solution of (8) is

$$y(x) = Ae^x + Be^{4x} + Ce^{-2x}. \quad \dots (9)$$

From (9), $y'(x) = Ae^x + 4Be^{4x} - 2Ce^{-2x}. \quad \dots (10)$

From (10), $y''(x) = Ae^x + 16Be^{4x} + 4Ce^{-2x}. \quad \dots (11)$

Putting $x = 0$ in (9), (10) and (11) and using (5) (6) and (7), we obtain

$$1 = A + B + C, \quad \dots (12)$$

$$1 = A + 4B - 2C, \quad \dots (13)$$

and $1 = A + 16B + 4C. \quad \dots (14)$

Solving (12), (13) and (14) for A, B, C, we obtain $A = 1, \quad B = 0, \quad C = 0$

With these values, (9) gives the required solution $y(x) = e^x$.

5.16. VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND.

Theorem. Volterra integral equation of the first kind can be converted to a Volterra integral equation of the second kind.

Proof. Let the given Volterra integral equation of the first kind be

$$\int_0^x K(x, t) y(t) dt = f(x), \quad \dots (1)$$

where $y(x)$ is the unknown function.

Suppose that the Kernel $K(x, t)$ and all its partial derivatives occurring in the problem are continuous. Then clearly condition $f(0) = 0$ is necessary for (1) to possess a continuous solution. Moreover, if the kernel $K(x, t)$ possess a continuous derivative $\partial K(x, t) / \partial x$, the first member of (1) also possess a continuous derivatives; the same must then be true of $f(x)$. Assuming that this condition is satisfied, we differentiate both sides of (1) w.r.t. 'x', and use Leibnitz's rule of differentiation under the sign of integration (refer Art. 1.13) to obtain

$$\int_0^x K'_x(x, t) y(t) dt + K(x, x) y(x) \frac{dx}{dx} - K(x, 0) y(0) \frac{d0}{dx} = f'(x), \text{ where } K'_x(x, t) = \frac{\partial K(x, t)}{\partial x}$$

or $\int_0^x K'_x(x, t) y(t) dt + K(x, x) y(x) = f'(x). \quad \dots (2)$

Conversely, every solution of (2) also satisfies (1), since the two members are zero at $x = 0$, and their derivatives are identical.

Let $K(x, x) \neq 0$ at any point of the basic interval $[0, a]$. Then dividing both sides of (2) by

$K(x, x)$, we get

$$\frac{1}{K(x, x)} \int_0^x K'_x(x, t) y(t) dt + y(x) = \frac{f'(x)}{K(x, x)}$$

or
$$y(x) = \frac{f'(x)}{K(x, x)} - \int_0^x \frac{K'_x(x, t)}{K(x, x)} y(t) dt. \quad \dots (3)$$

which is a Volterra integral equation of the second kind.

On the other hand, if $K(x, x) \equiv 0$, then we do not get (3). In fact, when $K(x, x) \equiv 0$, (2) is again an integral equation of the first kind, which can be dealt in a similarly fashion. Now, with $K(x, x) \equiv 0$, (2) becomes

$$\int_0^x K'_x(x, t) y(t) dt = f'(x). \quad \dots (4)$$

Suppose that the kernel $K(x, t)$ possesses continuous second order partial derivative $\partial^2 K(x, t) / \partial x^2$. Differentiating both sides of (4) w.r.t. 'x' and using again Leibnitz's rule, we have

$$\int_0^x K''_x(x, t) y(t) dt + K'_x(x, x) y(x) \frac{dx}{dx} - K'(x, 0) y(0) \frac{d0}{dx} = f''(x), \text{ where } K'_x(x, t) = \frac{\partial^2 K(x, t)}{\partial x^2}$$

or
$$\int_0^x K''_x(x, t) y(t) dt + K'_x(x, x) y(x) = f''(x). \quad \dots (5)$$

As before, if $K'_x(x, x) \neq 0$, then dividing (5) by $K'_x(x, x)$, we have

$$y(x) = \frac{f''(x)}{K'_x(x, x)} - \int_0^x \frac{K''_x(x, t)}{K'_x(x, x)} y(t) dt. \quad \dots (6)$$

which is again a Volterra integral equation of the second kind.

On the other hand if $K'_x(x, x) \equiv 0$, (5) reduces to integral equation of the first kind and use the same procedure again, and so on. In this manner we have a sequence to successive derivatives of $K(x, t)$ w.r.t. 'x', until we arrive at a derivative $K_x^{(p-1)}(x, t)$ which is not identically zero for $x = t$. In order that (1) have a continuous solution, it would be necessary for $f(x)$ to possess continuous derivatives $f'(x), f''(x), \dots, f^{(p-1)}(x)$ which are all zero for $x = 0$. In such a situation, the first $(p-1)$ equations obtained by differentiating both sides of (1) w.r.t. 'x' are satisfied for $x = 0$. If $K_x^{(p)}(x, t)$ is also continuous, $f^{(p)}(x)$ must also be continuous and, differentiating both sides w.r.t. x, one more times, we arrive at the integral equation, as before,

$$\int_0^x K_x^{(p)}(x, t) y(t) dt + K_x^{(p-1)}(x, x) y(x) = f^{(p)}(x). \quad \dots (7)$$

If $K_x^{(p-1)}(x, t) \neq 0$, then (7) reduces to as before

$$y(x) = \frac{f^{(p)}(x)}{K_x^{(p-1)}(x, x)} - \int_0^x \frac{K_x^{(p)}(x, t)}{K_x^{(p-1)}(x, x)} y(t) dt \quad \dots (8)$$

which is again a Volterra integral equation of the second kind. Now, (8) has a unique solution. In fact in going back step by step, we can easily prove that the solution so obtained satisfies all the

intermediate equations as well as (1).

5.17. SOLUTION OF VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND.

For this purpose we shall first convert the given volterra integral equation of the first kind into a Volterra integral equation of the second as explained in Art. 6.16. Then we adopt the usual methods of solving a Volterra integral equation of the second kind.

ILLUSTRATIVE SOLVED EXAMPLES.

Ex. 1. Solved the following Volterra integral equation of the first kind.

$$f(x) = \int_0^x e^{x-t} y(t) dt, \quad f(0) = 0.$$

Sol. Given
$$\int_0^x e^{x-t} y(t) dt = f(x) \quad \dots (1)$$

and
$$f(0) = 0. \quad \dots (2)$$

Here the Kernel e^{x-t} and its partial derivative $\partial e^{x-t} / \partial x$ are continuous. Condition (2) is necessary for (1) to possess a continuous solution. Again, since $\partial e^{x-t} / \partial x$ is continuous, the same must be true for $f(x)$.

Differentiating both sides of (1) w.r.t. 'x' and using Leibnitz's rule of differentiation under the sign of integration, we get

$$\int_0^x \frac{\partial(e^{x-t})}{\partial x} y(t) dt + e^{x-x} y(x) \frac{dx}{dx} - e^{x-0} y(0) \frac{d0}{dx} = f'(x)$$

or
$$y(x) = f'(x) - \int_0^x e^{x-t} y(t) dt, \quad \dots (3)$$

which is a Volterra integral equation of the second kind.

Comparing (3) with
$$y(x) = F(x) + \lambda \int_0^x K(x, t) y(t) dt,$$

we have
$$F(x) = f'(x), \quad \lambda = -1, \quad K(x, t) = e^{x-t} \quad \dots (4)$$

Let $K_m(x, t)$ be the m th iterated Kernel. Then

$$K_1(x, t) = K(x, t) \quad \dots (5)$$

and
$$K_m(x, t) = \int_t^x K(x, z) K_{m-1}(z, t) dz, \quad (m = 2, 3, \dots) \quad \dots (6)$$

From (4) and (5),
$$K_1(x, t) = K(x, t) = e^{x-t}. \quad \dots (7)$$

Putting $m = 2$ in (6), we have

$$\begin{aligned} K_2(x, t) &= \int_t^x K(x, z) K_1(z, t) dz = \int_t^x e^{x-z} e^{z-t} dz, \text{ by (7)} \\ &= e^{x-t} \int_t^x dz = e^{x-t} [z]_t^x = e^{x-t} (x - t). \end{aligned} \quad \dots (8)$$

Next, putting $m = 3$ in (6) and using (8), we have

$$K_3(x, t) = \int_t^x K(x, z) K_2(z, t) dz = \int_t^x e^{x-z} \cdot e^{z-t} (z - t) dz = e^{x-t} \int_t^x (z - t) dz$$

$$= e^{x-t} \left[\frac{(z-t)^2}{2} \right]_t^x = e^{x-t} \frac{(x-t)^2}{2!} \quad \dots (9)$$

Now, putting $m = 4$ in (6) and using (9), we have

$$\begin{aligned} K_4(x, t) &= \int_t^x K(x, z) K_3(z, t) dz = \int_t^x e^{x-z} e^{z-t} \frac{(z-t)^2}{2!} dz = \frac{e^{x-t}}{2!} \int_t^x (z-t)^2 dz \\ &= \frac{e^{x-t}}{2!} \left[\frac{(z-t)^3}{3} \right]_t^x = e^{x-t} \frac{(x-t)^3}{3!}, \end{aligned} \quad \dots (10)$$

and so on. So by mathematical induction, we have

$$K_m(x, t) = e^{x-t} \frac{(x-t)^{m-1}}{(m-1)!}, \quad m = 1, 2, 3, \dots \quad \dots (11)$$

Now, the resolvent kernel $R(x, t, \lambda)$ of (3) is given by

$$\begin{aligned} R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = \sum_{m=1}^{\infty} (-1)^{m-1} K_m(x, t) \quad [\because \lambda = -1, \text{ by (4)}] \\ &= K_1(x, t) - K_2(x, t) + K_3(x, t) - K_4(x, t) + \dots \\ &= e^{x-t} - e^{x-t}(x-t) + e^{x-t} \frac{(x-t)^2}{2!} - e^{x-t} \frac{(x-t)^3}{3!} + \dots \text{ad inf, using (11)} \\ &= e^{x-t} \left[1 - \frac{(x-t)}{1!} + \frac{(x-t)^2}{2!} - \frac{(x-t)^3}{3!} + \dots \text{ad inf} \right] = e^{x-t} e^{-(x-t)} = 1 \quad \dots (12) \end{aligned}$$

Hence, the required solution is given by

$$\begin{aligned} y(x) &= F(x) + \lambda \int_0^x R(x, t; \lambda) F(t) dt = f'(x) - \int_0^x f'(t) dt, \text{ by (4) and (12)} \\ &= f'(x) - [f(t)]_0^x = f'(x) - [f(x) - f(0)] = f'(x) - f(x), \text{ using (2)} \\ \therefore \quad y(x) &= f'(x) - f(x). \end{aligned}$$

Ex. 2. Solve $\int_0^x e^{x-t} y(t) dt = \sin x$.

Sol. Given $\int_0^x e^{x-t} y(t) dt = \sin x. \quad \dots (1)$

Taking $f(x) = \sin x$, we have $f(0) = \sin 0 = 0. \quad \dots (2)$

We proceed as in solved Ex. 1.

Differentiating both sides of (1) w.r.t. 'x', we get

$$\int_0^x e^{x-t} y(t) dt + y(x) = \cos x \quad \text{or} \quad y(x) = \cos x - \int_0^x e^{x-t} y(t) dt. \quad \dots (3)$$

Comparing (3) with $y(x) = F(x) + \lambda \int_0^x K(x, t) y(t) dt.$

here $F(x) = \cos x, \quad \lambda = -1, \quad K(x, t) = e^{x-t}. \quad \dots (4)$

As in solved Ex. 1, we have $R(x, t; \lambda) = 1.$

Hence the required solution of (1) is

$$y(x) = F(x) + \lambda \int_0^x R(x, t; \lambda) F(t) dt$$

$$\text{or} \quad y(x) = \cos x - \int_0^x \cos t dt \quad \text{or} \quad y(x) = \cos x - [\sin t]_0^x = \cos x - \sin x.$$

$$\text{Ex. 3. Solve } \int_0^x a^{x-t} y(t) dt = f(x), \quad f(0) = 0.$$

$$\text{Sol. Given} \quad \int_0^x a^{x-t} y(t) dt = f(x), \quad \dots (1)$$

$$\text{with} \quad f(0) = 0. \quad \dots (2)$$

Differentiating both sides of (1) w.r.t. 'x', we get

$$\int_0^x a^{x-t} \log_e a y(t) dt + y(x) = f'(x)$$

$$\text{or} \quad y(x) = f'(x) - \log_e a \int_0^x a^{x-t} y(t) dt. \quad \dots (3)$$

$$\text{Comparing (3) with} \quad y(x) = F(x) + \lambda \int_0^x K(x, t) y(t) dt,$$

$$\text{here} \quad F(x) = f'(x), \quad \lambda = -\log_e a, \quad K(x, t) = a^{x-t}. \quad \dots (4)$$

Let $K_m(x, t)$ be the m th iterated Kernel. Then

$$K_1(x, t) = K(x, t) \quad \dots (5)$$

$$\text{and} \quad K_m(x, t) = \int_t^x K(x, z) K_{m-1}(z, t) dz \quad \dots (6)$$

$$\text{From (4) and (5),} \quad K_1(x, t) = K(x, t) = a^{x-t}. \quad \dots (7)$$

Putting $m = 2$ in (6) and using (7), we have

$$\begin{aligned} K_2(x, t) &= \int_t^x K(x, z) K_1(z, t) dz = \int_t^x a^{x-z} a^{z-t} dz = a^{x-t} \int_t^x dz \\ &= a^{x-t} [z]_t^x = a^{x-t} (x-t). \end{aligned} \quad \dots (8)$$

Next, putting $m = 3$ in (6) and using (8), we have

$$\begin{aligned} K_3(x, t) &= \int_t^x K(x, z) K_2(z, t) dz = \int_t^x a^{x-z} a^{z-t} (z-t) dz \\ &= a^{x-t} \int_t^x (z-t) dz = a^{x-t} \left[\frac{(z-t)^2}{2} \right]_t^x = a^{x-t} \frac{(x-t)^2}{2!}. \end{aligned} \quad \dots (9)$$

Now, putting $m = 4$ in (6), we have

$$\begin{aligned} K_4(x, t) &= \int_t^x K(x, z) K_3(z, t) dz = \int_t^x a^{x-z} a^{z-t} \frac{(z-t)^2}{2!} dz, \text{ using (9)} \\ &= \frac{a^{x-t}}{2!} \int_t^x (z-t)^2 dz = \frac{a^{x-t}}{2!} \left[\frac{(z-t)^3}{3} \right]_t^x = a^{x-t} \frac{(x-t)^3}{3!}, \end{aligned} \quad \dots (10)$$

and so on. By, mathematical induction, we now obtain

$$K_m(x, t) = a^{x-t} \frac{(x-t)^{m-1}}{(m-1)!}, \quad (m = 1, 2, 3, \dots) \quad \dots (11)$$

Now, the resolvent kernel of (3) is given by

$$\begin{aligned} R(x, t; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = K_1(x, t) + \lambda K_2(x, t) + \lambda^2 K_3(x, t) + \dots \\ &= a^{x-t} - \log_e a \cdot a^{x-t} (x-t) + (\log_e a)^2 \cdot a^{x-t} \frac{(x-t)^2}{2!} + \dots \quad \text{ad inf, by (4) and (11)} \\ &= a^{x-t} \left[1 - \frac{(x-t) \log_e a}{1!} + \frac{[(x-t)^2 \log_e a]^2}{2!} + \dots \text{ad inf} \right] \\ &= a^{x-t} e^{-(x-t) \log_e a} = a^{x-t} a^{-(x-t)}. \quad [\because e^{m \log_e n} = n^m] \\ \therefore R(x, t; \lambda) &= 1. \quad \dots (12) \end{aligned}$$

Hence the required solution is

$$\begin{aligned} y(x) &= F(x) + \lambda \int_0^x R(x, t; \lambda) F(t) dt = f'(x) - \log_e a \int_0^x f'(t) dt, \quad \text{using (12)} \\ &= f'(x) - \log_e a [f(t)]_0^x = f'(x) - \log_e a [f(x) - f(0)] \\ \therefore y(x) &= f'(x) - f(x) \log_e a, \quad \text{by (2).} \end{aligned}$$

EXERCISE 5F

Solve the following Volterra integral equations of the first kind (1 to 5) by first reducing them to Volterra integral equations of the second kind :

1. $\int_0^x 3^{x-t} y(t) dt = x.$ **Ans.** $y(x) = 1 - x \log_e 3.$
2. $\int_0^x (1 - x^2 + t^2) y(t) dt = \frac{x^3}{2}.$ **Ans.** $y(x) = xe^{x^2}.$
3. $\int_0^x (2 + x^2 - t^2) y(t) dt = x^2.$ **Ans.** $y(x) = xe^{-x^2/2}.$
4. $\int_0^x \sin(x-t) y(t) dt = e^{x^2/2} - 1.$ **Ans.** $y(x) = e^{x^2/2}(x+2) - 1.$
5. $\int_0^x e^{x-t} y(t) dt = x$ [Kanpur 2006, 11; Meerut 2006, 07, 08, 11] **Ans.** $y(x) = 1 - x.$
6. Change the following Volterra integral equation of first kind into integral equation of second kind :
 $\int_0^x \cos(x-t) u(t) dt = x$ [Meerut 2008] **Ans.** $u(x) = 1 + \int_0^x \sin(x-t) u(t) dt.$

CHAPTER 6

Classical Fredholm Theory

6.1. INTRODUCTION.

In chapter 5, we obtained the solution of the Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt \quad \dots (1)$$

as a uniformly convergent power series in the parameter λ for $|\lambda|$ suitably small. Fredholm derived the solution of (1) in general form which is valid for all values of the parameter λ . He gave three important results which are known as Fredholm's first, second and third fundamental theorems. In the present chapter we propose to discuss these theorems.

6.2. FREDHOLM'S FIRST FUNDAMENTAL THEOREM.

[Meerut 2004]

The non-homogeneous Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt,$$

where the functions $f(x)$ and $y(t)$ are integrable, has a unique solution

$$y(x) = f(x) + \lambda \int_a^b R(x, t; \lambda) f(t) dt, \quad \dots (2)$$

where the resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda) \quad \dots (3)$$

with $D(\lambda) \neq 0$, is a meromorphic function of the complex variable λ , being the ratio of two entire functions defined by the series

$$D(x, t; \lambda) = K(x, t) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int \dots \int K \left(\begin{matrix} x, & z_1, \dots, z_p \\ t, & z_1, \dots, z_p \end{matrix} \right) dz_1 \dots dz_p \quad \dots (4)$$

$$\text{and} \quad D(\lambda) = 1 + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int \dots \int K \left(\begin{matrix} z_1, \dots, z_p \\ z_1, \dots, z_p \end{matrix} \right) dz_1 \dots dz_p, \quad \dots (5)$$

both of which converge for all values of λ . Also, note the following symbol for the determinant formed by the values of the kernel at all points (x_i, t_i)

$$\begin{vmatrix} K(x_1, t_1) & K(x_1, t_2) & \dots & K(x_1, t_n) \\ K(x_2, t_1) & K(x_2, t_2) & \dots & K(x_2, t_n) \\ \vdots & \vdots & \dots & \vdots \\ K(x_n, t_1) & K(x_n, t_2) & \dots & K(x_n, t_n) \end{vmatrix} = K \left(\begin{matrix} x_1, x_2, \dots, x_n \\ t_1, t_2, \dots, t_n \end{matrix} \right) \quad \dots (6)$$

which is known as the Fredholm determinant.

In particular, the solution of the Fredholm homogeneous equation

$$y(x) = \lambda \int_a^b K(x, t) y(t) dt \quad \dots (7)$$

is identically zero.

Proof. We divide the interval (a, b) into n equal parts,

$$x_1 = t_1 = a, \quad x_2 = t_2 = a + h, \quad \dots, \quad x_n = t_n = a + (n-1)h, \quad \dots (8)$$

where $h = (b - a) / n$. Thus, we get the approximate formula

$$\int_a^b K(x, t) y(t) dt \approx h \sum_{j=1}^n K(x, x_j) y(x_j). \quad \dots (9)$$

$$\text{Hence (1) reduces to} \quad y(x) \approx f(x) + \lambda h \sum_{j=1}^n K(x, x_j) y(x_j), \quad \dots (10)$$

which must hold for all values of x in the interval (a, b) . Using (10) at the n points of division x_i , $i = 1, 2, \dots, n$, we arrive at the system of equations

$$y(x_i) = f(x_i) + \lambda h \sum_{j=1}^n K(x_i, x_j) y(x_j), \quad i = 1, 2, \dots, n. \quad \dots (11)$$

Let us introduce the following symbols :

$$y(x_i) = y_i, \quad f(x_i) = f_i, \quad K(x_i, x_j) = K_{ij}. \quad \dots (12)$$

Then (11) gives an approximation for (1) in terms of the system of n linear equations

$$y_i - \lambda h \sum_{j=1}^n K_{ij} y_j = f_i, \quad i = 1, 2, \dots, n \quad \dots (13)$$

which contains n unknown quantities $y_1, y_2, \dots, y_n$.

Re-writing (13), we have

$$\left. \begin{aligned} (1 - \lambda h K_{11})y_1 - \lambda h K_{12}y_2 - \dots - \lambda h K_{1n}y_n &= f_1 \\ -\lambda h K_{21}y_1 + (1 - \lambda h K_{22})y_2 - \dots - \lambda h K_{2n}y_n &= f_2 \\ \dots &\dots \dots \dots \\ -\lambda h K_{n1}y_1 - \lambda h K_{n2}y_2 - \dots + (1 - \lambda h K_{nn})y_n &= f_n \end{aligned} \right\} \quad \dots (13)'$$

$$\text{Let} \quad D_n(\lambda) = \begin{vmatrix} 1 - \lambda h K_{11} & -\lambda h K_{12} & \dots & -\lambda h K_{1n} \\ -\lambda h K_{21} & 1 - \lambda h K_{22} & \dots & -\lambda h K_{2n} \\ \vdots & \vdots & \dots & \vdots \\ -\lambda h K_{n1} & -\lambda h K_{n2} & \dots & 1 - \lambda h K_{nn} \end{vmatrix} \quad \dots (14)$$

The solutions $y_1, y_2, \dots, y_n$ of the system of equations (13) or (13)' can be obtained as the ratios of certain determinants, with the determinant $D_n(\lambda)$ as the denominator provided $D_n(\lambda) \neq 0$.

We now expand the determinant $D_n(\lambda)$ in powers of the quantity $(-\lambda h)$. Clearly, the constant term is unity. The term containing $(-\lambda h)$ in the first power is the sum of all the determinants containing only one column $-\lambda h K_{\mu\nu}$, $\mu = 1, 2, \dots, n$. Taking the contribution from all the columns $\nu = 1, 2, \dots, n$, we find that the total contribution is

$$-\lambda h \sum_{\nu=1}^n K_{\nu\nu}.$$

The term containing $(-\lambda h)^2$ is the sum of all the determinants containing two columns with that factor. This gives rise to the determinants of the form

$$(-\lambda h)^2 \begin{vmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{vmatrix},$$

where (p, q) is an arbitrary pair of integers taken from the sequence $1, 2, \dots, n$ with $p < q$.

Next, the term containing the $(-\lambda h)^3$ is the sum of the determinants of the form

$$(-\lambda h)^3 \begin{vmatrix} K_{pp} & K_{pq} & K_{pr} \\ K_{qp} & K_{qq} & K_{qr} \\ K_{rp} & K_{rq} & K_{rr} \end{vmatrix},$$

where (p, q, r) is an arbitrary triplet of integers taken from the sequence $1, 2, \dots, n$ with $p < q < r$.

Proceeding likewise we may obtain the remaining terms in the expansion of $D_n(\lambda)$. This leads of the following expansion of the $D_n(\lambda)$:

$$\begin{aligned} D_n(\lambda) = & 1 - \lambda h \sum_{\nu=1}^n K_{\nu\nu} + \frac{(-\lambda h)^2}{2!} \sum_{p,q=1}^n \begin{vmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{vmatrix} + \frac{(-\lambda h)^3}{3!} \sum_{p,q,r=1}^n \begin{vmatrix} K_{pp} & K_{pq} & K_{pr} \\ K_{qp} & K_{qq} & K_{qr} \\ K_{rp} & K_{rq} & K_{rr} \end{vmatrix} + \dots \\ & \dots + \frac{(-\lambda h)^n}{n!} \sum_{p_1, p_2, \dots, p_n=1}^n \begin{vmatrix} K_{p_1 p_1} & K_{p_1 p_2} & \dots & K_{p_1 p_n} \\ K_{p_2 p_1} & K_{p_2 p_2} & \dots & K_{p_2 p_n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K_{p_n p_1} & K_{p_n p_2} & \dots & K_{p_n p_n} \end{vmatrix}. \quad \dots \quad (15) \end{aligned}$$

Using the symbol (6), (15) may be written as

$$D_n(\lambda) = 1 - \lambda h \sum_{p=1}^n K(x_p, x_p) + \frac{(-\lambda h)^2}{2!} \sum_{p,q=1}^n K \begin{pmatrix} x_p, x_q \\ x_p, x_q \end{pmatrix} + \frac{(-\lambda h)^3}{3!} \sum_{p,q,r=1}^n K \begin{pmatrix} x_p, x_q, x_r \\ x_p, x_q, x_r \end{pmatrix} + \dots \dots (16)$$

Since $h = (b-a)/n$, $n \rightarrow \infty \Rightarrow h \rightarrow 0$ and each term of the sum (16) tends to some single, double, triple integral etc. We thus obtain

$$D(\lambda) = 1 - \lambda \int_a^b K(x, x) dx + \frac{\lambda^2}{2!} \int_a^b \int_a^b K \begin{pmatrix} x_1, x_2 \\ x_1, x_2 \end{pmatrix} dx_1 dx_2 - \frac{\lambda^3}{3!} \int_a^b \int_a^b \int_a^b K \begin{pmatrix} x_1, x_2, x_3 \\ x_1, x_2, x_3 \end{pmatrix} dx_1 dx_2 dx_3 + \dots \quad (17)$$

(17) is known as the *Fredholm's first series*.

Hilbert has shown that the sequence $D_n(\lambda) \rightarrow D(\lambda)$ in the limit. Again Fredholm proved the convergence of the series (17) for all values of λ by using the fact that the kernel $K(x, t)$ is bounded and integrable function. Thus, $D(\lambda)$ is an entire function of the complex variable.

If $R(x, t; \lambda)$ be the resolvent kernel, then we wish to find the solution of (1) in the form

$$y(x) = f(x) + \lambda \int_a^b R(x, t; \lambda) f(t) dt, \quad (18)$$

where we expect $R(x, t; \lambda)$ to be the quotient

$$R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda) \quad (19)$$

where $D(x, t; \lambda)$ is the sum of certain functional series and is yet to be determined. We know that the resolvent kernel $R(x, t; \lambda)$ satisfies the following relation :

$$R(x, t; \lambda) = K(x, t) + \lambda \int_a^b K(x, z) R(z, t; \lambda) dz. \quad (20)$$

Using (19), (20) may be re-written as

$$\frac{D(x, t; \lambda)}{D(\lambda)} = K(x, t) + \lambda \int_a^b K(x, z) \frac{D(z, t; \lambda)}{D(\lambda)} dz$$

or

$$D(x, t; \lambda) = K(x, t) D(\lambda) + \lambda \int_a^b K(x, z) D(z, t; \lambda) dz. \quad (21)$$

From the form of the series (17) for $D(\lambda)$, it follows that we expect the solution of (21) in the form of a power series in the parameter λ :

$$D(x, t; \lambda) = B_0(x, t) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} B_p(x, t). \quad (22)$$

To this end, we re-write (17) as

$$D(\lambda) = 1 + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} C_p, \quad (23)$$

where

$$C_p = \int_a^b \dots \int_a^b K \begin{pmatrix} x_1, x_2, \dots, x_p \\ x_1, x_2, \dots, x_p \end{pmatrix} dx_1, \dots, dx_p. \quad (24)$$

Now, substituting the series for $D(x, t; \lambda)$ and $D(\lambda)$ from (22) and (23) in (21) and comparing the coefficients of equal powers of λ , leads us to the following important *recursion relations*

$$B_0(x, t) = K(x, t) \quad (25)$$

and

$$B_p(x, t) = C_p K(x, t) - p \int_a^b K(x, z) B_{p-1}(z, t) dz. \quad (26)$$

Now, we propose to prove that for each p , ($p = 1, 2, 3, \dots$)

$$B_p(x, t) = \int_a^b \dots \int_a^b K \begin{pmatrix} x, z_1, z_2, \dots, z_p \\ t, z_1, z_2, \dots, z_p \end{pmatrix} dz_1 \dots dz_p. \quad \dots (27)$$

First, observe that for $p = 1$, (26) taken the form

$$\begin{aligned} B_1(x, t) &= C_1 K(x, t) - \int_a^b K(x, z) B_0(z, t) dz \\ &= K(x, t) \int_a^b K(z, z) dz - \int_a^b K(x, z) K(z, t) dz, \text{ using (24) and (25)} \\ &= \int_a^b K \begin{pmatrix} x & z \\ t & z \end{pmatrix} dz, \end{aligned} \quad \dots (28)$$

showing that (27) holds for $p = 1$.

Now, we shall show that (27) holds for all the integral values. To this end, we expand the determinant under the integral sign in the relation :

$$K \begin{pmatrix} x, z_1, z_2, \dots, z_p \\ t, z_1, z_2, \dots, z_p \end{pmatrix} = \begin{vmatrix} K(x, t) & K(x, z_1) & \dots & K(x, z_p) \\ K(z_1, t) & K(z_1, z_1) & \dots & K(z_1, z_p) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K(z_p, t) & K(z_p, z_1) & \dots & K(z_p, z_p) \\ \dots & \dots & \dots & \dots \end{vmatrix}, \quad \dots (29)$$

with respect to the elements of the given row, transposing in turn the first column one place to the right, integrating both sides, and using (24); proof of (27) follows by mathematical induction.

Using (22), (25) and (27), we arrive at the so called *Fredholm second series* :

$$D(x, t; \lambda) = K(x, t) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_a^b \dots \int_a^b K \begin{pmatrix} x, z_1, \dots, z_p \\ t, z_1, \dots, z_p \end{pmatrix} dz_1 \dots dz_p \quad \dots (30)$$

The series (30) converges for all values of λ .

In view of (23) and (30) we observe that both terms of the quotient (19) have been determined and hence the existence of a solution to (1) is established for a bounded and integrable kernel $K(x, t)$, provided that $D(\lambda) \neq 0$. Furthermore since the both terms of the quotient (19) are entire functions of the parameter λ , we conclude that $R(x, t; \lambda)$ must be a meromorphic function of λ (i.e., an analytic function whose singularities may only be the poles, which in the present case are zeros of the divisor $D(\lambda)$).

In the end we propose to show that the solution in the form obtained by Fredholm is unique and is given by (18). Before doing this, we find that the integral equation (20) satisfied by $R(x, t; \lambda)$ is valid for all values of λ for which $D(\lambda) \neq 0$. From chapter 5, we already know that (20) holds for $|\lambda| < B^{-1}$, where

$$B = \left[\int_a^b \int_a^b |K(x, t)|^2 dx dt \right]^{1/2}$$

Since both sides of (20) are thus found to be meromorphic, the result follows. To establish the uniqueness of the solution of (1), we assume that $y(x)$ is a solution of (1) provided that $D(\lambda) \neq 0$.

$$\text{Re-writing (1),} \quad y(z) = f(z) + \lambda \int_a^b K(z, t) y(t) dt. \quad \dots (31)$$

Multiplying both sides of (31) by $R(x, t; \lambda)$ and then integrating both sides w.r.t. 'z' from a to b , we get

$$\int_a^b R(x, z; \lambda) y(z) dz = \int_a^b R(x, z; \lambda) f(z) dz + \lambda \int_a^b \left[\int_a^b R(x, z; \lambda) K(z, t) dz \right] y(t) dt. \quad \dots (32)$$

$$\text{Using (20), we have} \quad R(x, t; \lambda) = K(x, t) + \lambda \int_a^b R(x, z; \lambda) K(z, t) dz$$

$$\text{or} \quad \lambda \int_a^b R(x, z; \lambda) K(z, t) dz = R(x, t; \lambda) - K(x, t). \quad \dots (33)$$

Using (33), (32) reduces to

$$\int_a^b R(x, z; \lambda) y(z) dz = \int_a^b R(x, z; \lambda) f(z) dz + \int_a^b [R(x, t; \lambda) - K(x, t)] y(t) dt$$

$$\text{or} \quad \int_a^b R(x, t; \lambda) y(t) dt = \int_a^b R(x, t; \lambda) f(t) dt + \int_a^b R(x, t; \lambda) y(t) dt - \int_a^b K(x, t) y(t) dt$$

$$\text{or} \quad \int_a^b K(x, t) y(t) dt = \int_a^b R(x, t; \lambda) f(t) dt \quad \dots (34)$$

$$\text{From (1),} \quad \int_a^b K(x, t) y(t) dt = \frac{y(x) - f(x)}{\lambda}. \quad \dots (35)$$

Using (35), (34) reduces to

$$\frac{y(x) - f(x)}{\lambda} = \int_a^b R(x, t; \lambda) f(t) dt$$

$$\text{or} \quad y(x) = f(x) = \lambda \int_a^b R(x, t; \lambda) f(t) dt,$$

but this form is unique.

In particular, the solution of (7) is identically zero.

6.3. SOLVED EXMAPLES BASED ON FREDHOLM'S FIRST FUNDAMENTAL THEOREM.

We shall use the following results :

$$\text{For the Fredholm integral equation} \quad y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt, \quad \dots (1)$$

$$\text{the resolvent kernel is given by} \quad R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda) \quad \dots (2)$$

$$\text{where} \quad D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t) \quad \dots (3)$$

$$\text{and} \quad D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m. \quad \dots (4)$$

where

$$B_n(x, t) = \underbrace{\int_a^b \dots \int_a^b}_n \begin{vmatrix} K(x, t) & K(x, z_1) & \dots & K(x, z_n) \\ K(z_1, t) & K(z_1, z_1) & \dots & K(z_1, z_n) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K(z_n, t) & K(z_n, z_1) & \dots & K(z_n, z_n) \end{vmatrix} dz_1 dz_2 \dots dz_n, \quad \dots (5)$$

and

$$C_n = \underbrace{\int_a^b \dots \int_a^b}_n \begin{vmatrix} K(z_1, z_1) & K(z_1, z_2) & \dots & K(z_1, z_n) \\ K(z_2, z_1) & K(z_2, z_2) & \dots & K(z_2, z_n) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K(z_n, z_1) & K(z_n, z_2) & \dots & K(z_n, z_n) \end{vmatrix} dz_1 dz_2 \dots dz_n. \quad \dots (6)$$

The function $D(x, t; \lambda)$ is called the *Fredholm minor* and $D(\lambda)$ is called the *Fredholm determinant*.

Results (5) and (6) have been written in these forms with help of results (6), (24) and (27) of Art. 6.2.

Alternative procedure of calculating $B_m(x, t)$ and C_m . The following results will be used .

$$C_0 = 1 \quad \dots (7)$$

$$C_p = \int_a^b B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (8)$$

$$B_0(x, t) = K(x, t). \quad \dots (9)$$

$$B_p(x, t) = C_p K(x, t) - p \int_a^b K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1. \quad \dots (10)$$

After getting $R(x, t; \lambda)$, the required solution is given by

$$y(x) = f(x) + \lambda \int_a^b R(x, t; \lambda) f(t) dt \quad \dots (11)$$

Result (8) follows easily from results (5) and (6)

Ex. 1. Using Fredholm determinants, find the resolvent kernel, when

$$K(x, t) = xe^t, \quad a = 0, \quad b = 1. \quad [Kanpur 2007, Meerut 2001]$$

Sol. First write down all equations (1) to (6) given in Art 6.3.

Here $K(x, t) = xe^t. \quad \dots (7)$

From (5), $B_1(x, t) = \int_0^1 \begin{vmatrix} K(x, t) & K(x, z_1) \\ K(z_1, t) & K(z_1, z_1) \end{vmatrix} dz_1 = \int_0^1 \begin{vmatrix} xe^t & xe^{z_1} \\ z_1 e^t & z_1 e^{z_1} \end{vmatrix} dz_1$, by (7)

$= 0$, since the determinant under the integral sign vanishes

$$B_2(x, t) = \int_0^1 \int_0^1 \begin{vmatrix} K(x, t) & K(x, z_1) & K(x, z_2) \\ K(z_1, t) & K(z_1, z_1) & K(z_1, z_2) \\ K(z_2, t) & K(z_2, z_1) & K(z_2, z_2) \end{vmatrix} dz_1 dz_2$$

$$= \int_0^1 \int_0^1 \begin{vmatrix} xe^t & xe^{\bar{z}_1} & xe^{\bar{z}_2} \\ z_1 e^t & z_1 e^{\bar{z}_1} & z_1 e^{\bar{z}_2} \\ z_2 e^t & z_2 e^{\bar{z}_1} & z_2 e^{\bar{z}_2} \end{vmatrix} dz_1 dz_2$$

= 0, since the determinant under the integral sign vanishes

Since $B_1(x, t) = B_2(x, t) = 0$, it follows that $B_n(x, t) = 0$, for $n \geq 1$.

Now, from (6), we have

$$C_1 = \int_0^1 K(z_1, z_1) dz_1 = \int_0^1 z_1 e^{\bar{z}_1} dz_1 = \left[z_1 e^{\bar{z}_1} \right]_0^1 - \int_0^1 e^{\bar{z}_1} dz_1 = e - \left[e^{\bar{z}_1} \right]_0^1 = e - (e - 1) = 1.$$

$$C_2 = \int_0^1 \int_0^1 \begin{vmatrix} K(z_1, z_1) & K(z_1, z_2) \\ K(z_2, z_1) & K(z_2, z_2) \end{vmatrix} dz_1 dz_2 = \int_0^1 \int_0^1 \begin{vmatrix} z_1 e^{\bar{z}_1} & z_1 e^{\bar{z}_2} \\ z_2 e^{\bar{z}_1} & z_2 e^{\bar{z}_2} \end{vmatrix} dz_1 dz_2$$

= 0, since the determinant under the integral sign vanishes.

It follows that

$$C_m = 0 \quad \text{for all } m \geq 2.$$

Now, (3) gives

$$\begin{aligned} D(x, t; \lambda) &= K(x, t) - \lambda B_1(x, t) + (\lambda^2 / 2!) \times B_2(x, t) - \dots \\ &= xe^t, \text{ on putting values of } K(x, t), B_1(x, t) \text{ etc.} \end{aligned}$$

and (4) gives

$$\begin{aligned} D(\lambda) &= 1 - \lambda C_1 + (\lambda^2 / 2!) \times C_2 - \dots \\ &= 1 - \lambda, \text{ on putting values of } C_1, C_2, \text{ etc.} \end{aligned}$$

Hence (2) yields

$$R(x, t; \lambda) = \frac{D(x, t; \lambda)}{D(\lambda)} = \frac{xe^t}{1 - \lambda}.$$

Alternative method. We shall use results (7), (8), (9) and (10) of Art. 6.3 to compute $R(x, t; \lambda)$ as follows. First write down these results for complete solution yourself.

Here

$$C_0 = 1, \quad \dots (i)$$

and

$$B_0(x, t) = K(x, t) = xe^t. \quad \dots (ii)$$

$$\text{From (8), } C_1 = \int_0^1 B_0(s, s) ds = \int_0^1 se^s ds = \left[se^s \right]_0^1 - \int_0^1 e^s ds = e - \left[e^s \right]_0^1 = e - (e - 1) = 1.$$

... (iii)

From (10),

$$\begin{aligned} B_1(x, t) &= C_1 K(x, t) - \int_0^1 K(x, z) B_0(z, t) dz \\ &= xe^t - \int_0^1 xe^z ze^t dz = xe^t - xe^t \int_0^1 ze^z dz = xe^t - (xe^t) \times 1, \text{ as above} \end{aligned}$$

$\therefore$

$$B_1(x, t) = 0. \quad \dots (iv)$$

Then, from (8),

$$C_2 = \int_0^1 B_1(s, s) ds = 0, \text{ by (iv)} \quad \dots (v)$$

From (10), $B_2(x, t) = C_2 K(x, t) - 2 \int_0^1 K(x, z) B_1(z, t) dz = 0$, by (iv) and (v)

$$\therefore B_m(x, t) = 0 \quad \text{for all } m \geq 1$$

and

$$C_m = 0 \quad \text{for all } m \geq 2.$$

Now proceed as before to determine $R(x, t; \lambda)$.

Important Observation. The reader will find that the above alternative method is a short cut. However, he should find the required quantities strictly in the following order :

$$C_0, B_0(x, t), C_1, B_1(x, t), C_2, B_2(x, t) \text{ and so on.}$$

Ex.2. (a) Using Fredholm determinant, find the resolvent kernel of the integral equation $y(x) = f(x) + \lambda \int_0^1 x e^t y(t) dt, (\lambda \neq 1)$ and hence solve it.

(b) Explain the method of Fredholm determinant for the solution of Fredholm integral equation and hence solve the integral equation

$$y(x) = e^{-x} + \lambda \int_{-1}^1 x e^t y(t) dt \quad [\text{Meerut 2001, 03}]$$

Sol. (a) Given $y(x) = f(x) + \lambda \int_0^1 x e^t y(t) dt.$ Here $K(x, t) = x e^t.$

Proceed as in solved Ex. 1 and obtain $R(x, t; \lambda) = (x e^t)/(1 - \lambda)$

Hence the required solution is $y(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t) dt$

or $y(x) = f(x) + \lambda \int_0^1 \frac{x e^t}{1 - \lambda} f(t) dt$ or $y(x) = f(x) + \frac{\lambda x}{1 - \lambda} \int_0^1 e^t f(t) dt.$

(b) For the first part, refer Art. 6.2. Proceed as in Ex. 1 to prove that $R(x, t; \lambda) = (x e^t)/(1 - \lambda), \lambda \neq 1.$ Hence the required solution is given by

$$y(x) = e^{-x} + \lambda \int_{-1}^1 R(x, t; \lambda) e^{-t} dt \quad \text{or} \quad y(x) = e^{-x} + \lambda \int_{-1}^1 \frac{x e^t}{1 - \lambda} e^{-t} dt$$

or $y(x) = e^{-x} + \frac{\lambda x}{1 - \lambda} \left[t \right]_{-1}^1$ or $y(x) = e^{-x} + \frac{2\lambda x}{1 - \lambda}, \text{ if } \lambda \neq 1$

Ex.3. Find the resolvent kernel and solution of $y(x) = f(x) + \lambda \int_0^1 (x+t) y(t) dt.$

Sol. Given $y(x) = f(x) + \lambda \int_0^1 (x+t) y(t) dt. \quad \dots (1)$

Comparing (1) with $y(x) = f(x) + \lambda \int_0^1 K(x, t) y(t) dt,$

here $K(x, t) = x + t. \quad \dots (2)$

The resolvent kernel $R(x, t; \lambda)$ is given by $R(x, t; \lambda) = D(x, t; \lambda)/D(\lambda), \quad \dots (3)$

where $D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t) \quad \dots (4)$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (5)$$

$$C_0 = 1, \quad \dots (6)$$

$$B_0(x, t) = K(x, t) = x + t, \text{ by (2)} \quad \dots(7)$$

$$C_p = \int_0^1 B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (8)$$

and

$$B_p(x, t) = C_p K(x, t) - p \int_0^1 K(x, z) B_{p-1}(z, t) dz. \quad \dots (9)$$

$$\text{From (8),} \quad C_1 = \int_0^1 B_0(s, s) ds = \int_0^1 2s ds = 2 \left[\frac{s^2}{2} \right]_0^1 = 1. \quad \dots (10)$$

$$\begin{aligned} \text{From (9),} \quad B_1(x, t) &= C_1 K(x, t) - \int_0^1 K(x, z) B_0(z, t) dz \\ &= x + t - \int_0^1 (x + z)(z + t) dz, \text{ by (2) and (7)} \\ &= x + t - \int_0^1 [z^2 + z(x, t) + xt] dz = x + t - \left[\frac{1}{3} z^3 + \frac{z^2}{2} (x + t) + xtz \right]_0^1 \\ &= x + t - \frac{1}{3} - \frac{1}{2} (x + t) - xt = \frac{1}{2} (x + t) - xt - \frac{1}{3}. \end{aligned} \quad \dots (11)$$

$$\text{From (8),} \quad C_2 = \int_0^1 B_1(s, s) ds = \int_0^1 \left[\frac{1}{2} (s + s) - s^2 - \frac{1}{3} \right] ds = \left[\frac{s^2}{2} - \frac{s^3}{3} - \frac{1}{3} s \right]_0^1$$

$$\text{Thus,} \quad C_2 = (1/2) - (1/3) - (1/3) = -(1/6) \quad \dots (12)$$

$$\begin{aligned} \text{From (9),} \quad B_2(x, t) &= C_2 K(x, t) - 2 \int_0^1 K(x, z) B_1(z, t) dz \\ &= -\frac{1}{6} (x + t) - 2 \int_0^1 (x + z) \left[\frac{1}{2} (z + t) - zt - \frac{1}{3} \right] dz, \text{ by (2), (11) and (12)} \\ &= -\frac{1}{6} (x + t) - 2 \int_0^1 (x + z) \left[z \left(\frac{1}{2} - t \right) + \frac{1}{2} t - \frac{1}{3} \right] dz \\ &= -\frac{1}{6} (x + t) - 2 \int_0^1 \left[z^2 \left(\frac{1}{2} - t \right) + z \left(\frac{1}{2} t - \frac{1}{3} + \frac{x}{2} - xt \right) + x \left(\frac{1}{2} t - \frac{1}{3} \right) \right] dz \\ &= -\frac{1}{6} (x + t) - 2 \left[\frac{z^3}{3} \left(\frac{1}{2} - t \right) + \frac{z^2}{2} \left(\frac{1}{2} t - \frac{1}{3} + \frac{x}{2} - xt \right) + xz \left(\frac{1}{2} t - \frac{1}{3} \right) \right]_0^1 \\ &= -\frac{1}{6} (x + t) - 2 \left[\frac{1}{3} \left(\frac{1}{2} - t \right) + \frac{1}{2} \left(\frac{1}{2} t - \frac{1}{3} + \frac{x}{2} - xt \right) + x \left(\frac{1}{2} t - \frac{1}{3} \right) \right] \\ &= 0, \text{ on simplification.} \end{aligned} \quad \dots (13)$$

Since $B_2(x, t) = 0$, it follows from (8) and (9) that

$$C_3 = C_4 = C_5 = \dots = 0 \quad \dots (14)$$

and

$$B_3(x, t) = B_4(x, t) = B_5(x, t) = \dots = 0. \quad \dots (15)$$

Using the above values of C_p and $B_p(x, t)$, from (4) and (5), we have

$$D(x, t; \lambda) = K(x, t) - \lambda B_1(x, t) = x + t - \lambda \left[\frac{1}{2} (x + t) - xt - \frac{1}{3} \right]$$

and
$$D(\lambda) = 1 - \lambda C_1 + \frac{\lambda^2}{2!} C_2 = 1 - \lambda - \frac{1}{12} \lambda^2.$$

Putting these values in (3), we have

$$R(x, t; \lambda) = \frac{x + t - \lambda \left\{ (1/2) \times (x + t) - xt - 1/3 \right\}}{1 - \lambda - (\lambda^2/12)}.$$

The required solution of (1) is given by $y(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t) dt$

or
$$y(x) = f(x) + \lambda \int_0^1 \frac{x + t - \lambda \left\{ (1/2) \times (x + t) - xt - 1/3 \right\}}{1 - \lambda - (\lambda^2/12)} f(t) dt$$

Ex. 4. Solve $y(x) = 1 + \int_0^1 (1 - 3xt) y(t) dt$.

Sol. Given
$$y(x) = 1 + \int_0^1 (1 - 3xt) y(t) dt \quad \dots (1)$$

Comparing (1) with
$$y(x) = f(x) + \lambda \int_0^1 K(x, t) y(t) dt,$$

here $\lambda = 1, \quad f(x) = 1 \quad \text{and} \quad K(x, t) = 1 - 3xt. \quad \dots (2)$

The resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda), \quad \dots (3)$$

where
$$D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (4)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (5)$$

$$C_0 = 1 \quad \dots (6)$$

$$B_0(x, t) = K(x, t) = 1 - 3xt. \quad \dots (7)$$

$$C_p = \int_0^1 B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (8)$$

and
$$B_p(x, t) = C_p K(x, t) - p \int_0^1 K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1. \quad \dots (9)$$

From (7) and (8),
$$C_1 = \int_0^1 B_0(s, s) ds = \int_0^1 (1 - 3s^2) ds, \quad = \left[s - s^3 \right]_0^1 = 0. \quad \dots (10)$$

From (9),
$$B_1(x, t) = C_1 K(x, t) - \int_0^1 K(x, z) B_0(z, t) dz$$

$$= - \int_0^1 (1 - 3xz) (1 - 3zt) dz, \quad \text{by (2), (7) and (10)}$$

$$= - \int_0^1 [9xtz^2 - 3z(x+t) + 1] dz = - \left[3xt z^3 - \frac{3}{2} z^2 (x+t) + z \right]_0^1$$

$$\therefore B_1(x, t) = -\{3xt - (3/2) \times (x+t) + 1\}. \quad \dots (11)$$

$$\begin{aligned} \text{From (8), } C_2 &= \int_0^1 B_1(s, s) ds = -\int_0^1 (3s^2 - 3s + 1) ds, \text{ by (11)} \\ &= -\left[s^3 - \frac{3}{2}s^2 + s\right]_0^1 = -\left[1 - \frac{3}{2} + 1\right] = -\frac{1}{2} \end{aligned} \quad \dots (12)$$

$$\begin{aligned} \text{From (9), } B_2(x, t) &= C_2 K(x, t) - 2 \int_0^1 K(x, z) B_1(z, t) dz \\ &= -\frac{1}{2}(1-3xt) + 2 \int_0^1 (1-3xz) \left[3zt - \frac{3}{2}(z+t) + 1\right] dz, \text{ using (11) and (12)} \\ &= -\frac{1}{2}(1-3xt) + 2 \int_0^1 (1-3xz) \left\{3z \left(t - \frac{1}{2}\right) + 1 - \frac{3}{2}t\right\} dz \\ &= -\frac{1}{2}(1-3xt) + 2 \int_0^1 \left[-9x \left(t - \frac{1}{2}\right) z^2 + 3z \left(t - \frac{1}{2} - x + \frac{3}{2}xt\right) + 1 - \frac{3}{2}t\right] dz \\ &= \frac{1}{2}(1-3xt) + 2 \left[-3x \left(t - \frac{1}{2}\right) z^3 + \frac{3}{2} z^2 \left(t - \frac{1}{2} - x + \frac{3}{2}xt\right) + \left(1 - \frac{3}{2}t\right) z\right]_0^1 \\ &= \frac{1}{2}(1-3xt) + 2 \left[-3x \left(t - \frac{1}{2}\right) + \frac{3}{2} \left(t - \frac{1}{2} - x + \frac{3}{2}xt\right) + 1 - \frac{3}{2}t\right] \\ &= 0, \text{ on simplification.} \end{aligned} \quad \dots (13)$$

Since $B_2(x, t)$ vanishes, it follows that

$$B_p(x, t) = 0 \text{ for } p \geq 3 \quad \dots (14)$$

and

$$C_p = 0 \text{ for } p \geq 3. \quad \dots (15)$$

Putting the above values in (4) and (5), we have

$$\begin{aligned} D(x, t; \lambda) &= K(x, t) - B_1(x, t) \quad [\because \text{ here } \lambda = 1] \\ &= 1 - 3xt + 3xt - (3/2) \times (x+t) + 1 = 2 - (3/2) \times (x+t) \end{aligned} \quad \dots (16)$$

and

$$D(\lambda) = 1 - C_1 + (1/2!) \times C_2 \quad [\because \lambda = 1]$$

$$\therefore D(\lambda) = 1 - (1/4) = 3/4 \quad \dots (17)$$

$$\therefore \text{ From (3), } R(x, t; \lambda) = \frac{D(x, t; \lambda)}{D(\lambda)} = \frac{2 - (3/2) \times (x+t)}{(3/4)} = \frac{2}{3} [4 - 3(x+t)]. \quad \dots (18)$$

The required solution is given by

$$y(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t) dt$$

or

$$y(x) = 1 + \int_0^1 \frac{2}{3} [4 - 3(x+t)] dt, \text{ using (2) and (18)}$$

or

$$y(x) = 1 + \frac{2}{3} \left[4t - 3xt - \frac{3t^2}{2} \right]_0^1 = 1 + \frac{2}{3} \left[4 - 3x - \frac{3}{2} \right] = \frac{8-6x}{3}.$$

Ex. 5. Determine $D(\lambda)$ and $D(x, t; \lambda)$ and hence solve the integral equation

$$y(x) = e^x + \lambda \int_0^{10} xt y(t) dt. \quad [\text{Kanpur 2005, 08}]$$

Sol. Given

$$y(x) = e^x + \lambda \int_0^{10} xt y(t) dt.$$

$$\text{Comparing (1) with} \quad y(x) = f(x) + \lambda \int_0^{10} K(x, t) y(t) dt,$$

$$\text{we have} \quad f(x) = e^x \quad \text{and} \quad K(x, t) = xt \quad \dots (2)$$

The resolvent kernel is given by

$$R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda), \quad \dots (3)$$

$$\text{where} \quad D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (4)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (5)$$

$$C_0 = 1, \quad \dots (6)$$

$$B_0(x, t) = K(x, t) = xt, \text{ by (2)}, \quad \dots (7)$$

$$C_p = \int_0^{10} B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (8)$$

$$\text{and} \quad B_p(x, t) = C_p K(x, t) - p \int_0^{10} K(x, z) K_{p-1}(z, t) dz, \quad p \geq 1. \quad \dots (9)$$

$$\text{From (7) and (8)} \quad C_1 = \int_0^{10} B_0(s, s) ds = \int_0^{10} s^2 ds = \left[\frac{s^3}{3} \right]_0^{10} = \frac{10^3}{3}. \quad \dots (10)$$

$$\begin{aligned} \text{From (9),} \quad B_1(x, t) &= C_1 K(x, t) - \int_0^{10} K(x, z) B_0(z, t) dz \\ &= \frac{10^3}{3} xt - \int_0^{10} (xz)(zt) dz, \text{ by (2), (7) and (10)} \\ &= \frac{10^3}{3} xt - xt \left[\frac{z^3}{3} \right]_0^{10} = 0. \end{aligned} \quad \dots (11)$$

Since $B_1(x, t) = 0$, from (8) and (9), it follows that

$$C_p = 0 \quad \text{for} \quad p \geq 2, \quad \dots (12)$$

$$\text{and} \quad B_p(x, t) = 0 \quad \text{for} \quad p \geq 2. \quad \dots (13)$$

Putting the above values in (4) and (5), we have

$$D(x, t; \lambda) = K(x, t) - \lambda B_1(x, t) + (\lambda^2 / 2!) \times B_2(x, t) - \dots = xt,$$

$$\text{and} \quad D(\lambda) = 1 - \lambda C_1 + (\lambda^2 / 2!) \times C_2 - \dots = 1 - \lambda \times (10^3 / 3).$$

$$\therefore \text{ From (3),} \quad R(x, t; \lambda) = \frac{xt}{1 - \lambda \times (10^3 / 3)}. \quad \dots (14)$$

The required solution of (1) is given by $y(x) = f(x) + \lambda \int_0^{10} R(x, t; \lambda) f(t) dt$

or

$$\begin{aligned} y(x) &= e^x + \lambda \int_0^{10} \frac{xt}{1 - \lambda \times (10^3 / 3)} e^t dt, \text{ using (2) and (14)} \\ &= e^x + \frac{\lambda x}{1 - \lambda \times (10^3 / 3)} \int_0^{10} t e^t dt \\ &= e^x + \frac{\lambda x}{1 - \lambda \times (10^3 / 3)} \left[\left\{ t e^t \right\}_0^{10} - \int_0^{10} e^t dt \right], \text{ integrating by parts} \\ &= e^x + \frac{3\lambda x}{3 - 10^3 \lambda} \left\{ 10e^{10} - \left[e^t \right]_0^{10} \right\} = e^x + \frac{3\lambda x}{3 - 10^3 \lambda} (10e^{10} - e^{10} + 1) \end{aligned}$$

$$\therefore y(x) = e^x + \frac{3\lambda x}{3 - 10^3 \lambda} (1 + 9e^{10}).$$

Ex.6. Solve $y(x) = \sin x + \lambda \int_4^{10} x y(t) dt$.

Sol. Given $y(x) = \sin x + \lambda \int_4^{10} x y(t) dt$ (1)

Comparing (1) with $y(x) = f(x) + \lambda \int_4^{10} x y(t) dt$, we have

$$f(x) = \sin x \quad \text{and} \quad K(x, t) = x. \quad \dots (2)$$

The resolvent kernel is given by $R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda)$, ... (3)

where $D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t)$, ... (4)

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (5)$$

$$C_0 = 1, \quad \dots (6)$$

$$B_0(x, t) = K(x, t) = x, \text{ by (2)} \quad \dots (7)$$

$$C_p = \int_4^{10} B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (8)$$

and $B_p(x, t) = C_p K(x, t) - p \int_4^{10} K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1$... (9)

From (7) and (8), $C_1 = \int_4^{10} B_0(s, s) ds = \int_4^{10} s ds = \left[\frac{s^2}{2} \right]_4^{10} = \frac{1}{2} (10^2 - 4^2) = 42$ (10)

From (9), $B_1(x, t) = C_1 K(x, t) - \int_4^{10} K(x, z) K_0(z, t) dz$

$$\begin{aligned} &= 42x - \int_4^{10} xz dz, \text{ by (2), (7) and (10)} \\ &= 42x - x \left[\frac{z^2}{2} \right]_4^{10} = 42x - 42x = 0. \end{aligned} \quad \dots (11)$$

Since $B_1(x, t) = 0$, (8) and (9) show that

$$C_p = 0 \quad \text{for} \quad p \geq 2, \quad \dots (12)$$

and $B_p(x, t) = 0 \quad \text{for} \quad p \geq 2. \quad \dots (13)$

Putting the above values in (4) and (5), we have

$$D(x, t; \lambda) = K(x, t) - \lambda B_1(x, t) + (\lambda^2 / 2!) \times B_2(x, t) - \dots = x$$

and $D(\lambda) = 1 - \lambda C_1 + (\lambda^2 / 2!) \times C_2 - \dots = 1 - 42\lambda.$

$$\therefore \text{From (3),} \quad R(x, t; \lambda) = x / (1 - 42\lambda). \quad \dots (14)$$

The required solution of (1) is given by $y(x) = f(x) + \lambda \int_4^{10} R(x, t; \lambda) f(t) dt$

or $y(x) = \sin x + \lambda \int_4^{10} \frac{x}{1 - 42\lambda} \sin t dt, \text{ by (2) and (14)}$

$$= \sin x + \frac{\lambda x}{1 - 42\lambda} [-\cos t]_4^{10} = \sin x + \frac{\lambda x}{1 - 42\lambda} [\cos 4 - \cos 10]$$

$$\therefore y(x) = \sin x + \frac{2\lambda x \sin 7 \sin 3}{1 - 42\lambda}$$

Ex.7. Determine $D(\lambda)$ and $D(x, t; \lambda)$ and hence solve the integral equation

$$y(x) = e^x + \lambda \int_0^1 2e^x e^t y(t) dt \quad [\text{Meerut 2002, 06, 08, 10}]$$

Sol. Given $y(x) = e^x + \lambda \int_0^1 2e^x e^t y(t) dt. \quad \dots (1)$

Comparing (1) with $y(x) = f(x) + \lambda \int_0^1 K(x, t) y(t) dt,$

here $f(x) = e^x$ and $K(x, t) = 2e^x e^t. \quad \dots (2)$

The resolvent kernel is given by $R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda), \quad \dots (3)$

where $D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (4)$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (5)$$

$$C_0 = 1, \quad \dots (6)$$

$$B_0(x, t) = K(x, t) = 2e^x e^t, \quad \dots (7)$$

$$C_p = \int_0^1 B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (8)$$

and $B_p(x, t) = C_p K(x, t) - p \int_0^1 K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1 \quad \dots (9)$

From (7) and (8), $C_1 = \int_0^1 B_0(s, s) ds = \int_0^1 2e^s e^s ds = 2[e^{2s} / 2]_0^1 = e^2 - 1. \quad \dots (10)$

$$\begin{aligned}
\text{From (9) } B_1(x, t) &= C_1 K(x, t) - \int_0^1 K(x, z) B_0(z, t) dz \\
&= 2(e^2 - 1) e^x e^t - \int_0^1 (2e^x e^z) (2e^z e^t) dz = 2(e^2 - 1) e^x e^t - 4e^x e^t \int_0^1 e^{2z} dz \\
&= 2(e^2 - 1) e^x e^t - 4e^x e^t \left[\frac{e^{2z}}{2} \right]_0^1 = 2(e^2 - 1) e^x e^t - 2e^x e^t (e^2 - 1) = 0. \quad \dots (11)
\end{aligned}$$

Since $B_1(x, t) = 0$, (8) and (9) show that

$$C_p = 0 \quad \text{for} \quad p \geq 2. \quad \dots (12)$$

and

$$B_p(x, t) = 0 \quad \text{for} \quad p \geq 2. \quad \dots (13)$$

Putting the above values in (4) and (5), we have

$$D(x, t; \lambda) = K(x, t) - \lambda B_1(x, t) + (\lambda^2 / 2!) \times B_2(x, t) - \dots = 2e^x e^t.$$

and

$$D(\lambda) = 1 - \lambda C_1 + (\lambda^2 / 2!) \times C_2 - \dots = 1 - \lambda (e^2 - 1)$$

$$\therefore \text{ From (3),} \quad R(x, t; \lambda) = \frac{2e^x e^t}{1 - \lambda (e^2 - 1)}. \quad \dots (14)$$

The required solution is given by $y(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t) dt$

$$\text{or} \quad y(x) = e^x + \lambda \int_0^1 \frac{2e^x e^t}{1 - \lambda (e^2 - 1)} e^t dt, \text{ by (2) and (14)}$$

$$\text{or} \quad = e^x + \frac{2\lambda e^x}{1 - \lambda (e^2 - 1)} \int_0^1 e^{2t} dt = e^x + \frac{2\lambda e^x}{1 - \lambda (e^2 - 1)} \left[\frac{e^{2t}}{2} \right]_0^1$$

$$\text{or} \quad y(x) = e^x + \frac{\lambda e^x (e^2 - 1)}{1 - \lambda (e^2 - 1)} = \frac{e^x}{1 - \lambda (e^2 - 1)}.$$

Ex. 8. Solve $y(x) = 1 + \lambda \int_0^\pi \sin(x+t) y(t) dt$.

$$\text{Sol. Given} \quad y(x) = 1 + \lambda \int_0^\pi \sin(x+t) y(t) dt. \quad \dots (1)$$

$$\text{Comparing (1) with} \quad y(x) = f(x) + \lambda \int_0^\pi K(x, t) y(t) dt,$$

$$\text{here } f(x) = 1 \quad \text{and} \quad K(x, t) = \sin(x+t). \quad \dots (2)$$

$$\text{The resolvent kernel is given by} \quad R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda), \quad \dots (3)$$

$$\text{where} \quad D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (4)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (5)$$

$$C_0 = 1, \quad \dots (6)$$

$$B_0(x, t) = K(x, t) = \sin(x + t), \quad \dots (7)$$

$$C_p = \int_0^\pi B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (8)$$

and

$$B_p(x, t) = C_p K(x, t) - p \int_0^\pi K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1 \quad \dots (9)$$

$$\text{From (7) and (8),} \quad C_1 = \int_0^\pi B_0(s, s) ds = \int_0^\pi \sin 2s ds = \left[-\frac{\cos 2s}{2} \right]_0^\pi = 0. \quad \dots (10)$$

$$\begin{aligned} \text{From (9), } B_1(x, t) &= C_1 K(x, t) - \int_0^\pi K(x, z) B_0(z, t) dz \\ &= - \int_0^\pi \sin(x + z) \sin(z + t) dz \quad \text{by (2), (7) and (10)} \\ &= - \frac{1}{2} \int_0^\pi \{ \cos(x - t) - \cos(x + t + 2z) \} dz = - \frac{1}{2} \left[z \cos(x - t) - \frac{1}{2} \sin(x + t + 2z) \right]_0^\pi \\ &= - \frac{1}{2} \left[\pi \cos(x - t) - \frac{1}{2} \sin(x + t) + \frac{1}{2} \sin(x + t) \right] = - \frac{\pi}{2} \cos(x - t). \quad \dots (11) \end{aligned}$$

$$\text{From (8) and (11),} \quad C_2 = \int_0^\pi B_1(s, s) ds = \int_0^\pi \left[-\frac{\pi}{2} \right] ds = -\frac{\pi}{2} [s]_0^\pi = -\frac{\pi^2}{2}. \quad \dots (12)$$

$$\begin{aligned} \text{From (9), } B_2(x, t) &= C_2 K(x, t) - 2 \int_0^\pi K(x, z) B_1(z, t) dz. \\ &= -\frac{\pi^2}{2} \sin(x + t) - 2 \int_0^\pi \sin(x + z) \left\{ -\frac{\pi}{2} \cos(z - t) \right\} dz, \quad \text{using (2), (11) and (12)} \\ &= -\frac{\pi^2}{2} \sin(x + t) + \frac{\pi}{2} \int_0^\pi \{ \sin(2z + x - t) + \sin(x + t) \} dz. \\ &= -\frac{\pi^2}{2} \sin(x + t) + \frac{\pi}{2} \left[-\frac{\cos(2z + x - t)}{2} + z \sin(x + t) \right]_0^\pi \\ &= -\frac{\pi^2}{2} \sin(x + t) + \frac{\pi}{2} \left[-\frac{1}{2} \cos(x - t) + \frac{1}{2} \cos(x - t) + \pi \sin(x + t) \right] \\ &\therefore B_2(x, t) = 0 \quad \dots (13) \end{aligned}$$

Since $B_2(x, t) = 0$, (8) and (9) show that

$$C_p = 0 \quad \text{for} \quad p \geq 3 \quad \dots (14)$$

and

$$B_p(x, t) = 0 \quad \text{for} \quad p \geq 3. \quad \dots (15)$$

Putting the above values in (4) and (5), we have

$$\begin{aligned} D(x, t; \lambda) &= K(x, t) - \lambda B_1(x, t) + \frac{\lambda^2}{2!} B_2(x, t) - \frac{\lambda^3}{3!} B_3(x, t) + \dots \\ &= \sin(x + t) + (\pi\lambda/2) \cos(x - t) \end{aligned}$$

and

$$D(\lambda) = 1 - \lambda C_1 + \frac{\lambda^2}{2!} C_2 - \frac{\lambda^3}{3!} C_3 + \dots = 1 - \frac{\pi^2 \lambda^2}{4}.$$

$$\therefore \text{ From (3), } R(x, t; \lambda) = \frac{\sin(x+t) + (\lambda\pi/2) \times \cos(x-t)}{1 - (\pi^2/4) \times \lambda^2} \quad \dots (16)$$

The required solution is given by $y(x) = f(x) + \lambda \int_0^\pi R(x, t; \lambda) f(t) dt$

$$\text{or } y(x) = 1 + \lambda \int_0^\pi \frac{\sin(x+t) + (\lambda\pi/2) \times \cos(x-t)}{1 - (\pi^2/4) \times \lambda^2} dt, \text{ using (2) and (16)}$$

$$\begin{aligned} &= 1 + \frac{4\lambda}{4 - \pi^2 \lambda^2} \left[-\cos(x+t) - \frac{\lambda\pi}{2} \sin(x-t) \right]_0^\pi \\ &= 1 + \frac{4\lambda}{4 - \pi^2 \lambda^2} \left[\cos x + \frac{1}{2} \pi \lambda \sin x + \cos x + \frac{1}{2} \pi \lambda \sin x \right] \end{aligned}$$

$$\text{or } y(x) = 1 + \frac{4\lambda}{4 - \pi^2 \lambda^2} (2 \cos x + \pi \lambda \sin x).$$

Ex. 9. Solve $y(x) = x + \lambda \int_0^1 \{xt + (xt)^{1/2}\} y(t) dt$.

$$\text{Sol. Given } y(x) = x + \lambda \int_0^1 \{xt + (xt)^{1/2}\} y(t) dt. \quad \dots (1)$$

$$\text{Comparing (1) with } y(x) = f(x) + \lambda \int_0^1 K(x, t) y(t) dt,$$

$$\text{here } f(x) = x \quad \text{and} \quad K(x, t) = xt + (xt)^{1/2}. \quad \dots (2)$$

$$\text{The resolvent kernel } R(x, t; \lambda) \text{ is given by } R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda), \quad \dots (3)$$

$$\text{where } D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (4)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (5)$$

$$C_0 = 1, \quad \dots (6)$$

$$B_0(x, t) = K(x, t) = xt + (xt)^{1/2}, \quad \dots (7)$$

$$C_p = \int_0^1 B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (8)$$

$$\text{and } B_p(x, t) = C_p K(x, t) - p \int_0^1 K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1. \quad \dots (9)$$

$$\text{From (7) and (8), } C_1 = \int_0^1 B_0(s, s) ds = \int_0^1 (s^2 + s) ds = \left[s^3/3 + s^2/2 \right]_0^1 = 5/6. \quad \dots (10)$$

$$\text{From (9), } B_1(x, t) = C_1 K(x, t) - \int_0^1 K(x, z) B_0(z, t) dz$$

$$= \frac{5}{6} \{xt + (xt)^{1/2}\} - \int_0^1 \{xz + (xz)^{1/2}\} \{zt + (zt)^{1/2}\} dz, \text{ by (2) and (7)}$$

$$\begin{aligned}
&= \frac{5}{6} \{xt + (xt)^{1/2}\} - \int_0^1 [xtz^2 + x\sqrt{t} z^{3/2} + t\sqrt{x} z^{3/2} + z (xt)^{1/2}] dz \\
&= \frac{5}{6} \{xt + (xt)^{1/2}\} - \left[\frac{xt z^3}{3} + (x\sqrt{t} + t\sqrt{x}) \frac{z^{5/2}}{(5/2)} + \frac{z^2}{2} (xt)^{1/2} \right]_0^1 \\
&= \frac{5}{6} \{xt + (xt)^{1/2}\} - \left[\frac{xt}{3} + \frac{2}{5} (x\sqrt{t} + t\sqrt{x}) + \frac{1}{2} (xt)^{1/2} \right] \\
&= \frac{1}{2} xt + \frac{1}{3} (xt)^{1/2} - \frac{2}{5} (x\sqrt{t} + t\sqrt{x}). \quad \dots (11)
\end{aligned}$$

From (8), $C_2 = \int_0^1 B_1(s, s) ds = \int_0^1 \left[\frac{1}{2} s^2 + \frac{1}{3} s - \frac{2}{5} (s\sqrt{s} + s\sqrt{s}) \right] ds$, by (11)

$$\begin{aligned}
&= \int_0^1 \left(\frac{1}{2} s^2 + \frac{1}{3} s - \frac{4}{5} s^{3/2} \right) ds = \left[\frac{s^3}{6} + \frac{s^2}{6} - \frac{4}{5} \times \frac{s^{5/2}}{(5/2)} \right]_0^1 = \frac{1}{6} + \frac{1}{6} - \frac{8}{25} \\
&= 1/75. \quad \dots (12)
\end{aligned}$$

From (9), $B_2(x, t) = C_2 K(x, t) - 2 \int_0^1 K(x, z) B_1(z, t) dz$

$$\begin{aligned}
&= \frac{1}{75} \{xt + (xt)^{1/2}\} - 2 \int_0^1 \{xz + (xz)^{1/2}\} \left\{ \frac{1}{2} zt + \frac{1}{3} (zt)^{1/2} - \frac{2}{5} (z\sqrt{t} + t\sqrt{z}) \right\} dz \\
&\quad \text{[using (2), (11) and (12)]} \\
&= \frac{1}{75} \{xt + (xt)^{1/2}\} - 2 \int_0^1 \left\{ \frac{1}{2} xt z^2 + \frac{x\sqrt{t}}{3} z^{3/2} - \frac{2x\sqrt{t}}{5} z^2 - \frac{2xt}{5} z^{3/2} \right. \\
&\quad \left. + \frac{1}{2} t\sqrt{x} z^{3/2} + \frac{1}{3} z(xt)^{1/2} - \frac{2}{5} (xt)^{1/2} z^{3/2} - \frac{2}{5} t\sqrt{x} z \right\} dz \\
&= \frac{1}{75} \{xt + (xt)^{1/2}\} - 2 \left[\frac{xt z^3}{6} + \frac{2x\sqrt{t} z^{5/2}}{15} - \frac{2x\sqrt{t} z^3}{15} - \frac{4xt z^{5/2}}{25} \right. \\
&\quad \left. + \frac{t\sqrt{x} z^{5/2}}{5} + \frac{(xt)^{1/2} z^2}{6} - \frac{4(xt)^{1/2} z^{5/2}}{25} - \frac{t\sqrt{x} z^2}{5} \right]_0^1 \\
&= \frac{1}{75} \{xt + (xt)^{1/2}\} - 2 \left\{ \frac{xt}{6} + \frac{2x\sqrt{t}}{15} - \frac{2x\sqrt{t}}{15} - \frac{4xt}{25} + \frac{t\sqrt{x}}{5} + \frac{(xt)^{1/2}}{6} \right. \\
&\quad \left. - \frac{4}{25} (xt)^{1/2} - \frac{t\sqrt{x}}{5} \right\} = 0 \quad \dots (13)
\end{aligned}$$

Since $B_2(x, t) = 0$, (8) and (9) show that

$$C_p = 0 \quad \text{for} \quad p \geq 3 \quad \dots (14)$$

and $B_p(x, t) = 0 \quad \text{for} \quad p \geq 3. \quad \dots (15)$

Putting the above values in (4) and (5), we get

$$D(x, t; \lambda) = K(x, t) - \lambda B_1(x, t) + (\lambda^2 / 2!) \times B_2(x, t) - \dots$$

$$= xt + (xt)^{1/2} - \lambda \left\{ \frac{1}{2} xt + \frac{1}{3} (xt)^{1/2} - \frac{2}{5} (x\sqrt{t} + t\sqrt{x}) \right\}$$

and
$$D(\lambda) = 1 - \lambda C_1 + \frac{\lambda^2}{2!} C_2 - \dots = 1 - \frac{5\lambda}{6} + \frac{\lambda^2}{150}.$$

$\therefore$

from (3),
$$R(x, t; \lambda) = \frac{xt + (xt)^{1/2} - \lambda \left\{ (1/2) \times xt + (1/3) \times (xt)^{1/2} - (2/5) \times (x\sqrt{t} + t\sqrt{x}) \right\}}{1 - (5/6) \times \lambda + (1/150) \times \lambda^2}$$
 ... (16)

Hence the required solution of (1) is
$$y(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t) dt$$

or
$$y(x) = x + \lambda \int_0^1 \frac{xt + (xt)^{1/2} - \lambda \left\{ (1/2) \times xt + (1/3) \times (xt)^{1/2} - (2/5) \times (x\sqrt{t} + t\sqrt{x}) \right\}}{1 - (5/6) \times \lambda + (1/150) \times \lambda^2} t dt$$
 [using (2) and (16)]

$$= x + \frac{\lambda}{1 - (5/6) \times \lambda + (1/150) \times \lambda^2} \left[\frac{xt^3}{3} + \frac{2\sqrt{x} t^{5/2}}{5} - \frac{\lambda xt^3}{6} - \frac{2\sqrt{x} \lambda t^{5/2}}{15} - \frac{4x \lambda t^{5/2}}{25} - \frac{2\sqrt{x} \lambda t^3}{15} \right]_0^1$$

$$= x + \frac{\lambda}{1 - (5/6) \times \lambda + (1/150) \times \lambda^2} \left[\frac{x}{3} + \frac{2\sqrt{x}}{5} - \frac{\lambda x}{6} - \frac{2\lambda \sqrt{x}}{15} - \frac{4x \lambda}{25} - \frac{2\lambda \sqrt{x}}{15} \right]$$

Thus,
$$y(x) = \frac{150x + \lambda(60\sqrt{x} - 75x) + 21x\lambda^2}{\lambda^2 - 125\lambda + 150}, \text{ on simplification.}$$

Ex.10. Using Fredholm determinants, find the resolvent kernel of the following kernels :

(i) $K(x, t) = 2x - t, 0 \leq x \leq 1, \quad 0 \leq t \leq 1.$

(ii) $K(x, t) = x^2 t - xt^2, 0 \leq x \leq 1, \quad 0 \leq t \leq 1.$

(iii) $K(x, t) = 1 + 3xt, 0 \leq x \leq 1, \quad 0 \leq t \leq 1.$ [Kanpur 2006]

(iv) $K(x, t) = \sin x \cos t, 0 \leq x \leq 2\pi, \quad 0 \leq t \leq 2\pi.$

(v) $K(x, t) = \sin x - \sin t, 0 \leq x \leq 2\pi, \quad 0 \leq t \leq 2\pi.$

Sol. (i) Here
$$K(x, t) = 2x - t \quad \dots(1)$$

$$\begin{aligned} \therefore B_1(x, t) &= \int_0^1 \begin{vmatrix} K(x, t) & K(x, z_1) \\ K(z_1, t) & K(z_1, z_1) \end{vmatrix} dz_1 = \int_0^1 \begin{vmatrix} 2x - t & 2x - z_1 \\ 2z_1 - t & 2z_1 - z_1 \end{vmatrix} dz_1 \\ &= \int_0^1 [z_1(2x - t) - (2x - z_1)(2z_1 - t)] dz_1 = \int_0^1 [z_1(2x - t) - (4xz_1 - 2xt - 2z_1^2 + z_1 t)] dz_1 \\ &= \int_0^1 [2z_1^2 + z_1(2x - t - 4x - t) + 2xt] dz_1 = \left[\frac{2z_1^3}{3} + \frac{z_1^2}{2}(-2x - t) + 2xt z_1 \right]_0^1 \end{aligned}$$

$\therefore B_1(x, t) = (2/3) - x - t + 2xt. \quad \dots(2)$

$$\begin{aligned} B_2(x, t) &= \int_0^1 \int_0^1 \begin{vmatrix} K(x, t) & K(x, z_1) & K(x, z_2) \\ K(z_1, t) & K(z_1, z_1) & K(z_1, z_2) \\ K(z_2, t) & K(z_2, z_1) & K(z_2, z_2) \end{vmatrix} dz_1 dz_2 = \int_0^1 \int_0^1 \begin{vmatrix} 2x - t & 2x - z_1 & 2x - z_2 \\ 2z_1 - t & 2z_1 - z_1 & 2z_1 - z_2 \\ 2z_2 - t & 2z_2 - z_1 & 2z_2 - z_2 \end{vmatrix} dz_1 dz_2 \\ &= 0 \text{ (simplify yourself)} \end{aligned}$$

Hence $B_p(x, t) = 0$ for $p \geq 3$ (3)

$$C_1 = \int_0^1 K(z_1, z_1) dz_1 = \int_0^1 (2z_1 - z_1) dz_1 = \left[\frac{1}{2} z_1^2 \right]_0^1 = \frac{1}{2}. \quad \dots (4)$$

$$C_2 = \int_0^1 \int_0^1 \begin{vmatrix} K(z_1, z_1) & K(z_1, z_2) \\ K(z_2, z_1) & K(z_2, z_2) \end{vmatrix} dz_1 dz_2 = \int_0^1 \int_0^1 \begin{vmatrix} 2z_1 - z_1 & 2z_1 - z_2 \\ 2z_2 - z_1 & 2z_2 - z_2 \end{vmatrix} dz_1 dz_2 = \frac{1}{3}, \quad \dots (5)$$

[on simplification]

$$C_3 = \int_0^1 \int_0^1 \int_0^1 \begin{vmatrix} K(z_1, z_1) & K(z_1, z_2) & K(z_1, z_3) \\ K(z_2, z_1) & K(z_2, z_2) & K(z_2, z_3) \\ K(z_3, z_1) & K(z_3, z_2) & K(z_3, z_3) \end{vmatrix} dz_1 dz_2 dz_3$$

$$= \int_0^1 \int_0^1 \int_0^1 \begin{vmatrix} 2z_1 - z_1 & 2z_1 - z_2 & 2z_1 - z_3 \\ 2z_2 - z_1 & 2z_2 - z_2 & 2z_2 - z_3 \\ 2z_3 - z_1 & 2z_3 - z_2 & 2z_3 - z_3 \end{vmatrix} dz_1 dz_2 dz_3 = 0, \text{ on simplification} \quad \dots (6)$$

Hence $C_p = 0$ for all $p \geq 4$

Now, $D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t) = K(x, t) - \lambda B_1(x, t)$

$$= 2x - t - \lambda(2/3 - x - t + 2xt)$$

and $D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m = 1 - \lambda C_1 + \frac{\lambda^2}{2!} C_2 = 1 - \frac{1}{2} \lambda - \frac{\lambda^2}{6}$

$$\therefore R(x, t; \lambda) = \frac{D(x, t; \lambda)}{D(\lambda)} = \frac{2x - t - \lambda(2/3 - x - t + 2xt)}{1 - \lambda/2 - \lambda^2/6}.$$

Part (ii) Hint. Student is adviced to complete the solution.

Here $K(x, t) = x^2 t - x t^2$.

$$B_1(x, t) = \int_0^1 \begin{vmatrix} x^2 t - x t^2 & x^2 z_1 - x z_1^2 \\ z_1^2 t - z_1 t^2 & z_1^2 z_1 - z_1 z_1^2 \end{vmatrix} dz_1 = -xt \left(\frac{x+t}{4} - \frac{xt}{3} - \frac{1}{5} \right), \text{ on simplification}$$

Also, $B_2(x, t) = 0$. (verify yourself)

$\therefore B_p(x, t) = 0$ for all $p \geq 3$.

Again, $C_1 = \int_0^1 K(z_1, z_1) dz_1 = \int_0^1 (z_1^3 - z_1^3) dz_1 = 0$.

$$C_2 = \int_0^1 \int_0^1 \begin{vmatrix} 0 & z_1^2 z_2 - z_1 z_2^2 \\ z_2^2 z_1 - z_2 z_1^2 & z_2^2 z_2 - z_2 z_2^2 \end{vmatrix} dz_1 dz_2 = \frac{1}{20}, \text{ on simplifying}$$

Also, $C_3 = 0$ (verify yourself)

$\therefore C_p = 0$ for all $p \geq 4$.

$$\therefore D(x, t; \lambda) = K(x, t) - \lambda B_1(x, t) + \dots = x^2 t - x t^2 + \lambda x t \left(\frac{x+t}{4} - \frac{x t}{3} - \frac{1}{5} \right)$$

and
$$D(\lambda) = 1 - \lambda C_1 + \frac{\lambda}{2!} C_2 - \dots = 1 - \frac{\lambda^2}{240}$$

$$\therefore R(x, t; \lambda) = \frac{D(x, t; \lambda)}{D(\lambda)} = \frac{x^2 t - x t^2 + \lambda x t \left(\frac{x+t}{4} - \frac{x t}{3} - \frac{1}{5} \right)}{1 - (\lambda^2 / 240)}$$

Part (iii) Hint. Student is adviced to complete the solution.

Here $K(x, t) = 1 + 3xt$... (1)

$$B_1(x, t) = \int_0^1 \left| \begin{array}{cc} 1+3xt & 1+3xz_1 \\ 1+3z_1 t & 1+3z_1^2 \end{array} \right| dz_1 = - \left[\frac{3(x+t)}{2} - 3xt - 1 \right], \text{ (on simplification) ... (2)}$$

Also, $B_2(x, t) = 0$. (verify yourself).

$\therefore B_p(x, t) = 0$ for all $p \geq 3$ (3)

Next,
$$C_1 = \int_0^1 K(z_1, z_1) dz_1 = \int_0^1 (1 + 3z_1^2) dz_1 = 2.$$

$$C_2 = \int_0^1 \int_0^1 \left| \begin{array}{cc} 1+3z_1^2 & 1+3z_1 z_2 \\ 1+3z_2 z_1 & 1+3z_2^2 \end{array} \right| dz_1 dz_2 = \frac{1}{2}. \quad \text{(simplify yourself)}$$

Also, $C_3 = 0$. (verify yourself)

$\therefore C_p = 0$ for all $p \geq 4$.

$$\therefore D(x, t; \lambda) = K(x, t) - \lambda K(x, t) + \dots = 1 + 3xt + \lambda [(3/2) \times (x+t) - 3xt - 1]$$

and
$$D(\lambda) = 1 - \lambda C_1 - (\lambda^2 / 2!) \times C_2 - \dots = 1 - 2\lambda + (\lambda^2 / 4).$$

$$\therefore R(x, t; \lambda) = \frac{D(x, t; \lambda)}{D(\lambda)} = \frac{1 + 3xt + \lambda [(3/2) \times (x+t) - 3xt - 1]}{1 - 2\lambda - (1/4) \times \lambda^2}$$

Part (iv) Here $K(x, t) = \sin x \cos t$ (1)

$$B_1(x, t) = \int_0^{2\pi} \left| \begin{array}{cc} K(x, t) & K(x, z_1) \\ K(z_1, t) & K(z_1, z_1) \end{array} \right| dz_1 = \int_0^{2\pi} \left| \begin{array}{cc} \sin x \cos t & \sin x \cos z_1 \\ \sin z_1 \cos t & \sin z_1 \cos z_1 \end{array} \right| dz_1, \text{ by (1)}$$

= 0, since the determinant under the integral sign vanishes

$$\therefore B_p(x, t) = 0 \quad \text{for all } p \geq 2.$$

Next,
$$C_1 = \int_0^{2\pi} K(z_1, z_1) dz_1 = \int_0^{2\pi} \sin z_1 \cos z_1 dz_1 = 0.$$

$$\therefore C_p = 0 \quad \text{for all } p \geq 2.$$

Now,
$$D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(\lambda)^m}{m!} B_m(x, t)$$

$$= K(x, t) - \lambda B_1(x, t) + \dots = \sin x \cos t, \text{ using the above values}$$

and
$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m = 1 - \lambda C_1 + \dots = 1, \text{ using the above values}$$

$\therefore R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda) = \sin x \cos t.$

Prove (v) Hint. Students are required to complete the solution.

Here
$$K(x, t) = \sin x - \sin t. \quad \dots (1)$$

$$B_1(x, t) = \int_0^{2\pi} \begin{vmatrix} \sin x - \sin t & \sin x - \sin z_1 \\ \sin z_1 - \sin t & 0 \end{vmatrix} dz_1$$

$$= \pi (1 + 2 \sin x \sin t), \text{ (simplify yourself)}$$

Also,
$$B_2(x, t) = 0. \text{ (verify yourself)}$$

$\therefore B_p(x, t) = 0 \text{ for all } p \geq 3.$

Again,
$$C_1 = \int_0^{2\pi} K(z_1, z_1) dz_1 = \int_0^{2\pi} (\sin z_1 - \sin z_1) dz_1 = 0.$$

$$C_2 = \int_0^{2\pi} \int_0^{2\pi} \begin{vmatrix} \sin z_1 - \sin z_1 & \sin z_1 - \sin z_2 \\ \sin z_2 - \sin z_1 & \sin z_2 - \sin z_2 \end{vmatrix} dz_1 dz_2 = 4\pi^2 \text{ (simplify yourself).}$$

$\therefore C_p = 0 \text{ for all } p \geq 3.$

$\therefore D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t) = K(x, t) - \lambda B_1(x, t)$

$$= \sin x - \sin t - \lambda \pi (1 + 2 \sin x \sin t).$$

and
$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m = 1 - \lambda C_1 + \frac{\lambda^2}{2!} C_2 \dots = 1 + 4\lambda^2 \pi^2.$$

$\therefore R(x, t; \lambda) = \frac{D(x, t; \lambda)}{D(\lambda)} = \frac{\sin x - \sin t - \lambda \pi (1 + 2 \sin x \sin t)}{1 + 4\pi^2 \lambda^2}$

Ex.11. Evaluate the resolvent kernels of the following integrals :

- (i) $K(x, t) = \sin x \cos t; 0 \leq x \leq 2\pi, 0 \leq t \leq 2\pi.$
- (ii) $K(x, t) = x - 2t; 0 \leq x \leq 1, 0 \leq t \leq 1$ [Meerut 2011, 2012; Kanpur 2005, 09]
- (iii) $K(x, t) = x + t + 1; -1 \leq x \leq 1, -1 \leq t \leq 1$
- (iv) $K(x, t) = e^{x-t}; 0 \leq x \leq 1, 0 \leq t \leq 1$
- (v) $K(x, t) = 4xt - x^2; 0 \leq x \leq 1, 0 \leq t \leq 1$
- (vi) $K(x, t) = \sin(x + t); 0 \leq x \leq 2\pi, 0 \leq t \leq 2\pi$
- (vii) $K(x, t) = x - \sinh t; -1 \leq x \leq 1, -1 \leq t \leq 1$

$$(viii) \quad K(x, t) = x^2 t - xt^2; \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1$$

$$(ix) \quad K(x, t) = \sin x - \sin t; \quad 0 \leq x \leq 2\pi, \quad 0 \leq t \leq 2\pi$$

$$(x) \quad K(x, t) = 1 + 3xt; \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1.$$

$$\text{Sol. (i) Here} \quad K(x, t) = \sin x \cos t. \quad \dots (1)$$

$$\text{The resolvent kernel } R(x, t; \lambda) \text{ is given by} \quad R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda) \quad \dots (2)$$

$$\text{where} \quad D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (3)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (4)$$

$$C_0 = 1, \quad \dots (5)$$

$$B_0(x, t) = K(x, t) = \sin x \cos t, \quad \dots (6)$$

$$C_p = \int_0^{2\pi} B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (7)$$

$$\text{and} \quad B_p(x, t) = C_p K(x, t) - p \int_0^{2\pi} K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1 \quad \dots (8)$$

$$\text{From (7), } C_1 = \int_0^{2\pi} B_0(s, s) ds = \int_0^{2\pi} \sin s \cos s ds = \frac{1}{2} \int_0^{2\pi} \sin 2s ds = \frac{1}{2} \left[-\frac{\cos 2s}{2} \right]_0^{2\pi} = 0. \quad \dots (9)$$

$$\begin{aligned} \text{From (8), } B_1(x, t) &= C_1 K(x, t) - \int_0^{2\pi} K(x, z) B_0(z, t) dz = - \int_0^{2\pi} (\sin x \cos z) (\sin z \cos t) dz \\ &= - \sin x \cos t \int_0^{2\pi} \sin z \cos z dz = 0. \end{aligned} \quad \dots (10)$$

Since $B_1(x, t) = 0$, (7) and (8) show that

$$B_p(x, t) = 0 \quad \text{and} \quad C_p = 0 \text{ for all } p \geq 2. \quad \dots (11)$$

Putting the above values in (3) and (4), we have

$$D(x, t; \lambda) = K(x, t) = \sin x \cos t \quad \text{and} \quad D(\lambda) = 1.$$

$$\therefore \text{ From (2)} \quad R(x, t; \lambda) = \sin x \cos t.$$

$$\text{Part (ii) Given} \quad K(x, t) = x - 2t. \quad \dots (1)$$

$$\text{The resolvent kernel is given by} \quad R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda) \quad \dots (2)$$

$$\text{where} \quad D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (3)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (4)$$

$$C_0 = 1, \quad \dots (5)$$

$$B_0(x, t) = K(x, t) = x - 2t. \quad \dots (6)$$

$$C_p = \int_0^1 B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (7)$$

$$B_p(x, t) = C_p K(x, t) - p \int_0^1 K(x, z) B_{p-1}(z, t) dz \quad \dots (8)$$

$$\text{From (7),} \quad C_1 = \int_0^1 B_0(s, s) ds = \int_0^1 (s - 2s) ds = -\frac{1}{2} \quad \dots (9)$$

$$\begin{aligned} \text{From (8),} \quad B_1(x, t) &= C_1 K(x, t) - \int_0^1 K(x, z) B_0(z, t) dz \\ &= -\frac{1}{2}(x - 2t) - \int_0^1 (x - 2z)(z - 2t) dz, \text{ using (1), (6) and (9)} \\ &= -\frac{1}{2}(x - 2t) - \int_0^1 \{-2z^2 + z(x + 4t) - 2xt\} dz = -\frac{1}{2}(x - 2t) - \left[-\frac{2z^3}{3} + \frac{z^2}{2}(x + 4t) - 2xtz \right]_0^1 \\ &= -\frac{1}{2}(x - 2t) - \left[-\frac{2}{3} + \frac{1}{2}(x + 4t) - 2xt \right] \\ \therefore \quad B_1(x, t) &= (2/3) + 2xt - x - t. \quad \dots (10) \end{aligned}$$

$$\begin{aligned} \text{From (7),} \quad C_2 &= \int_0^1 B_1(s, s) ds = \int_0^1 \left(\frac{2}{3} + 2s^2 - 2s \right) ds = \left[\frac{2}{3}s + \frac{2s^3}{3} - s^2 \right]_0^1, \text{ by (3)} \\ \therefore \quad C_2 &= (2/3) + (2/3) - 1 = 1/3. \quad \dots (11) \end{aligned}$$

$$\begin{aligned} \text{From (8),} \quad B_2(x, t) &= C_2 K(x, t) - 2 \int_0^1 K(x, z) B_1(z, t) dz \\ &= \frac{1}{3}[(x - 2t) - 2] \int_0^1 (x - 2z) \left(\frac{2}{3} + 2zt - z - t \right) dz, \text{ using (1), (10) and (11)} \\ &= \frac{1}{3}(x - 2t) - 2 \int_0^1 \left[\frac{2x}{3} + 2xzt - xz - xt - \frac{4}{3}z - 4z^2t + 2z^2 + 2zt \right] dz \\ &= \frac{1}{3}(x - 2t) - 2 \left[\frac{2xz}{3} + xtz^2 - \frac{xz^2}{2} - xtz - \frac{2z^2}{3} - \frac{4z^2t}{3} + \frac{2z^3}{3} + z^2t \right]_0^1 \\ &= \frac{1}{3}(x - 2t) - \frac{4x}{3} - 2xt + x + 2xt + \frac{4}{3} + 4t - \frac{4}{3} - 2t = 0. \quad \dots (12) \end{aligned}$$

Since $B_2(x, t) = 0$, (7) and (8) show that

$$B_p(x, t) = 0 \quad \text{and} \quad C_p = 0 \quad \text{for all} \quad p \geq 3$$

Putting the above values in (3) and (4), we have

$$D(x, t; \lambda) = K(x, t) - \lambda B_1(x, t) + \dots = x - 2t - \lambda(2/3 + 2xt - x - t)$$

$$\text{and} \quad D(\lambda) = 1 - \lambda C_1 + (\lambda^2 / 2!) \times C_2 - \dots = 1 + (\lambda / 2) + (\lambda^2 / 6).$$

$$\therefore \text{ from (2),} \quad R(x, t; \lambda) = \frac{x - 2t - \lambda(2/3 + 2xt - x - t)}{1 + (\lambda / 2) + (\lambda^2 / 6)}.$$

$$\text{Part (iii) Given} \quad K(x, t) = x + t + 1. \quad \dots (1)$$

$$\text{The resolvent kernel is given by} \quad R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda) \quad \dots (2)$$

where

$$D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (3)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (4)$$

$$C_0 = 1 \quad \dots (5)$$

$$B_0(x, t) = K(x, t) = x + t + 1. \quad \dots (6)$$

$$C_p = \int_{-1}^1 B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (7)$$

and

$$B_p(x, t) = C_p(x, t) - p \int_{-1}^1 K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1 \quad \dots (8)$$

$$\text{From (6) and (7),} \quad C_1 = \int_{-1}^1 B_0(s, s) ds = \int_{-1}^1 (1 + 2s) ds = \left[s + s^2 \right]_{-1}^1 = 2. \quad \dots (9)$$

$$\begin{aligned} \text{From (8),} \quad B_1(x, t) &= C_1(x, t) - \int_{-1}^1 K(x, z) B_0(z, t) dz \\ &= 2(x + t + 1) - \int_{-1}^1 (x + z + 1)(z + t + 1) dz, \text{ by (2), (6) and (9)} \\ &= 2(x + t + 1) - \int_{-1}^1 [z^2 + z(t + 2 + x) + (x + 1)(t + 1)] dz \\ &= 2(x + t + 1) - \left[\frac{z^3}{3} + \frac{z^2}{2}(t + 2 + x) + (x + 1)(t + 1)z \right]_{-1}^1 \\ &= 2(x + t + 1) - \left\{ 2/3 + 2(x + 1)(t + 1) \right\} \\ &= 2(x + t + 1) - 2/3 - 2(xt + x + t + 1) = -2(xt + 1/3). \quad \dots (10) \end{aligned}$$

$$\text{From (7) and (10),} \quad C_2 = \int_{-1}^1 B_1(s, s) ds = -2 \int_{-1}^1 \left(s^2 + \frac{1}{3} \right) ds = -2 \left[\frac{s^3}{3} + \frac{1}{3}s \right]_{-1}^1 = -\frac{8}{3}. \quad \dots (11)$$

$$\begin{aligned} \text{From (8),} \quad B_2(x, t) &= C_2 K(x, t) - 2 \int_{-1}^1 K(x, z) B_1(z, t) dz \\ &= -\frac{8}{3}(x + t + 1) - 2 \int_{-1}^1 (x + z + 1) \left\{ -2 \left(zt + \frac{1}{3} \right) \right\} dz, \text{ using (1), (10) and (11)} \\ &= -\frac{8}{3}(x + t + 1) + 4 \int_{-1}^1 \left[z^2 t + z \left(\frac{1}{3} + xt + t \right) + \frac{1}{3}(x + 1) \right] dz \\ &= -\frac{8}{3}(x + t + 1) + 4 \left[\frac{z^3 t}{3} + \frac{z^2}{2} \left(\frac{1}{3} + xt + 1 \right) + \frac{1}{3}(x + 1)z \right]_{-1}^1 \\ &= -\frac{8}{3}(x + t + 1) + 4 \left[\frac{2t}{3} + \frac{2}{3}(x + 1) \right] = 0. \quad \dots (12) \end{aligned}$$

Since $B_2(x, t) = 0$, (7) and (8) show that

$$B_p(x, t) = 0 \quad \text{and} \quad C_p = 0 \quad \text{for all} \quad p \geq 2. \quad \dots (13)$$

Putting these values in (3) and (4), we get

$$D(x, t; \lambda) = K(x, t) - \lambda B_1(x, t) = x + t + 1 + 2\lambda(xt + 1/3)$$

and

$$D(\lambda) = 1 - \lambda C_1 + (\lambda^2 / 2!) \times C_2 = 1 - 2\lambda - (4\lambda^2 / 3)$$

$$R(x, t; \lambda) = \frac{x + t + 1 + 2\lambda(xt + 1/3)}{1 - 2\lambda - (4\lambda^2 / 3)}.$$

Part (iv). Given

$$K(x, t) = e^{x-t}. \quad \dots (1)$$

The resolvent kernel $R(x, t; \lambda)$ is given by $R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda)$

$$\dots (2)$$

where

$$D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t) \quad \dots (3)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (4)$$

$$C_0 = 1, \quad \dots (5)$$

$$B_0(x, t) = K(x, t) = e^{x-t}, \quad \dots (6)$$

$$C_p = \int_0^1 B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (7)$$

and

$$B_p(x, t) = C_p K(x, t) - p \int_0^1 K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1. \quad \dots (8)$$

$$\text{From (7),} \quad C_1 = \int_0^1 B_0(s, s) ds = \int_0^1 e^{s-s} ds = \int_0^1 ds = 1. \quad \dots (9)$$

$$\text{From (8),} \quad B_1(x, t) = C_1 K(x, t) - \int_0^1 K(x, z) B_0(z, t) dz$$

$$= e^{x-t} - \int_0^1 e^{x-z} e^{z-t} dz, \quad \text{using (1), (6) and (9)}$$

$$= e^{x-t} - e^{x-t} \int_0^1 dz = e^{x-t} - e^{x-t} = 0. \quad \dots (10)$$

Since $B_1(x, t) = 0$, (7) and (8) show that

$$B_p(x, t) = 0 \quad \text{and} \quad C_p = 0 \quad \text{for all} \quad p \geq 2. \quad \dots (11)$$

Putting these values in (3) and (4), we get

$$D(x, t; \lambda) = K(x, t) = e^{x-t} \quad \text{and} \quad D(\lambda) = 1 - \lambda C_1 = 1 - \lambda.$$

$\therefore$ From (2),

$$R(x, t; \lambda) = e^{x-t} / (1 - \lambda).$$

Part (v) Given

$$K(x, t) = 4xt - x^2. \quad \dots (1)$$

The resolvent kernel $R(x, t; \lambda)$ is given by $R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda)$,

$$\dots (2)$$

where

$$D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (3)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (4)$$

$$C_0 = 1, \quad \dots (5)$$

$$B_0(x, t) = K(x, t) = 4xt - x^2, \quad \dots (6)$$

$$C_p = \int_0^1 B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (7)$$

and $B_p(x, t) = C_p K(x, t) - p \int_0^1 K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1. \quad \dots (8)$

From (6) and (7), $C_1 = \int_0^1 B_0(s, s) ds = \int_0^1 (4s^2 - s^2) ds = \left[s^3 \right]_0^1 = 1. \quad \dots (9)$

From (8), $B_1(x, t) = C_1 K(x, t) - \int_0^1 K(x, z) B_0(z, t) dz$

$$= 4xt - x^2 - \int_0^1 (4xz - x^2) (4zt - z^2) dz, \text{ by (1), (6) and (9)}$$

$$= 4xt - x^2 - \int_0^1 [-4xz^3 + z^2(x^2 + 16xt) - 4x^2 tz] dz$$

$$= 4xt - x^2 - \left[-xz^4 + (z^3/3) \times (x^2 + 16xt) - 2x^2 z \right]_0^1$$

$$= 4xt - x^2 - \left[-x + \frac{1}{3}(x^2 + 16xt) - 2x^2 t \right] = 2x^2 t - \frac{4}{3}x^2 + x - \frac{4}{3}xt. \quad \dots (10)$$

From (7), $C_2 = \int_0^1 B_1(s, s) ds = \int_0^1 \left(2s^3 - \frac{4}{3}s^2 + s - \frac{4}{3}s^2 \right) ds$, using (10)

$$= \left[\frac{s^4}{2} - \frac{4}{9}s^3 + \frac{s^2}{2} - \frac{4s^3}{9} \right]_0^1 = \frac{1}{2} - \frac{4}{9} + \frac{1}{2} - \frac{4}{9} = \frac{1}{9}. \quad \dots (11)$$

From (8), $B_2(x, t) = C_2 K(x, t) - 2 \int_0^1 K(x, z) B_1(z, t) dz$

$$= \frac{1}{9}(4xt - x^2) - 2 \int_0^1 (4xz - x^2) \left(2z^2 t - \frac{4}{3}z^2 + z - \frac{4}{3}zt \right) dz$$

[using (1), (10) and (11)]

$$= \frac{1}{9}(4xt - x^2) - 2 \int_0^1 (4xz - x^2) \left[z^2 \left(2t - \frac{4}{3} \right) + z \left(1 - \frac{4t}{3} \right) \right] dz$$

$$= \frac{1}{9}(4xt - x^2) - 2 \int_0^1 \left[4x \left(2t - \frac{4}{3} \right) z^3 + z^2 \left\{ 4x \left(1 - \frac{4t}{3} \right) - x^2 \left(2t - \frac{4}{3} \right) \right\} - x^2 \left(1 - \frac{4t}{3} \right) z \right] dz$$

$$= \frac{1}{9}(4xt - x^2) - 2 \left[x \left(2t - \frac{4}{3} \right) z^4 + \frac{z^3}{3} \left\{ 4x \left(1 - \frac{4t}{3} \right) - x^2 \left(2t - \frac{4}{3} \right) \right\} - \frac{x^2}{2} \left(1 - \frac{4t}{3} \right) z^2 \right]_0^1$$

$$= \frac{1}{9}(4xt - x^2) - 2 \left[x \left(2t - \frac{4}{3} \right) + \frac{1}{3} \left\{ 4x \left(1 - \frac{4t}{3} \right) - x^2 \left(2t - \frac{4}{3} \right) \right\} - \frac{x^2}{2} \left(1 - \frac{4t}{3} \right) \right] = 0.$$

Since $B_2(x, t) = 0$, (7) and (8) show that

$$B_p(x, t) = 0 \quad \text{and} \quad C_p = 0 \quad \text{for all } p \geq 2. \quad \dots (12)$$

Putting these values in (3) and (4), we have

$$D(x, t; \lambda) = K(x, t) - \lambda B_1(x, t) = 4xt - x^2 - \lambda \left(2x^2t - \frac{4}{3}x^2 + x - \frac{4}{3}xt \right)$$

and

$$D(\lambda) = 1 - \lambda C_1 + (\lambda^2 / 2!) \times C_2 = 1 - \lambda + (\lambda^2 / 18).$$

$\therefore$ from (2),

$$R(x, t; \lambda) = \frac{4xt - x^2 - \lambda \left(2x^2t - \frac{4}{3}x^2 + x - \frac{4}{3}xt \right)}{1 - \lambda + (\lambda^2 / 18)}.$$

Part (vi). Try yourself

$$\text{Ans. } R(x, t; \lambda) = \frac{\sin(x+t) + \pi\lambda \cos(x-t)}{1 - \pi^2\lambda^2}.$$

Part (vii). Try yourself

$$\text{Ans. } R(x, t; \lambda) = \frac{x - \sinh t - 2\lambda(e^{-1} + x \sinh t)}{1 + 4e^{-1}\lambda^2}.$$

Part (viii). Try yourself

$$\text{Ans. } R(x, t; \lambda) = \frac{x^2t - xt^2 + x + \lambda \left(\frac{x+t}{4} - \frac{xt}{3} - \frac{1}{5} \right)}{1 + (\lambda^2 / 240)}.$$

Part (ix). Try yourself

$$\text{Ans. } R(x, t; \lambda) = \frac{\sin x - \sin t - \pi\lambda(1 + 2 \sin x \sin t)}{1 + 2\pi^2\lambda^2}.$$

Part (x). Try yourself

$$\text{Ans. } R(x, t; \lambda) = \frac{1 + 3xt + \lambda \left\{ \frac{3(x+t)}{2} - 3xt - 1 \right\}}{1 - 2\lambda + (\lambda^2 / 4)}.$$

Ex.12. For the integral equation $y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt$,

compute $D(\lambda)$ and $D(x, t; \lambda)$ for the following kernels for the specified limits a and b .

(i) $K(x, t) = 1, a = 0, b = 1$. [Kanpur 2007] (ii) $K(x, t) = -1, a = 0, b = 1$.

(iii) $K(x, t) = \sin x, a = 0, b = \pi$.

(iv) $K(x, t) = xt, a = 0, b = 10$. [Kanpur 2006]

(v) $K(x, t) = t, a = 0, b = 10$.

(vi) $K(x, t) = x, a = 4, b = 10$. [Meerut 2009]

(vii) $K(x, t) = 2e^x e^t, a = 0, b = 1$.

(viii) $K(x, t) = g(x), a = a, b = b$.

(ix) $K(x, t) = g(t), a = a, b = b$.

(x) $K(x, t) = x - t, a = 0, b = 1$.

Sol. (i) Given

$$K(x, t) = 1. \quad \dots (1)$$

We know that

$$D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (2)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (3)$$

$$C_0 = 1, \quad \dots (4)$$

$$B_0(x, t) = K(x, t) = 1, \quad \dots (5)$$

$$C_p = \int_0^1 B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (6)$$

and
$$B_p(x, t) = C_p K(x, t) - p \int_0^1 K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1 \quad \dots (7)$$

From (5) and (6),
$$C_1 = \int_0^1 B_0(s, s) ds = \int_0^1 ds = 1. \quad \dots (8)$$

From (1), (5), (7) and (8),
$$B_1(x, t) = C_1 K(x, t) - \int_0^1 K(x, z) B_0(z, t) dz = 1 - \int_0^1 dz = 0. \quad \dots (9)$$

Since $B_1(x, t) = 0$, (6) and (7) show that

$$B_p(x, t) = 0 \quad \text{and} \quad C_p = 0 \quad \text{for all } p \geq 2. \quad \dots (10)$$

Putting the above values in (2) and (3), we get

$$D(x, t; \lambda) = K(x, t) = 1 \quad \text{and} \quad D(\lambda) = 1 - \lambda.$$

Part (ii). Ans : Proceed as in part (i). **Ans.** $D(x, t; \lambda) = -1, D(\lambda) = 1 + \lambda.$

Part (iii) Given
$$K(x, t) = \sin x. \quad \dots (1)$$

We know that
$$D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (2)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m \quad \dots (3)$$

$$C_0 = 1 \quad \dots (4)$$

$$B_0(x, t) = K(x, t) = \sin x. \quad \dots (5)$$

$$C_p = \int_0^{\pi} B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (6)$$

and
$$B_p(x, t) = C_p K(x, t) - p \int_0^{\pi} K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1 \quad \dots (7)$$

From (5) and (6),
$$C_1 = \int_0^1 B_0(s, s) ds = \int_0^{\pi} \sin s ds = [-\cos s]_0^{\pi} = 2. \quad \dots (8)$$

From (7)
$$\begin{aligned} B_1(x, t) &= C_1 K(x, t) - \int_0^{\pi} K(x, z) B_0(z, t) dz \\ &= 2 \sin x - \int_0^{\pi} \sin x \sin z dz, \quad \text{using (1), (5) and (8)} \\ &= 2 \sin x - \sin x [-\cos z]_0^{\pi} = 2 \sin x - 2 \sin x = 0. \end{aligned} \quad \dots (9)$$

Since $B_1(x, t) = 0$, (6) and (7) show that

$$B_p(x, t) = 0 \quad \text{and} \quad C_p = 0 \quad \text{for all } p \geq 2. \quad \dots (10)$$

Putting the above values in (2) and (3), we get

$$D(x, t; \lambda) = K(x, t) = \sin x \quad \text{and} \quad D(\lambda) = 1 - 2\lambda.$$

Part (iv). Proceed as in part (i) **Ans.** $D(x, t; \lambda) = xt, D(\lambda) = 1 - (10^3 / 3)\lambda.$

Part (v). Given
$$K(x, t) = t. \quad \dots (1)$$

We know that
$$D(x, t; \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t), \quad \dots (2)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m, \quad \dots (3)$$

$$C_0 = 1, \quad \dots (4)$$

$$B_0(x, t) = K(x, t) = t, \quad \dots (5)$$

$$C_p = \int_0^{10} B_{p-1}(s, s) ds, \quad p \geq 1 \quad \dots (6)$$

and

$$B_p(x, t) = C_p K(x, t) - p \int_0^{10} K(x, z) B_{p-1}(z, t) dz, \quad p \geq 1. \quad \dots (7)$$

$$\text{From (6),} \quad C_1 = \int_0^{10} s ds = \left[s^2 / 2 \right]_0^{10} = 50. \quad \dots (8)$$

$$\begin{aligned} \text{From (7), } B_1(x, t) &= C_1 K(x, t) - \int_0^{10} K(x, z) B_0(z, t) dz = 50t - \int_0^{10} zt dz, \text{ by (1), (5) and (8)} \\ &= 50t - t \left[z^2 / 2 \right]_0^{10} = 50t - 50t = 0. \end{aligned} \quad \dots (9)$$

Since $B_1(x, t) = 0$, (6) and (7) show that

$$B_p(x, t) = 0 \quad \text{and} \quad C_p = 0 \quad \text{for all } p \geq 2. \quad \dots (10)$$

Putting the above values in (2) and (3), we get $D(x, t; \lambda) = K(x, t) = t$, $D(\lambda) = 1 - 50\lambda$.**Part (vi). Try yourself**

$$\text{Ans. } D(x, t; \lambda) = x \text{ and } D(\lambda) = 1 - 42\lambda.$$

Part (vii). Try yourself

$$\text{Ans. } D(x, t; \lambda) = 2e^x e^t \text{ and } D(\lambda) = 1 - \lambda(e^2 - 1).$$

Part (viii). Try yourself

$$\text{Ans. } D(\lambda) = 1 - \lambda \int_a^b g(t) dt.$$

Part (ix). Try yourself

$$\text{Ans. } D(\lambda) = 1 - \lambda \int_a^b g(t) dt.$$

EXERCISES

1. State and prove Fredholm's first fundamental theorem. [Meerut 2004]
2. State and prove first and second series for the non-homogeneous integral equation of second kind.
3. Use Fredholm determinants to find the resolvent kernel.

$$R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda)$$

of the kernel $K(x, t) = xe^t$ under the limits of integration $a = 0$, $b = 1$. Hence solve the equation

$$\phi(x) = e^{-x} + \lambda \int_0^1 xe^t \phi(t) dt. \quad \text{Ans. } R(x, t; \lambda) = \frac{xe^t}{1 - \lambda}; \text{ solution is } \phi(x) = e^{-x} - \frac{\lambda xt}{1 - \lambda}$$

$$4. \text{ Solve : } y(x) = \sec^2 x + \lambda \int_0^1 y(t) dt \quad \text{Ans. } y = \sec^2 x + \frac{\lambda}{1 - \lambda} \tan 1$$

5. Determine the resolvent kernel and hence solve the following integral equations :

$$(i) \quad y(x) = 1 + \lambda \int_0^{2\pi} \sin(x+t), y(t) dt. \quad \text{Ans. } y(x) = 1.$$

$$(ii) \quad y(x) = \cos 2x + \int_0^{2\pi} \sin x \cos t \, y(t) \, dt. \quad \text{Ans. } y(x) = \cos 2x.$$

$$(iii) \quad y(x) = e^x - \int_0^1 e^{x-t} y(t) \, dt. \quad \text{Ans. } y(x) = \frac{1}{2} e^x.$$

$$(iv) \quad y(x) = \frac{x}{6} + \lambda \int_0^1 (2x-t) y(t) \, dt. \quad \text{Ans. } y(x) = \frac{1}{6} \left[x + \frac{\lambda (6x-2) - \lambda^2 x}{\lambda^2 - 3\lambda + 6} \right].$$

$$(v) \quad y(x) = x + \lambda \int_0^1 (4xt - x^2) y(t) \, dt. \quad \text{Ans. } y(x) = \frac{3x(2\lambda - 3\lambda x + 6)}{\lambda^2 - 18\lambda + 18}.$$

6.4. FREDHOLM'S SECOND FUNDAMENTAL THEOREM.

If λ_0 is a zero of multiplicity m of the function $D(\lambda)$, then the homogeneous integral equation

$$y(x) = \lambda_0 \int_a^b K(x, t) y(t) \, dt \quad \dots (1)$$

possesses at least one, and the most m , linearly independent solutions

$$y_i(x) = D_r \left(\begin{array}{c} x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_r \\ t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_r \end{array} \middle| \lambda_0 \right), \quad i = 1, 2, \dots, r; \quad 1 \leq r \leq m. \quad \dots (2)$$

not identically zero. Any other solution of this equation is a linear combination of these solutions. Here, we have to remember the following definition of the Fredholm minor

$$D_n \left(\begin{array}{c} x_1, x_2, \dots, x_n \\ t_1, t_2, \dots, t_n \end{array} \right) = K \left(\begin{array}{c} x_1, x_2, \dots, x_n \\ t_1, t_2, \dots, t_n \end{array} \right) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_a^b \dots \int_a^b K \left(\begin{array}{c} x_1, \dots, x_n, z_1, \dots, z_p \\ t_1, \dots, t_n, z_1, \dots, z_p \end{array} \right) dz_1 \dots dz_p, \quad \dots (3)$$

where $\{x_i\}$ and $\{t_i\}$, $i = 1, 2, \dots, n$, are two sequences of arbitrary variables. Series (3) converges for all values of λ and hence it is an entire function of λ .

Proof. We shall first show that every zero of $D(\lambda)$ is a pole of the resolvent kernel $R(x, t; \lambda)$ given by

$$R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda), \quad \dots (4)$$

the order of this pole being at most equal to the order of the zero of $D(\lambda)$.

The Fredholm's first series is

$$D(\lambda) = 1 + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_a^b \dots \int_a^b K \left(\begin{array}{c} x_1, \dots, x_p \\ x_1, \dots, x_p \end{array} \right) dx_1 \dots dx_p \quad \dots (5)$$

and the Fredholm's second series is

$$D(x, t; \lambda) = K(x, t) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_a^b \dots \int_a^b K \left(\begin{array}{c} x, x_1, \dots, x_p \\ t, x_1, \dots, x_p \end{array} \right) dx_1 \dots dx_p. \quad \dots (6)$$

Differentiating both sides of (5) w.r.t. ' λ ' and interchanging the indices of the variables of integration, we obtain

$$D'(\lambda) = - \int_a^b D(x, x; \lambda) dx \quad \dots (7)$$

showing that if λ_0 is a zero of order k of $D(\lambda)$, then it is a zero of order $k-1$ of $D'(\lambda)$ and hence λ_0 may be zero of order at most $k-1$ of the entire function $D(x, t; \lambda)$. It follows that λ_0 must be the pole of the quotient (4) of order at most k . In particular, if λ_0 is a simple pole of $D(\lambda)$, then $D(\lambda_0) = 0$, $D'(\lambda_0) \neq 0$, and λ_0 is a simple pole of the resolvent kernel. Again, (7) shows that $D(x, t; \lambda) \neq 0$. Keeping in mind this particular case and the following equation.

$$D(x, t; \lambda) = K(x, t) D(\lambda) + \lambda \int_a^b K(x, z) D(z, t; \lambda) dz,$$

we find that if $D(\lambda) = 0$ and $D(x, t; \lambda) \neq 0$, then $D(x, t; \lambda)$, as a function of x , is solution of (1) and so is $\alpha D(x, t; \lambda)$, α being an arbitrary constant.

Next, we take the general case when λ is a zero of an arbitrary multiplicity m , that is,

$$\text{when } D(\lambda_0) = 0, \quad D'(\lambda_0) = 0, \quad \dots, \quad D^{(r)}(\lambda_0) = 0, \quad \dots, \quad D^{(m)}(\lambda_0) \neq 0, \quad \dots (8)$$

where the superscript r stands for the differential of order r , $r = 1, 2, \dots, m-1$.

Differentiating series (5) n times and comparing it with the series (3), we obtain the following relation

$$\frac{d^n D(\lambda)}{d\lambda^n} = (-1)^n \int_a^b \dots \int_a^b D_n \left(\begin{matrix} x_1, \dots, x_n \\ x_1, \dots, x_n \end{matrix} \middle| \lambda \right) dx_1 \dots dx_n, \quad \dots (9)$$

showing that, if λ_0 is a zero of multiplicity m of the function $D(\lambda)$, then the following condition is valid for the Fredholm minor of order m for that value of λ_0 :

$$D_m \left(\begin{matrix} x_1, x_2, \dots, x_m \\ t_1, t_2, \dots, t_m \end{matrix} \middle| \lambda_0 \right) \neq 0. \quad \dots (10)$$

In this connection note that we may get minors of order lower than m which also do not identically vanish.

We now determine the relation among the minors that corresponds to the resolvent formula

$$R(x, t; \lambda) = K(x, t) + \lambda \int_a^b K(x, z) R(z, t; \lambda) dz. \quad \dots (11)$$

Expanding the determinant under the integral sign in (3), we obtain

$$\begin{vmatrix} K(x_1, t_1) & K(x_1, t_2) & \dots & K(x_1, t_n) & K(x_1, z_1) & \dots & K(x_1, z_p) \\ K(x_2, t_1) & K(x_2, t_2) & \dots & K(x_2, t_n) & K(x_2, z_1) & \dots & K(x_2, z_p) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ K(x_n, t_1) & K(x_n, t_2) & \dots & K(x_n, t_n) & K(x_n, z_1) & \dots & K(x_n, z_p) \\ K(z_1, t_1) & K(z_1, t_2) & \dots & K(z_1, t_n) & K(z_1, z_1) & \dots & K(z_1, z_p) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ K(z_p, t_1) & K(z_p, t_2) & \dots & K(z_p, t_n) & K(z_p, z_1) & \dots & K(z_p, z_p) \end{vmatrix} \quad \dots (12)$$

by elements of the first row and integrating p times with respect to $z_1, z_2, \dots, z_p$ for $p \geq 1$, we get

$$\begin{aligned} & \int_a^b \dots \int_a^b K \begin{pmatrix} x_1, \dots, x_n, z_1, \dots, z_p \\ t_1, \dots, t_n, z_1, \dots, z_p \end{pmatrix} dz_1 \dots dz_p \\ &= \sum_{h=1}^n (-1)^{h+1} K(x_1, t_h) \int_a^b \dots \int_a^b K \begin{pmatrix} x_2, \dots, x_h, \dots, x_n, z_1, \dots, z_p \\ t_1, \dots, t_{h-1}, t_{h+1}, \dots, t_n, z_1, \dots, z_p \end{pmatrix} dz_1 dz_2 \dots dz_p \\ &+ \sum_{h=1}^p (-1)^{h+n-1} \int_a^b \dots \int_a^b K(x_1, z_h) K \begin{pmatrix} x_2, \dots, x_n, z_1, z_2, \dots, z_h, \dots, z_p \\ t_1, \dots, t_{n-1}, t_n, z_1, \dots, z_{h-1}, z_{h+1}, \dots, z_p \end{pmatrix} dz_1 \dots dz_p \dots (13) \end{aligned}$$

Observe carefully that the symbols for the determinant K on the right hand side of (13) do not contain the variable x_1 in the upper sequence and the variables t_h or z_h in the lower sequence. Again, by transposing the variable x_h in the upper sequence to the first place by means of $h + n - 2$ transpositions, it follows that all the components of the second sum on the right side are equal. Accordingly, (13) can be re-written as

$$\begin{aligned} & \int_a^b \dots \int_a^b K \begin{pmatrix} x_1, \dots, x_n, z_1, \dots, z_p \\ t_1, \dots, t_n, z_1, \dots, z_p \end{pmatrix} dz_1 \dots dz_p \\ &= \sum_{h=1}^n (-1)^{h+1} K(x_1, t_h) \int_a^b \dots \int_a^b K \begin{pmatrix} x_2, \dots, x_n, z_1, \dots, z_p \\ t_1, \dots, t_{h-1}, t_{h+1}, \dots, t_n, z_1, \dots, z_p \end{pmatrix} dz_1 \dots dz_p \\ &- p \int_a^b K(x_1, z) \left[\int_a^b \dots \int_a^b K \begin{pmatrix} z, x_2, \dots, x_n, z_1, \dots, z_{p-1} \\ t_1, t_2, \dots, t_n, z_1, \dots, z_{p-1} \end{pmatrix} dz_1 \dots dz_{p-1} \right] dz \dots (14) \end{aligned}$$

where we have omitted the subscript h from z .

Substituting (14) in (13), we see that Fredholm minor satisfies the following integral equation:

$$D_n \begin{pmatrix} x_1, \dots, x_n \\ t_1, \dots, t_n \end{pmatrix} \lambda = \sum_{h=1}^n (-1)^{h+1} K(x_1, t_h) D_{n-1} \begin{pmatrix} x_2, \dots, x_n \\ t_1, \dots, t_{h-1}, t_{h+1}, \dots, t_n \end{pmatrix} + \lambda \int_a^b K(x_1, z) D_n \begin{pmatrix} z, z_2, \dots, x_n \\ t_1, t_2, \dots, t_n \end{pmatrix} \lambda dz \dots (15)$$

It may be observed that the expansion by the elements of any other row also leads to a similar identity, with z placed at the corresponding place.

Expanding the determinant (12) with respect to the first column and proceeding as above, we obtain the integral equation

$$D_n \begin{pmatrix} x_1, \dots, x_n \\ t_1, \dots, t_n \end{pmatrix} \lambda = \sum_{h=1}^n (-1)^{h+1} K(x_h, t_1) D_{n-1} \begin{pmatrix} x_1, \dots, x_{h-1}, x_{h+1}, \dots, x_n \\ t_2, \dots, t_n \end{pmatrix} + \lambda \int_a^b K(z, t_1) D_n \begin{pmatrix} x_1, \dots, x_n \\ z, t_2, \dots, t_n \end{pmatrix} dz \dots (16)$$

and a similar result can be obtained by expanding by any other column

The relations (15) and (16) hold for all values of λ . We now show that (15) can be used to find the solution of (1) for the special case when $\lambda = \lambda_0$ is an eigenvalue. Suppose that $\lambda = \lambda_0$ is a zero of multiplicity m of $D(\lambda)$. Then the minor $D_m \neq 0$ and even the minors $D_1, D_2, \dots, D_{m-1}$

may not identically vanish. Suppose D_r is the first minor in the sequence $D_1, D_2, \dots, D_{m-1}$ such that $D_r \neq 0$. Then the number r must be between 1 and m and is the index of the eigenvalue. Further, it follows that $D_{r-1} = 0$. Then (15) shows that

$$y_1(x) = D_r \begin{vmatrix} x_1, x_2, \dots, x_r \\ t_1, t_2, \dots, t_r \end{vmatrix} \lambda_0 \quad \dots (17)$$

is a solution of (1). Substituting x at different points of the upper sequence in the minor D_r , we obtain r nontrivial solutions $y_i(x)$, $i = 1, 2, \dots, r$, of (1), which are usually written as

$$\phi_i(x) = \frac{D_r \begin{vmatrix} x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_r \\ t_1, \dots, t_r \end{vmatrix} \lambda_0}{D_r \begin{vmatrix} x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_r \\ t_1, \dots, t_r \end{vmatrix} \lambda_0}, i = 1, 2, \dots, r \quad \dots (18)$$

in which the denominator is non-zero.

We now establish that solutions ϕ_i given by (18) are linearly independent. To this end note that if we put two of the arguments x_i equal in the determinant (12), this is equivalent to putting two rows equal, and hence the determinant vanishes. It follows that, in (18), $\phi_k(x_i) = 0$ for $i \neq k$, whereas $\phi_k(x_k) = 1$. Now, if we have a relation of the form $\sum_k C_k \phi_k \equiv 0$, then putting $x = x_i$, we obtain $C_i \equiv 0$ and so the solutions ϕ_i are linearly independent. This system of solutions ϕ_i is known as *fundamental system* of eigenfunctions of λ_0 and any linear combination of these functions gives a solution of (1).

Conversely, we can prove that any solution of (1) is a linear combination of $\phi_1(x), \phi_2, \dots, \phi_r(x)$. For this purpose we define kernel $H(x, t; \lambda)$ as follows* :

$$H(x, t; \lambda) = D_{r+1} \begin{vmatrix} x, x_1, \dots, x_r \\ t, t_1, \dots, t_r \end{vmatrix} \lambda_0 \bigg/ D_r \begin{vmatrix} x_1, \dots, x_r \\ t_1, \dots, t_r \end{vmatrix} \lambda_0. \quad \dots (19)$$

Putting $n = r$ and adding extra arguments x and t in (10), we arrive at

$$\begin{aligned} D_{r+1} \begin{vmatrix} x, x_1, \dots, x_r \\ t, t_1, \dots, t_r \end{vmatrix} \lambda_0 &= K(x, t) D_{r+1} \begin{vmatrix} x_1, \dots, x_r \\ t_1, \dots, t_r \end{vmatrix} \lambda_0 \\ &+ \sum_{h=1}^r (-1)^h K(x_h, t) D_r \begin{vmatrix} x, x_1, \dots, x_{h-1}, x_{h+1}, \dots, x_r \\ t_1, t_2, \dots, t_r \end{vmatrix} \lambda_0 \\ &+ \lambda_0 \int_a^b K(z, t) D_{r+1} \begin{vmatrix} x, x_1, \dots, x_r \\ z, t_1, \dots, t_r \end{vmatrix} \lambda_0 dz. \quad \dots (20) \end{aligned}$$

* The Kernel $H(x, t; \lambda)$ will correspond to the resolvent kernel $R(x, t; \lambda)$ of Art. 6.1.

Transposing the variable x from the first place to the place between the variables x_{h-1} and x_{h+1} in every minor D_r in (20) and then dividing both sides of (20) by the constant

$$D_r \left(\begin{matrix} x_1, \dots, x_r \\ t_1, \dots, t_r \end{matrix} \middle| \lambda_0 \right) \neq 0, \text{ we have}$$

$$H(x, t; \lambda) - K(x, t) - \lambda_0 \int_a^b H(x, z, \lambda) K(z, t) dz = - \sum_{h=1}^r K(x_h, t) \phi_h(x). \quad \dots (21)$$

Suppose that $y(x)$ is an arbitrary solution of (1). Multiplying both sides of (21) by $y(t)$ and then integrating both sides w.r.t. ' t ' from a to b , we obtain

$$\int_a^b y(t) H(x, t; \lambda) dt - \frac{y(x)}{\lambda_0} - \int_a^b y(z) R(x, z; \lambda) dz = - \sum_{h=1}^r \frac{y(x_h)}{\lambda_0} \phi_h(x), \quad \dots (22)$$

where we have used (1) in all terms except the first ; we have also assumed that

$$\lambda_0 \int_a^b K(x_h, t) y(t) dt = y(x_h).$$

Cancelling the equal terms in (22), we obtain

$$y(x) = \sum_{h=1}^r y(x_h) \phi_h(x) \quad \dots (23)$$

This completes the proof of the Fredholm's second fundamental theorem.

6.5 FREDHOLM THIRD FUNDAMENTAL THEOREM.

For an inhomogeneous equation

$$y(x) = f(x) + \lambda_0 \int_a^b K(x, t) y(t) dt. \quad \dots (1)$$

to possess a solution in the case $D(\lambda_0) = 0$, it is necessary and sufficient that the given function $f(x)$ be orthogonal to all the eigenfunctions $z_i(x), i = 1, 2, \dots, v$, of the transposed homogenous equation corresponding to the eigenvalue λ_0 . The general solution has the form

$$y(x) = f(x) + \lambda \int_a^b \frac{D_{r+1} \left(\begin{matrix} x, x_1, x_2, \dots, x_r \\ t, t_1, t_2, \dots, t_r \end{matrix} \middle| \lambda_0 \right)}{D_r \left(\begin{matrix} x_1, x_2, \dots, x_r \\ t_1, t_2, \dots, t_r \end{matrix} \middle| \lambda_0 \right)} f(t) dt + \sum_{h=1}^r C_h \phi_h(x), \quad \dots (2)$$

where $\phi_i(x)$ are given by

$$\phi_i(x) = \frac{D_r \left(\begin{matrix} x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_r \\ t_1, \dots, t_r \end{matrix} \middle| \lambda_0 \right)}{D_r \left(\begin{matrix} x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_r \\ t_1, \dots, t_r \end{matrix} \middle| \lambda_0 \right)}, \quad i = 1, 2, \dots, r \quad \dots (2)'$$

Proof. Consider

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt. \quad \dots (1)'$$

We know that the trasponse (or adjoint) of (1) is given by

$$z(x) = f(x) + \lambda \int_a^b K(t, x) z(t) dt. \quad \dots (3)$$

Then for the transposed equation (3), Fredholm's first series $D(\lambda)$ and the Fredholm's second series $D(t, x; \lambda)$ are given by

$$D(\lambda) = 1 + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_a^b \dots \int_a^b K \left(\begin{matrix} z_1, \dots, z_p \\ z_1, \dots, z_p \end{matrix} \right) dz_1 \dots dz_p \quad \dots (4)$$

and
$$D(t, x; \lambda) = K(t, x) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_a^b \dots \int_a^b K \left(\begin{matrix} t, z_1, \dots, z_p \\ x, z_1, \dots, z_p \end{matrix} \right) dz_1 \dots dz_p \quad \dots (5)$$

From, the above fact, it follows that the kernels of (1) and (2) possess the same eigenvalues. Moreover, the resolvent kernel of (2) is given by

$$R(t, x; \lambda) = D(t, x; \lambda) / D(\lambda) \quad \dots (6)$$

and hence the solution of (2) is

$$z(x) = f(x) + \lambda \int_a^b \frac{D(t, x; \lambda)}{D(\lambda)} f(t) dt, \quad \dots (7)$$

provided λ is not an eigenvalue.

Again it is obvious that not only the transposed kernel has the eigenvalues as the original kernel of (1)', but also the index r of each of the eigen values is equal. Furthermore the eigenfunctions of the transposed equation for an eigenvalue λ_0 are given by

$$z_i(t) = \frac{D_r \left(\begin{matrix} x_1, \dots, x_r \\ t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_r \end{matrix} \middle| \lambda_0 \right)}{D_r \left(\begin{matrix} x_1, \dots, x_r \\ t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_r \end{matrix} \middle| \lambda_0 \right)}, \quad \dots (8)$$

where the values $(x_1, \dots, x_r)$ and $(t_1, \dots, t_r)$ are so chosen that the denominator in (8) does not vanish. Substituting r in different places in the lower sequence of this formula, we obtain a linearly independent system of r eigenfunctions. Again, we know that each ϕ_i is orthogonal to each z_i with different eigenvalues.

Suppose that a solution $y(x)$ of (1)' exists. Then multiplying (1)' by each member $z_k(x)$ of the above-mentioned system of functions and integrating w.r.t. x from a to b , we get

$$\begin{aligned} \int_a^b f(x) z_k(x) dx &= \int_a^b y(x) z_k(x) dx - \lambda \int_a^b \int_a^b K(x, t) y(t) z_k(x) dx dt \\ &= \int_a^b y(x) dx [z_k(x) - \lambda \int_a^b K(t, x) z_k(t) dt] = 0. \end{aligned} \quad \dots (9)$$

Since $z_{\kappa}(x)$ is an eigenfunctions of the transposed equation, we have

$$z_{\kappa}(x) = \lambda \int_a^b K(t, x) z_{\kappa}(t) dt \quad \dots (10)$$

Using (10), (9) reduces to
$$\int_a^b f(x) z_{\kappa}(x) dx = 0, \quad \dots (11)$$

showing that a necessary condition for (1)' to have a solution is that the inhomogeneous term $f(x)$ be orthogonal to each solution of the transposed homogeneous equation.

Conversely, we shall prove that the condition (11) of orthogonality is sufficient for the existence of a solution. In what follows, we shall also obtain an explicit solution in that case. At this stage we need to define a kernel $H(x, t; \lambda)$ as follows :

$$H(x, t; \lambda) = D_{r+1} \left(\begin{matrix} x, x_1, \dots, x_r \\ t, t_1, \dots, t_r \end{matrix} \middle| \lambda_0 \right) / D_r \left(\begin{matrix} x_1, \dots, x_r \\ t_1, \dots, t_r \end{matrix} \middle| \lambda_0 \right), \quad \dots (12)$$

where we have assumed that $D_r \neq 0$ and that r is the index of the eigenvalue λ_0 .

To prove the required result we shall show that if the orthogonality condition is satisfied, then the function

$$y_0(x) = f(x) + \lambda_0 \int_a^b H(x, t; \lambda) f(t) dt \quad \dots (13)$$

is a solution. To this end, putting the above value of $y(x)$ in (1)', we have

$$f(x) + \lambda_0 \int_a^b H(x, t; \lambda) dt = f(x) + \lambda_0 \int_a^b K(x, t) [f(t) + \lambda_0 \int_a^b H(t, z; \lambda) f(z) dz] dt$$

or
$$\int_a^b f(t) dt [H(x, t; \lambda) - K(x, t) - \lambda_0 \int_a^b K(x, z) H(z, t; \lambda) dz] = 0. \quad \dots (14)$$

Recall the procedure of getting the equation (21) of Art. 6.4. Proceeding like-wise, we can obtain its "transpose".

$$H(x, t; \lambda) - K(x, t) - \lambda_0 \int_a^b K(x, z) H(z, t; \lambda) dz = - \sum_{h=1}^r K(x, t_h) z_h(t). \quad \dots (15)$$

Substituting this in (14) and making use of the orthogonality condition, we shall arrive at an identity and thus we prove what we wished to prove.

Now, the difference of any two solutions of (1) is a solution of the homogeneous equation. Therefore, the most general solution of (1)' is

$$y(x) = f(x) + \lambda_0 \int_a^b H(x, t; \lambda) f(t) dt + \sum_{h=1}^r C_h \phi_h(x).$$

MISLLANEOUS EXERCISE

1. Prove that every zero of Fredholm function $D(\lambda)$ is a pole of the resolvent kernel

$$R(x, t; \lambda) = D(x, t; \lambda) / D(\lambda).$$

Prove also that the order of this pole is at most equal to the order of the zero of the denominator $D(\lambda)$. [Meerut 2007]

Hint : Proof is contained in Fredholm's second fundamental theorem.

2. State and prove Fredholm's second fundamental theorem.

3. State and prove Fredholm's third fundamental theorem.
4. Use the method of this chapter to find the resolvent kernels for the following integral equations

$$(i) \quad y(x) = f(x) + \lambda \int_0^1 |x - t| y(t) dt \qquad (ii) \quad y(x) = f(x) + \lambda \int_0^1 e^{-|x-t|} y(t) dt$$

$$(iii) \quad y(x) = f(x) + \lambda \int_0^{2\pi} \cos(x+t) y(t) dt$$

5. Show by using the method of this chapter that the resolvent kernel for the integral equation with kernel $K(x, t) = 1 - 3xt$ in the integral $(0, 1)$ is

$$R(x, t; \lambda) = [4 / (4 - \lambda^2)] \times [1 + \lambda - (1/2) \times (x + t) - 3(1 - \lambda)xt], \lambda \neq \pm 2.$$

6. Solve the following homogeneous equations :

$$(i) \quad y(x) = \frac{1}{2} \int_0^\pi \sin(x+t) y(t) dt \qquad (ii) \quad y(x) = \frac{1}{e^2 - 1} \int_0^1 2e^x e^t y(t) dt$$

7. State and prove Fredholm first and second series for non-homogeneous Fredholm integral equation of the second kind. [Meerut 2000, 02]

Hint : Refer results (17) and (30) of Art. 6.2 for Fredholm first and second series respectively.

Integral equations with symmetric kernels

7.1. INTRODUCTION

7.1 (a) Symmetric kernels.

A kernel is called *symmetric* if it coincides with its own complex conjugate. Such a kernel is characterized by the identity

$$K(x, t) = \bar{K}(t, x),$$

where the bar denotes the complex conjugate.

If the kernel is real, then its *symmetry* is defined by the identity $K(x, t) = K(t, x)$.

An integral equation with a symmetric kernel is called a *symmetric equation*.

Remark. We have already discussed eigenvalues and eigenfunctions for integral equations in chapter 3, 4 and 6. We have established that the eigenvalues of an integral equations are the zeros of certain determinant. In this process we have seen that there are many kernels for which there are no eigenvalues (see solved examples 3 and 4 of Art 3.3, chapter 3).

However in this chapter we shall prove that for a symmetric kernel that is not identically zero, at least one eigenvalue will always exist. This is an important characteristic of symmetric kernels.

An eigenvalue is *simple* if there is only one corresponding eigenfunction, otherwise the eigenvalues are *degenerate*. The *spectrum* of the kernel $K(x, t)$ is the set of all its eigenvalues. Thus, as discussed above the *spectrum of a symmetric kernel is never empty*.

7.1 (b). Regularity conditions.

In our study we shall mainly deal with functions which are either continuous, or integrable or square-integrable. When an integral sign is used, the Lebesgue integral is understood. Also, note that if a function is Riemann-integrable, it is also Lebesgue-integrable. By a square-

integrable function $f(x)$, we mean that
$$\int_a^b |f(x)|^2 dx < \infty \quad \dots (1)$$

A square-integrable function $f(x)$ is called an $\mathcal{L}_2$ -function. The regularity conditions on the kernel $K(x, t)$ are identical. It is an $\mathcal{L}_2$ -function if the following three conditions are satisfied :

$$(i) \int_a^b \int_a^b |K(x, t)|^2 dx dt < \infty, \forall x \in [a, b], \forall t \in [a, b], \quad \dots (2)$$

$$(ii) \int_a^b |K(x, t)|^2 dt < \infty, \forall x \in [a, b], \quad \dots (3)$$

$$(iii) \int_a^b |K(x, t)|^2 dx < \infty, \forall t \in [a, b]. \quad \dots (4)$$

Here $\forall$ stands for every (or for all).

7.1. (c) The inner or scalar product of two functions.

The *inner or scalar product* of two complex $\mathcal{L}_2$ -functions ϕ and ψ of real variable x , $a \leq x \leq b$, is denoted by (ϕ, ψ) and is defined as

$$(\phi, \psi) = \int_a^b \phi(x) \bar{\psi}(x) dx. \quad \dots (5)$$

where the bar denotes the complex conjugate.

Two functions are called *orthogonal* if their inner product is zero, that is, ϕ and ψ are orthogonal if

$$(\phi, \psi) = 0 \quad i.e., \quad \int_a^b \phi(x) \bar{\psi}(x) dx = 0. \quad \dots (6)$$

The *norm* of a function $\phi(x)$ is denoted by $\|\phi(x)\|$ and is given by the relation.

$$\|\phi(x)\| = \left[\int_a^b \phi(x) \bar{\phi}(x) dx \right]^{1/2} = \left[\int_a^b |\phi(x)|^2 dx \right]^{1/2}. \quad \dots (7)$$

A function $\phi(x)$ is said to be *normalized* if $\|\phi(x)\| = 1$. It follows that a nonnull function (whose norm is not zero) can always be normalized by dividing it by its norm.

7.1. (d) Schwarz inequality. If $\phi(x)$ and $\psi(x)$ are $\mathcal{L}_2$ -functions, then

$$|(\phi, \psi)| \leq \|\phi\| \|\psi\| \quad \dots (8)$$

Minkowski inequality. If $\phi(x)$ and $\psi(x)$ are $\mathcal{L}_2$ -functions, then

$$\|\phi + \psi\| \leq \|\phi\| + \|\psi\|. \quad \dots (9)$$

7.1. (e) Complex Hilbert space.

(Kanpur 2011)

We present review of some important properties of the complex Hilbert space $\mathcal{L}_2^-(a, b)$. The same discussion will remain applicable to real $\mathcal{L}_2$ space as a particular case.

A linear space of infinite dimension with inner product (or scalar product) (x, y) which is a complex number is called a complex *Hilbert space* if it satisfies the following three axioms :

- (i) the definiteness axiom $(x, x) > 0$ for $x \neq 0$,
- (ii) the linearity axiom $(\alpha x_1 + \beta x_2, y) = \alpha(x_1, y) + \beta(x_2, y)$,

where α and β are arbitrary complex numbers.

- (iii) the axiom of (Hermitian) symmetry $(y, x) = \overline{(x, y)}$,

where the bar denotes the complex conjugate.

Let H be the set of complex-valued functions $\phi(x)$ defined in the interval (a, b) such that

$$\int_a^b |\phi(x)|^2 dx < \infty.$$

Then H is linear and complex Hilbert space $\mathcal{L}_2^-(a, b)$ (or $\mathcal{L}_2^-$). The norm of function generates the *natural metric*

$$d(\phi, \psi) = \|\phi - \psi\| = (\phi - \psi, \phi - \psi)^{1/2}. \quad \dots (10)$$

A metric space is called *complete* if every Cauchy sequence of functions in this space is a convergent sequence. A Hilbert space is an inner-product linear space that is complete in its natural metric. The completeness of $\mathcal{L}_2^-$ spaces plays an important role in the theory of linear operators such as the *Fredholm operator* K , defined as

$$K\phi = \int_a^b K(x, t) \phi(t) dt. \quad \dots (11)$$

$$\text{The operator adjoint to } K \text{ is} \quad \bar{K}\psi = \int_a^b \bar{K}(t, x) \psi(t) dt. \quad \dots (12)$$

$$\text{The operators (11) and (12) are connected as follows :} \quad (K\phi, \psi) = (\phi, \bar{K}\psi). \quad \dots (13)$$

which can be easily proved as follows :

L.H.S. of (13)

$$\begin{aligned} &= (K\phi, \psi) = \int_a^b \left\{ \int_a^b K(x, t) \phi(t) dt \right\} \bar{\psi}(x) dx, \text{ using results (5) and (11)} \\ &= \int_a^b \left\{ \int_a^b K(x, t) \bar{\psi}(x) dx \right\} \phi(t) dt, \text{ changing the order of integration} \\ &= \int_a^b \left\{ \int_a^b K(t, x) \bar{\psi}(t) dt \right\} \phi(x) dx \\ &\quad \text{[Re-naming the variables } x \text{ and } t \text{ as } t \text{ as } x \text{ respectively]} \\ &= \int_a^b \left\{ \overline{\int_a^b \bar{K}(t, x) \psi(t) dt} \right\} \phi(x) dx, \text{ where the bar denotes the complex conjugate} \\ &= (\phi, \bar{K}\psi), \text{ using results (5) and (11)} \\ &= \text{R.H.S. of (13)} \end{aligned}$$

$$\text{For a symmetric kernel, (13) reduces to} \quad (K\phi, \psi) = (\phi, K\psi), \quad \dots (14)$$

that is, a symmetric operator is self-adjoint. Further, we find that permutation of factors in a scalar product is equivalent to taking the complex conjugate, that is, $(\phi, K\phi) = \overline{(K\phi, \phi)}$. Combining this fact with (14), we find that, for a symmetric kernel, the inner product $(K\phi, \phi)$ is always real. The converse of this is also true.

7.1 (f) An orthonormal system of functions.

A finite or an infinite set $\{\phi_k(x)\}$ defined on an interval $a \leq x \leq b$ is said to be an orthogonal set if

$$(\phi_i, \phi_j) = 0 \quad \text{or} \quad \int_a^b \phi_i(x) \phi_j(x) dx = 0, \quad i \neq j. \quad \dots (15)$$

If none of the elements of this set is a zero vector, then it is called a *proper orthogonal* set. The set $\{\phi_i(x)\}$ is *orthonormal* if

$$(\phi_i, \phi_j) = \int_a^b \phi_i(x) \phi_j(x) dx = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \quad \dots (16)$$

Any function $\phi(x)$ for which $\|\phi(x)\| = 1$ is said to be *normalized*.

Given a finite or an infinite (denumerable) independent set of functions $\{\psi_1, \psi_2, \dots, \psi_k, \dots\}$, we can construct an orthonormal set $\{\phi_1, \phi_2, \dots, \phi_k, \dots\}$ by *Gram-Schmidt method* as follows.

$$\text{Let} \quad \phi_1 = \psi_1 / \|\psi_1\|.$$

$$\text{To obtain } \phi_2, \text{ we define} \quad w_2(x) = \psi_2(x) - (\psi_2, \phi_1) \phi_1.$$

Then function w_2 is orthogonal to ϕ_1 . Hence ϕ_2 can be constructed by setting

$\phi_2 = w_2 / \|w_2\|$. Proceeding in this manner, we obtain

$$w_k(x) = \psi_k(x) - \sum_{i=1}^{k-1} (\psi_k, \phi_i) \phi_i, \text{ where } \phi_k = w_k / \|w_k\|.$$

Again, if we are given a set of orthogonal functions, we can convert it into an orthonormal set simply by dividing each function by its norm.

Starting from an arbitrary orthonormal system, it is possible to construct the theory of *Fourier series*. Suppose we wish to obtain the best approximation of an arbitrary function $\psi(x)$ in terms of a linear combination of an orthonormal set $(\phi_1, \phi_2, \dots, \phi_n)$. By the best approximation, we mean that we choose the coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ such as to minimize

$$\left\| \psi - \sum_{i=1}^n \alpha_i \phi_i \right\|, \quad \text{i.e.,} \quad \left\| \psi - \sum_{i=1}^n \alpha_i \phi_i \right\|^2$$

Now, for any $\alpha_1, \alpha_2, \dots, \alpha_n$, we have

$$\left\| \psi - \sum_{i=1}^n \alpha_i \phi_i \right\|^2 = \left\| \psi \right\|^2 + \sum_{i=1}^n |(\psi, \phi_i) - \alpha_i|^2 - \sum_{i=1}^n |(\psi, \phi_i)|^2. \quad \dots (17)$$

Clearly, the minimum can be attained by setting $\alpha_i = (\psi, \phi_i) = a_i$ (say). The numbers a_i are known as *Fourier coefficients* of the functions $\psi(x)$ relative to the orthonormal system $\{\phi_i\}$. In that case, the relation (17) may be written as

$$\left\| \psi - \sum_{i=1}^n \alpha_i \phi_i \right\|^2 = \left\| \psi \right\|^2 - \sum_{i=1}^n |a_i|^2. \quad \dots (18)$$

Since the quantity on the L.H.S. of (18) is nonnegative, we obtain

$$\sum_{i=1}^n |a_i|^2 \leq \left\| \psi \right\|^2,$$

which, for the infinite set $\{\phi_i\}$, leads to the *Bessel inequality*

Assuming that we are given an infinite orthonormal system $\{\phi_i(x)\}$ in $\mathcal{L}_2$ space and a sequence of constants $\{\alpha_i\}$, then the convergence of the series $\sum_{n=1}^{\infty} |\alpha_n|^2$ is clearly a necessary condition for the existence of an $\mathcal{L}_2$ -function $f(x)$ whose Fourier coefficients with respects to the system ϕ_i are α_i . It is to be noted that this condition also turns out to be a sufficient condition as proved in the Riesz Fischer theorem given below.

7.1 (g) Riesz-Fischer Theorem.

[Meerut 2004]

If $\{\phi_i(x)\}$ is a given orthonormal system of functions in $\mathcal{L}_2$ space and $\{\alpha_i\}$ is a given sequence of complex numbers such that the series $\sum_{i=1}^{\infty} |\alpha_i|^2$ converges, then there exists a unique function $f(x)$ for which α_i are the Fourier coefficients with respect to the orthonormal system $\{\phi_i(x)\}$ and to which the Fourier series converges in the mean, that is,

$$\left\| f(x) - \sum_{i=1}^n \alpha_i \phi_i(x) \right\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

Proof. Consider the sequence of partial sums

$$S_n(x) = \sum_{i=1}^n \alpha_i \phi_i(x). \quad \dots (1)$$

and
$$\int_a^b |S_{n+m}(x) - S_n(x)|^2 dx = |\alpha_{n+1}^2| + |\alpha_{n+2}^2| + \dots + |\alpha_{n+m}^2|. \quad \dots (2)$$

Given that the series $\sum_{i=1}^{\infty} |\alpha_i^2|$ converges. Hence, corresponding to each $\epsilon > 0$, there must exist δ_ϵ such that

$$\|S_{n+m}(x) - S_n(x)\| < \epsilon, \quad n > \delta_\epsilon, \quad m \text{ (arbitrary)} \quad \dots (3)$$

Moreover, given that α_i are the Fourier coefficients of the function $f(x)$ with regard to given orthonormal system of functions $\{\phi_i(x)\}$. Hence we may write

$$\|f(x) - \sum_{i=1}^n \alpha_i \phi_i(x)\| = \|f(x)\|^2 + \sum_{i=1}^n |\alpha_i - C_i|^2 \rightarrow 0 \text{ as } n \rightarrow \infty. \quad \dots (5)$$

Now, by Bessel's inequality,
$$\alpha_i = C_i = \int_a^b f(x) \bar{\phi}_i(x) dx. \quad \dots (6)$$

It follows that the Fourier series $\sum_{i=1}^{\infty} C_i \phi_i(x)$ of the function $f(x)$ with regard to the given sequence of system of functions $\{\phi_i(x)\}$ is convergent in the mean to that function, that is,

$$\|f(x) - \sum_{i=1}^n C_i \phi_i(x)\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

7.1 (h) Some useful results.

If an orthonormal system of functions ϕ_i exist in $\mathcal{L}_2$ space in such a manner that every other element of this space can be represented linearly in terms of this system, then it is known as an *orthonormal basis*. It has been proved that the concepts of an orthonormal basis and a complete system of orthogonal functions are equivalent. In fact, if any of following conditions are satisfied, the orthonormal set $\{\phi_1, \phi_2, \phi_n, \dots\}$ is complete.

(i) For every function ψ in $\mathcal{L}_2$ space,
$$\psi = \sum_i (\psi, \phi_i) \phi_i = \sum_i a_i \phi_i \quad [\because a_i = (\psi, \phi_i)]$$

(ii) For every function ψ in $\mathcal{L}_2$ space,
$$\|\psi\|^2 = \sum_{i=1}^{\infty} |(\psi, \phi_i)|^2,$$

which is known as *Parseval's identity*.

(iii) The only function ψ in $\mathcal{L}_2$ space for which the Fourier coefficients vanish is the trivial (or zero) function.

(iv) There exists no function ψ in $\mathcal{L}_2$ space such that $\{\psi, \phi_1, \phi_2, \dots, \phi_n, \dots\}$ is an orthonormal set.

The equivalence of the above four conditions can be easily proved.

7.1. (i) Fourier series of a general character.

In practice, we have to deal with Fourier series of a somewhat more general character. Let $r(t)$ be a continuous, real and nonnegative function in the interval (a, b) . We shall say that the functions $\phi(t)$ and $\psi(t)$ are orthogonal with weight $r(t)$ if

$$\int_a^b r(t) \phi(t) \psi(t) dt = 0.$$

The set of function $\{\phi_1(t), \phi_2(t), \dots, \phi_n(t), \dots\}$ will be said to be orthonormal with weight $r(t)$

if its members satisfy the relation

$$\int_a^b r(t) \phi_i(t) \phi_j(t) dt = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

The Fourier expansions in terms of such functions are dealt with by introducing a new inner product as follows :

$$(\phi, \psi) = \int_a^b r(t) \phi(t) \bar{\psi}(t) dt$$

with the corresponding norm

$$\|\phi\|_r = \left\{ \int_a^b r(t) \phi(t) \bar{\psi}(t) dt \right\}^{1/2}$$

The space of functions for which $\|\phi\|_r < \infty$ is a Hilbert space and all the above results of Art. 7.3 and 7.4 hold.

7.1 (j) Some examples of the complete orthogonal and orthonormal systems.

(i) The system $\phi_n(x) = (2\pi)^{-1/2} e^{inx}$, where n takes every integer value from $-\infty$ to ∞ , is orthonormal in the interval $(-\pi, \pi)$.

(ii) The functions $1, \cos x, \cos 2x, \cos 3x, \dots$ form an orthogonal system in the interval $(0, \pi)$. Again the functions $\sin x, \sin 2x, \sin 3x, \dots$ also form an orthogonal system in $(0, \pi)$.

(iii) The Legendre polynomials given by

$$P_0(x) = 1, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n}, \quad n = 1, 2, 3, \dots$$

are orthogonal in the interval $(-1, 1)$. It can be shown that

$$\int_{-1}^1 P_m(x) P_n(x) dx = \begin{cases} 0, & \text{if } m \neq n \\ 2/(2n+1), & \text{if } m = n \end{cases}$$

(iv) The Chebychev polynomials $T_n(x) = 2^{1-n} \cos(n \cos^{-1} x)$, $n = 0, 1, 2, 3, \dots$ are orthogonal with weight $r(x) = 1/(1-x^2)^{1/2}$ in the interval $(-1, 1)$. They can be normalized by multiplying $T_n(x)$ by the quality $(2^{2n-1}/\pi)^{1/2}$.

(v) Let $J_n(x)$ denote the Bessel function of the first kind and order n , and let $\alpha_{i,n}$ denote its positive zeros, $i = 1, 2, 3, \dots$. Let us assume that $n > -1$. The system of functions $J_n(\alpha_{i,n}x)$, $i = 1, 2, 3, \dots$ is orthogonal with weight function $r(x) = x$ in the interval $(0, 1)$. These functions satisfy the

relation

$$\int_0^1 x J_n(\alpha_{i,n}x) J_n(\alpha_{j,n}x) dx = \begin{cases} 0, & \text{if } i \neq j \\ J_{n+1}^2(\alpha_{i,n}) & \text{if } i = j \end{cases}$$

The systems (i) – (v) are complete. The systems mentioned above play an important role in many practical applications.

7.1 (k) A complete two-dimensional orthonormal set over the rectangle $a \leq x \leq b, c \leq t \leq d$.

Theorem. Let $\{\phi_i(x)\}$ be a complete orthonormal set over $a \leq x \leq b$, and let $\{\psi_i(t)\}$ be a complete orthonormal set over $c \leq t \leq d$. Then, the set $\phi_1(x) \psi_1(t), \phi_1(x) \psi_2(t), \dots, \phi_2(x) \psi_1(t), \dots$ is a complete two-dimensional orthonormal set over the rectangle $a \leq x \leq b, c \leq t \leq d$.

Proof. In order to show that the sequence of two dimensional functions $\{\phi_i(x) \psi_j(t)\}$ in an orthonormal set, one has to integrate over the rectangle $a \leq x \leq b, c \leq t \leq d$. The completeness can be established by showing that every continuous function $f(x, t)$ with finite norm $\|f(x, t)\|$, whose Fourier coefficients with respect to the set $\{\phi_i(x) \psi_j(t)\}$ are all zero vanishes identically over the rectangle $a \leq x \leq b, c \leq t \leq d$.

7.2. SOME FUNDAMENTAL PROPERTIES OF EIGENVALUES AND EIGENFUNCTIONS FOR SYMMETRIC KERNELS.

Theorem I. If a kernel is symmetric then all its iterated kernels are also symmetric.

Proof. Let kernel $K(x, t)$ be symmetric. Then, by definition

$$K(x, t) = \bar{K}(t, x). \quad \dots (1)$$

By definition, the iterated kernels $K_n(x, t)$ $n = 1, 2, 3, \dots$ are defined as follows :

$$K_1(x, t) = K(x, t), \quad \dots (2)$$

$$K_n(x, t) = \int_a^b K(x, z) K_{n-1}(z, t) dz, \quad n = 2, 3, \dots \quad \dots (3)$$

and

$$K_n(x, t) = \int_a^b K_{n-1}(x, z) K(z, t) dz, \quad n = 2, 3, \dots \quad (3)'$$

We shall use mathematical induction to prove the required result.

$$\text{Now, } K_2(x, t) = \int_a^b K(x, z) K_1(z, t) dz, \text{ by (3)}$$

$$= \int_a^b K(x, z) K(z, t) dz, \text{ by (2)} = \int_a^b \bar{K}(z, x) \bar{K}(t, z) dz, \text{ by (1)}$$

$$= \int_a^b \bar{K}(t, z) \bar{K}(z, x) dz = \int_a^b \bar{K}(t, z) \bar{K}_1(z, x) dz, \text{ by (2)}$$

$$= \bar{K}_2(t, x), \text{ by (3)}$$

Thus,

$$K_2(x, t) = \bar{K}_2(t, x),$$

showing that $K_2(x, t)$ is symmetric. Hence the required result is true for $n = 1$ and $n = 2$.

Let $K_n(x, t)$ be symmetric for $n = m$. Then by definition, we have

$$K_m(x, t) = \bar{K}_m(t, x). \quad \dots (4)$$

We shall now prove that $K_n(x, t)$ is also symmetric for $n = m + 1$, i.e.,

$$K_{m+1}(x, t) = \bar{K}_{m+1}(t, x). \quad \dots (5)$$

$$\text{L.H.S. of (5)} = \int_a^b K(x, z) K_m(z, t) dz, \text{ by (3)} = \int_a^b \bar{K}(z, x) \bar{K}_m(t, z) dz, \text{ by (1) and (4)}$$

$$= \int_a^b \bar{K}_m(t, z) \bar{K}(z, x) dz = \bar{K}_{m+1}(t, x), \text{ by (3)' = R. H. S. of (5).}$$

Thus iterated Kernel $K_n(x, t)$ is symmetric for $n = 1$ and $n = 2$. Moreover, $K_n(x, t)$ is symmetric for $n = m + 1$ whenever it is symmetric for $n = m$. Hence, by the mathematical induction, $K_n(x, t)$ is symmetric for $n = 1, 2, 3, \dots$

Remark. Since $K(x, x) = \bar{K}(x, x)$, it follows that the trace $K(x, x)$ of a symmetric kernel is always real. Likewise, we can prove that the traces of all iterated kernels are also real.

Theorem II Hilbert Theorem. Every symmetric kernel with a norm not equal to zero has at least one eigenvalue. [Meerut 2000, 01, 03, 05, 06]

OR

If the kernel $K(x, t)$ is symmetrical and not identically equal to zero, then it has at least one eigen value. [Kanpur 2008]

There are several proofs of this important theorem, each of them serves as the basis of a certain method of calculating approximately the eigen value. One such method will be discussed later on. We shall show that theorem II is a corollary of Art. 7.11 (f) on page 7.47

However, for the present, we shall present the proof the above theorem II for a special case of real kernel. This is stated and proved as follows.

Theorem II A. If the kernel $K(x, t)$ is real, symmetrical and not identically equal zero, then it has at least one eigenvalue. [Meerut 1998, 2001]

***Proof.** From Fredholm's first fundamental theorem if $D(\lambda) \neq 0$, then Fredholm integral

$$\text{equation,} \quad y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt \quad \dots (1)$$

$$\text{has a unique solution given by} \quad y(x) = f(x) + \lambda \int_a^b \frac{D(x, t; \lambda)}{D(\lambda)} f(t) dt \quad \dots (2)$$

To begin with we consider the expansion of $D(x, t; \lambda)/D(\lambda)$ as a power series in λ . Since $D(0) = 1 \neq 0$, it follows that we can determine a constant ρ such that if $|\lambda| < \rho$, then

$$D(\lambda) \neq 0, \quad |\lambda| < \rho. \quad \dots (3)$$

Now, since $D(\lambda)$ is *permanently convergent*, it can be expanded into a power series which will be convergent for $|\lambda| < \rho$. Thus, we have

$$1/D(\lambda) = d_0 + d_1\lambda + d_2\lambda^2 + \dots, \quad |\lambda| < \rho. \quad \dots (4)$$

Since $D(x, t; \lambda)$ is a permanently convergent power series in λ , it follows that the product $\{1/D(\lambda)\} \times D(x, t; \lambda)$ can also be expanded into a power series in λ .

$$\therefore \quad D(x, t; \lambda)/D(\lambda) = \sum_{n=1}^{\infty} g_n(x, t) \lambda^{n-1}. \quad \dots (5)$$

The series in (5) is uniformly convergent with respect to x and t in rectangle $a \leq x \leq b, \quad a \leq t \leq b, \quad \text{for} \quad |\lambda| < \rho$.

Substituting (5) in (2), we have

$$y(x) = f(x) + \lambda \int_a^b \left[\sum_{n=1}^{\infty} \lambda^{n-1} g_n(x, t) \right] f(t) dt, \quad |\lambda| < \rho. \quad \dots (6)$$

Since the series $\sum_{n=1}^{\infty} g_n(x, t) \lambda^n$ is uniformly convergent, the order of the integration and summation in (6) can be interchanged. We, therefore, have

*Reader is advised to consult chapter 6 in order to understand the proof that follows.

$$y(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b g_n(x, t) f(t) dt, \quad |\lambda| < \rho. \quad \dots (7)$$

We known that by the method of successive appronimations (refer Art. 5.7, Chapter 5) the solution of (1) is given by

$$y(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b K_n(x, t) f(t) dt, \quad \dots (8)$$

where $K_n(x, t)$ are the iterated kernels given by

$$\left. \begin{aligned} K_1(x, t) &= K(x, t), \\ K_n(x, t) &= \int_a^b K(x, z) K_{n-1}(z, t) dz. \end{aligned} \right\} \quad \dots (9)$$

and

Then the coefficients $g_n(x, t)$ can be obtained by comparing (7) and (8).

But the solution (8) is valid if $|\lambda| < 1/\{M(b-a)\}$, where M is the maximum value of $|K(x, t)|$ on rectangle $R: a \leq x \leq b, a \leq t \leq b$.

$$\text{Let} \quad r = \min \left(\rho, \frac{1}{M(b-a)} \right). \quad \dots (10)$$

Then, clearly, (7) and (8) will be valid simultaneously for $|\lambda| < r$. Since (1) has unique solution, it follows that (7) and (8) represent the same function. It follows that the coefficients of corresponding powers of λ must be equal.

$$\therefore \quad \int_a^b g_n(x, t) f(t) dt = \int_a^b K_n(x, t) f(t) dt, \quad (x) \quad \dots (11)$$

where the notation (x) is used to mean uniformly as to x .

$$\therefore \quad \int_a^b \{K_n(x, t) - g_n(x, t)\} f(t) dt = 0, \quad (x) \quad \dots (12)$$

Now, $K_n(x, t)$ and $g_n(x, t)$ are independent of $f(x)$ and, for a given value of x ,

$$K_n(x, t) - g_n(x, t) = M(t) \quad \dots (13)$$

is real and continuous function of t . Moreover, since (12) is true for an arbitrary continuous function $f(x)$, we may take $f(t) = M(t)$. Then (12) reduces to

$$\int_a^b \{M(t)\}^2 dt = 0. \quad \dots (14)$$

(14) shows that

$$M(t) \equiv 0 \text{ on } [a, b]. \quad \dots (15)$$

$\therefore$

$$g_n(x, t) = K_n(x, t), \quad (x, t) \quad \dots (16)$$

where the notation (x, t) is used to mean uniformly as to x and t .

$$\text{Using (16), (5) becomes} \quad D(x, t; \lambda) / D(\lambda) = \sum_{n=1}^{\infty} K_n(x, t) \lambda^{n-1}, \quad |\lambda| < r. \quad \dots (17)$$

Since (17) is valid for $t = x$. We, therefore, have

$$D(x, x; \lambda) / D(\lambda) = \sum_{n=1}^{\infty} K_n(x, x) \lambda^{n-1}, \quad |\lambda| < r \quad \dots (18)$$

and $\sum K_n(x, x) \lambda^{n-1}$ is uniformly convergent on $[a, b]$.

Integrating both sides of (18) w.r.t. 'x' from a to b , we have

$$\frac{1}{D(\lambda)} \int_a^b D(x, x; \lambda) dx = \int_a^b \left[\sum_{n=1}^{\infty} K_n(x, x) \lambda^{n-1} \right] dx. \quad \dots (19)$$

Interchanging the order of integration and summation in (19) on account of the uniform convergence of $\sum_{n=1}^{\infty} K_n(x, x) \lambda^{n-1}$, we have

$$\frac{1}{D(\lambda)} \int_a^b D(x, x; \lambda) dx = \sum_{n=1}^{\infty} \lambda^n \int_a^b K_n(x, x) dx \quad \dots (20)$$

In what follows, we shall use the following symbol.

$$\int_a^b K_n(x, x) dx = U_n, \text{ a constant.} \quad \dots (21)$$

We know that (refer Eq. (7) of Art 6.4 in Chapter 6) $\int_a^b D(x, x; \lambda) dx = -D'(\lambda).$ $\dots (22)$

Using (21) and (22), (20) becomes

$$-\frac{D'(\lambda)}{D(\lambda)} = \sum_{n=1}^{\infty} U_n \lambda^n \quad \text{or} \quad \frac{D'(\lambda)}{D(\lambda)} = -\sum_{n=1}^{\infty} U_n \lambda^{n-1}$$

or
$$-\frac{D'(\lambda)}{D(\lambda)} = -\sum_{n=0}^{\infty} U_{n+1} \lambda^n, |\lambda| < r \quad \dots (23)$$

If possible, suppose that $K(x, t)$ has no eigenvalue, that is, that $D(\lambda) = 0$ has no root, real or imaginary. Then the quotient $D'(\lambda)/D(\lambda)$ can be directly expanded into a power series

$$\frac{D'(\lambda)}{D(\lambda)} = \sum_{n=0}^{\infty} C_n \lambda^n, \quad \dots (24)$$

which is clearly a permanently convergent series.

But (23) is valid for $|\lambda| < r$. Hence
$$\sum_{n=0}^{\infty} C_n \lambda^n = \sum_{n=0}^{\infty} U_{n+1} \lambda^n, |\lambda| < r, \quad \dots (25)$$

so that
$$C_n = U_{n+1} \quad \dots (26)$$

and hence $\sum_{n=0}^{\infty} U_{n+1} \lambda^n$ is permanently convergent and so $\sum_{n=0}^{\infty} |U_{n+1}| |\lambda|^n$ is permanently convergent

and also $\sum_{n=0}^{\infty} |U_{n+1}| |\lambda|^{n+1}$ is permanently convergent. $\dots (27)$

It follows that the series formed by omitting any number of terms from (27) is also

permanently convergent. Accordingly,
$$\sum_{n=0}^{\infty} |U_{2n}| |\lambda|^{2n} \quad \dots (28)$$

is permanently convergent.

In the above discussion we have proved that (28) is permanently convergent by assuming that $K(x, t)$ had no eigen value. It follows that if, for a given kernel $K(x, t)$ we can show that the corresponding series (28) is not permanently convergent, we will have proved that $K(x, t)$ has at least one eigenvalue.

Now, we know that if $K(x, t)$ is symmetric, then

$$K_n(x, t) \equiv 0 \text{ on } R; a \leq x \leq b, a \leq t \leq b. \quad \dots (29)$$

Now,
$$U_{2n} = \int_a^b K_{2n}(x, x) dx. \quad \dots (30)$$

Also, we know that $K_{p+q}(x, t) = \int_a^b K_p(x, z) K_q(z, t) dz$ (31)

$$\begin{aligned} \therefore K_{2n}(x, x) &= K_{n+n}(x, x) = \int_a^b K_n(x, z) K_n(z, x) dz, \text{ by (31)} \\ &= \int_a^b K_n(x, z) K_n(x, z) dz, \\ [\because K(x, t) \text{ is symmetric} \Rightarrow K_n(x, t) \text{ is symmetric} \Rightarrow K_n(z, x) &= K_n(x, z)] \\ &= \int_a^b [K_n(x, z)]^2 dz = \int_a^b [K_n(x, t)]^2 dt. \end{aligned}$$

$$\therefore (30) \text{ reduces to } U_{2n} = \int_a^b \int_a^b [K_n(x, t)]^2 dt dx. \quad \dots (32)$$

Again, we have

$$\begin{aligned} K_{2n}(x, x) &= K_{(n-1)+(n+1)}(x, x) = \int_a^b K_{n-1}(x, t) K_{n+1}(t, x) dt, \text{ by (31)} \\ &= \int_a^b K_{n-1}(x, t) K_{n+1}(x, t) dt \\ [\because K_{n+1}(t, x) \text{ is symmetric} \Rightarrow K_{n+1}(t, x) &= K_{n+1}(x, t)] \end{aligned}$$

$$\therefore \text{ from (30), } U_{2n} = \int_a^b \int_a^b K_{n-1}(x, t) K_{n+1}(x, t) dt dx. \quad \dots (33)$$

Now, By Schwarz's inequality*, we have

$$\left[\int_a^b \int_a^b K_{n-1}(x, t) K_{n+1}(x, t) dt dx \right]^2 \leq \left(\int_a^b \int_a^b [K_{n-1}(x, t)]^2 dt dx \right) \times \left(\int_a^b \int_a^b [K_{n+1}(x, t)]^2 dt dx \right). \dots (34)$$

$$\text{Using (31) and (33), (34) becomes } U_{2n}^2 \leq U_{2n-2} U_{2n+2}. \quad \dots (35)$$

Since $U_{2n-2} \neq 0, U_{2n} \neq 0$, dividing both sides of (35) by $U_{2n-2} U_{2n}$, we have

$$\frac{U_{2n+2}}{U_{2n}} \geq \frac{U_{2n}}{U_{2n-2}}. \quad \dots (36)$$

Putting successively $n = 2, 3, \dots, n$ is (36), we obtain

$$\frac{U_{2n+2}}{U_{2n}} \geq \frac{U_{2n}}{U_{2n-2}} \geq \dots \geq \frac{U_6}{U_4} \geq \frac{U_4}{U_2}$$

$$\text{so that } \frac{U_{2n+2}}{U_{2n}} \geq \frac{U_4}{U_2}, (n) \quad \dots (37)$$

where the notation (n) is used to mean uniformly as to n

$$\text{Let } \sum_{n=1}^{\infty} U_{2n} |\lambda|^{2n} \equiv \sum_{n=1}^{\infty} V_n, \quad \dots (38)$$

$$\text{so that } V_n = U_{2n} |\lambda|^{2n} \quad \dots (39)$$

*Schwarz's inequality. If $\phi(x, y)$ and $\psi(x, y)$ are real and continuous on rectangle $R; a \leq x \leq b, a \leq y \leq b$,

$$\text{then } \left[\int_a^b \int_a^b \phi \psi dx dy \right]^2 \leq \left(\int_a^b \int_a^b \phi^2 dx dy \right) \times \left(\int_a^b \int_a^b \psi^2 dx dy \right)$$

We now apply ratio test to the series (38) in order to evaluate the radius of convergence in terms of λ . To, this end, we note that

$$\frac{V_{n+1}}{V_n} = \frac{U_{2n+2}}{U_{2n}} |\lambda|^2 \geq \frac{U_4}{U_2} |\lambda|^2, (n), \text{ using (37)}$$

Hence the series (38) is divergent when

$$\frac{U_4}{U_2} |\lambda|^2 > 1 \quad \text{or} \quad |\lambda| > \sqrt{\frac{U_2}{U_4}}.$$

It follows that the series (38) is not a permanently convergent power series in λ . Then as explained earlier, we have proved the theorem.

Theorem. III. *The eigenvalues of a symmetric kernel are real.*

[Meerut, 2010, 12; Kanpur 2008]

OR

If $K(x, t)$ is real, symmetric, continuous and identically not equal to zero, then all the characteristic constants (eigenvalues) are real. [Kanpur 2009, 10; Meerut 2001, 07, 11]

Proof. Let λ and $\phi(x)$ be an eigenvalue and a corresponding eigenfunction of the kernel $K(x, t)$. Then, by definition

$$\phi(x) = \lambda \int_a^b K(x, t) \phi(t) dt \quad \dots (1)$$

Multiplying (1) by $\bar{\phi}(x)$ and integrating with respect to x from a to b .

$$\int_a^b \phi(x) \bar{\phi}(x) dx = \lambda \int_a^b \left\{ \int_a^b K(x, t) \phi(t) dt \right\} \bar{\phi}(x) dx \quad \dots (2)$$

By definition of Fredholm operator K , we have

$$K\phi = \int_a^b K(x, t) \phi(t) dt \quad \dots (3)$$

$$\text{Also,} \quad \text{norm of } \phi(x) = \|\phi(x)\| = \left\{ \int_a^b \phi(x) \bar{\phi}(x) dx \right\}^{1/2} \quad \dots (4)$$

Using (3), (4) and the definition of inner product (see Art. 7.1 (c)), (2) reduces to

$$\|\phi(x)\|^2 = \lambda (K\phi, \phi) \quad \text{so that} \quad \lambda = \|\phi(x)\|^2 / (K\phi, \phi)$$

Since both the numerator and denominator are real, it follows that λ is also real and thus the required result is proved.

Theorem IV. *The eigenfunctions of a symmetric kernel, corresponding to different eigenvalues are orthogonal*

[Meerut 2001, 08, 09, 10]

OR

The fundamental functions (i.e. eigenfunctions) $\phi_m(x)$ and $\phi_n(x)$ of the symmetric kernel $K(x, t)$ for corresponding eigenvalues λ_m and λ_n ($\lambda_m \neq \lambda_n$) are orthogonal in the domain (a, b) .

(Meerut 2011)

Proof. Since $\phi_m(x)$ and $\phi_n(x)$ are eigenfunctions corresponding to eigenvalues λ_m and λ_n respectively, where $\lambda_m \neq \lambda_n$. Then, by definition, we have

$$\phi_m(x) = \lambda_m \int_a^b K(x, t) \phi_m(t) dt \quad \dots (1)$$

and
$$\phi_n(x) = \lambda_n \int_a^b K(x, t) \phi_n(t) dt \quad \dots (2)$$

Since λ_n is real, (2) may be re-written as
$$\bar{\phi}_n(x) = \lambda_n \int_a^b \bar{K}(x, t), \bar{\phi}_n(t) dt \quad \dots (3)$$

Since $K(x, t)$ is symmetric, we have
$$\bar{K}(x, t) = K(t, x) \quad \dots (4)$$

Using (4), (3) may be re-written as
$$\bar{\phi}_n(x) = \lambda_n \int_a^b K(t, x) \bar{\phi}_n(t) dt \quad \dots (5)$$

Interchanging x and t in (5), we have
$$\bar{\phi}_n(t) = \lambda_n \int_a^b K(x, t) \bar{\phi}_n(x) dx \quad \dots (6)$$

Multiplying both sides of (1) by $\bar{\phi}_n(x)$ and then integrating the both sides w.r.t. 'x' from a to b , we have

$$\begin{aligned} \int_a^b \phi_m(x) \bar{\phi}_n(x) dx &= \lambda_m \int_a^b \left\{ \int_a^b K(x, t) \phi_m(t) dt \right\} \bar{\phi}_n(x) dx \\ &= \lambda_m \int_a^b \left\{ \int_a^b K(x, t) \bar{\phi}_n(x) dx \right\} \phi_m(t) dt \\ &\quad \text{[on changing the order of integration]} \\ &= (\lambda_m / \lambda_n) \int_a^b \phi_m(t) \bar{\phi}_n(t) dt, \text{ by (6)} \end{aligned}$$

$$\therefore \lambda_n \int_a^b \phi_m(x) \bar{\phi}_n(x) dx = \lambda_m \int_a^b \phi_m(x) \bar{\phi}_n(x) dx$$

or
$$(\lambda_n - \lambda_m) \int_a^b \phi_m(x) \bar{\phi}_n(x) dx = 0, \quad i.e., \quad (\lambda_n - \lambda_m) (\phi_m, \phi_n) = 0 \quad \dots (7)$$

Since $\lambda_n \neq \lambda_m$, $(\lambda_n - \lambda_m) \neq 0$ and so (7) reduces to $(\phi_m, \phi_n) = 0$, showing that the eigenfunctions ϕ_m and ϕ_n are orthogonal.

Theorem V. *The multiplicity of any nonzero eigenvalue is finite for every symmetric kernel for which $\int_a^b \int_a^b |K(x, t)|^2 dx dt$ is finite.*

Proof. Let the functions $\phi_{1\lambda}(x), \phi_{2\lambda}(x), \dots, \phi_{n\lambda}(x) \dots$ be the linearly independent eigenfunctions which correspond to a non-zero eigenvalue λ . Using the Gram-Schmidt procedure, we can find linear combinations of these functions which form an orthonormal system $\{u_{k\lambda}(x)\}$. Then, the corresponding complex conjugate system $\{\bar{u}_{k\lambda}(x)\}$ also forms an orthonormal system. Let

$$K(x, t) \sim \sum_i a_i \bar{u}_{i\lambda}(t), \quad \text{where} \quad a_i = \int_a^b K(x, t) u_{i\lambda}(t) dt = \lambda^{-1} u_{i\lambda}(x), \quad \dots (1)$$

be the series associated with kernel $K(x, t)$ for a fixed x . Applying Bessel's inequality to this

series, we get
$$\int_a^b |K(x, t)|^2 dt \geq \sum_i \frac{1}{\lambda^2} |u_{i\lambda}(x)|^2, \quad \dots (2)$$

Integrating both sides of (2) with respect to x , we have

$$\int_a^b \int_a^b |K(x, t)|^2 dx dt \geq \sum_i \frac{1}{\lambda_i^2} \quad \text{or} \quad \int_a^b \int_a^b |K(x, t)|^2 dx dt \geq \frac{m}{\lambda^2}, \quad \dots (3)$$

where m is the multiplicity of λ . Since the L.H.S. of (3) is finite, it follows that m is finite.

Theorem VI. *The sequence of eigenfunctions of a symmetric kernel can be made orthonormal.*

Proof. Corresponding to a certain eigenvalue, let there be m linearly independent eigenfunctions. Using the linearity property of the integral operator, every linear combination of these functions is also an eigenfunction. Hence, on applying the well known Gram-Schmidt procedure, we can get equally numerous eigenfunctions which are orthonormal. Again, for different eigenvalues, the corresponding eigenfunctions are orthogonal and can be easily normalized. Combining these two facts, the complete proof of the Theorem VI follows.

Theorem VII. *The eigenvalues of a symmetric $\mathcal{L}_2$ -kernel form a finite or an infinite sequence $\{\lambda_n\}$ with no finite limit point.*

Proof. On including each eigenvalue in the sequence a number of times equal to its multiplicity, we have

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \leq \int_a^b \int_a^b |K(x, t)|^2 dx dt. \quad \dots (1)$$

Let $\{u_k(x)\}$ be the orthonormal eigenfunctions corresponding to different (nonzero) eigenvalues λ_i . Then proceeding as in theorem V and using the Bessel inequality, we obtain

$$\sum_i \frac{1}{\lambda_i^2} \leq \int_a^b \int_a^b |K(x, t)|^2 dx dt < \infty. \quad \dots (2)$$

Hence, if there exists an enumerable infinity of λ_i , then we must have

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty, \quad \dots (3)$$

Therefore $\lim (1/\lambda_i) \rightarrow 0$ and ∞ is the only limit point of the eigenvalues.

Theorem VIII *The set of eigenvalues of the second iterated kernel coincide with the set of squares of the eigenvalues of the given kernel.*

Proof. Let λ be an eigenvalue of kernel $K(x, t)$ corresponding to the eigenfunction $\phi(x)$. Then, by definition, we have

$$\phi = \lambda K \phi \quad \text{or} \quad (I - \lambda K) \phi = 0, \quad \dots (1)$$

where I is the identity operator.

Operating both sides of (1) with the operator $(I + \lambda K)$, we obtain

$$(I - \lambda^2 K^2) \phi = 0 \quad \text{or} \quad \phi(x) = \lambda^2 \int_a^b K_2(x, t) \phi(t) dt, \quad \dots (2)$$

showing that λ^2 is an eigenvalue of the Kernel $K_2(x, t)$.

Conversely, let $\mu = \lambda^2$ be an eigenvalue of the kernel $K_2(x, t)$ with $\phi(x)$ as the corresponding eigenfunction. Then, we have

$$(I - \lambda^2 K^2) \phi = 0 \quad \text{or} \quad (I - \lambda K)(I + \lambda K) \phi = 0. \quad \dots (3)$$

If λ is an eigenvalue of kernel K , then the theorem VIII is proved. If not, let us suppose that

$$(I + \lambda K) \phi = \Psi(x) \quad \dots (4)$$

$$\text{Using (4), (3) reduces to} \quad (I - \lambda K) \Psi(x) = 0. \quad \dots (5)$$

Since λ is not an eigenvalue of K by our assumption, (5) shows that $\Psi(x) \equiv 0$, or equivalently $(I + \lambda K)\phi = 0$. Thus, $-\lambda$ is an eigenvalue of the kernel K and thus the theorem is proved.

Remark I. The result can be extended to the n th iterate. The set of eigenvalues of the kernel $K_n(x, t)$ coincide with the set of n th powers of the eigenvalues of the kernel $K(x, t)$.

Remark 2. While proving above theorem VIII, the symmetry of the kernel $K(x, t)$ has not been assumed.

Theorem IX. If λ_1 is the smallest eigenvalue of the kernel K , then

$$\frac{1}{|\lambda_1|} = \max \left[\frac{(K\phi, \phi)}{\|\phi\|^2} \right] = \max \frac{\left| \int_a^b \int_a^b K(x, t) \phi(t) \bar{\phi}(x) dt dx \right|}{\|\phi\|^2} \quad \dots (17)$$

$$\text{or, equivalently,} \quad 1/\|\lambda_1\| = \max (K\phi, \phi), \quad \|\phi\| = 1. \quad \dots (18)$$

Proof. The maximum value is attained when $\phi(x)$ is an eigenfunction of the symmetric $\mathcal{L}_2$ -kernel corresponding to the smallest eigenvalue of the kernel K as shown in Art. 7.11 (f) on page 7.47.

7.3. EXPANSION IN EIGENFUNCTIONS AND BILINEAR FORM

In this article we propose to deal with results concerning the expansion of a symmetric kernel and of functions represented in some sense by the kernel, in terms of its eigenfunctions and the eigenvalues.

Let $K(x, t)$ be a nonnull, symmetric kernel which has a finite or an infinite number of eigenvalues (always real and nonzero). We order them in the sequence

$$\lambda_1, \lambda_2, \dots, \lambda_n \quad \dots (1)$$

in such a manner that each eigenvalue is repeated as many times as its multiplicity. Also, we denominate these eigenvalues in the order that corresponds to their absolute values, that is,

$$0 \leq |\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_n| \leq |\lambda_{n+1}| \leq \dots$$

$$\text{Let} \quad \phi_1(x), \phi_2(x), \dots, \phi_n(x), \dots \quad \dots (2)$$

be the sequence of eigenfunctions corresponding to the eigenvalues gives by the sequence (1). We further agree to arrange these eigenfunctions in such a manner that they are not repeated and are linearly independent in each group corresponding to the same eigenvalue. It follows that to each eigenvalues λ_k in (1) there corresponds just one eigenfunction $\phi_k(x)$ in (2). Suppose that these eigenfunctions have been orthonormalized (refer theorem VI of Art. 7.2).

Suppose that a symmetric $\mathcal{L}_2$ -kernel has at least one eigenvalue, say λ_1 . Then $\phi_1(x)$ is the corresponding eigenfunction. It follows that the *second* “truncated” symmetric kernel

$$K^{(2)}(x, t) = K(x, t) - \frac{\phi_1(x) \bar{\phi}_1(t)}{\lambda_1} \quad \dots (3)$$

is nonnull and it will also possess at least one eigenvalue λ_2 (we select the smallest in case there are more than one eigenvalues) with corresponding normalized eigenfunction $\phi_2(x)$. The function $\phi_1(x) \neq \phi_2(x)$ even if $\lambda_1 = \lambda_2$, since

$$\int_a^b K^{(2)}(x, t) \phi_1(t) dt = \int_a^b K(x, t) \phi_1(t) dt - \frac{\phi_1(x)}{\lambda_1} \int_a^b \phi_1(t) \bar{\phi}_1(t) dt = \frac{\phi_1(x)}{\lambda_1} - \frac{\phi_1(x)}{\lambda_1} = 0.$$

Similarly, the third “truncated” symmetric kernel

$$K^{(3)}(x, t) = K^{(2)}(x, t) - \frac{\phi_2(x) \bar{\phi}_2(t)}{\lambda_2} = K(x, t) - \sum_{k=1}^2 \frac{\phi_k(x) \bar{\phi}_k(t)}{\lambda_k} \quad \dots (4)$$

yields the third eigenvalue λ_3 and the corresponding normalized eigenfunction $\phi_3(x)$.

Proceeding likewise we finally arrive at the two following possibilities :

(i) the above process terminates after n steps, that is, $K^{(n+1)}(x, t) \equiv 0$, and the kernel $K(x, t)$ is

a degenerate kernel, given by
$$K(x, t) = \sum_{k=1}^n \frac{\phi_k(x) \bar{\phi}_k(t)}{\lambda_k}, \quad \dots (5)$$

(ii) the above process can be continued indefinitely and there are infinite number of eigenvalues and eigenfunctions.

Remark 1. We have denoted the least eigenvalue and the corresponding eigenfunction of $K^{(n)}(x, t)$ as λ_n and ϕ_n , which are the n th eigenvalue and the n th eigenfunction in the sequences (1) and (2). This fact can be justified with help of theorem II which is given below.

Remark 2. We shall investigate in the next article whether the bilinear form (5) is valid for the case when the kernel $K(x, t)$ has infinite eigenvalues and eigenfunctions.

Theorem I. Let the sequence $\{\phi_k(x)\}$ be all the eigenfunctions of a symmetric $\mathcal{L}_2$ -kernel, with $\{\lambda_k\}$ as the sequence of corresponding eigenvalues. Then, the series $\sum_{n=1}^{\infty} \frac{|\phi_n(x)|^2}{\lambda_n^2}$ converges

and its sum is bounded by C_1^2 , which is an upper bound of the integral $\int_a^b |K^2(x, t)| dt$.

Proof. The Fourier coefficients a_n of the function $K(x, t)$ with fixed x , with respect to the orthonormal system $\{\bar{\phi}_n(x)\}$ are given by

$$a_n = \int_a^b K(x, t) \phi_n(t) dt = \frac{\phi_n(x)}{\lambda_n}. \quad \dots (6)$$

Using Bessel's inequality, we now obtain

$$\sum_{n=1}^{\infty} \frac{|\phi_n(x)|^2}{\lambda_n^2} \leq \int_a^b |K(x, t)|^2 dt \leq C_1^2. \quad \dots (7)$$

Theorem II. Let the sequence $\phi_n(x)$ be all the eigenfunctions of a symmetric kernel $K(x, t)$, with $\{\lambda_n\}$ as the corresponding eigenvalues. Then, the truncated kernel

$$K^{(n+1)}(x, t) = K(x, t) - \sum_{m=1}^n \frac{\phi_m(x) \bar{\phi}_m(t)}{\lambda_m}$$

has the eigenvalues $\lambda_{n+1}, \lambda_{n+2}(x), \dots$, to which correspond the eigenfunctions $\phi_{n+1}(x), \phi_{n+2}(x), \dots$. The kernel $K^{(n+1)}(x, t)$ has no other eigenvalues or eigenfunctions.

Proof. (i) We begin with fact that the integral equation

$$\phi(x) - \lambda \int_a^b K^{(n+1)}(x, t) \phi(t) dt = 0 \quad \dots (8)$$

is equivalent to

$$\phi(x) - \lambda \int_a^b K(x, t) \phi(t) dt + \lambda \sum_{m=1}^n \frac{\phi_m(x)}{\lambda_m} (\phi, \phi_m) dt = 0 \quad \dots (9)$$

Writing $\lambda = \lambda_j$ and $\phi(x) = \phi_j(x)$, $j \geq n+1$ on L.H.S. of (9) and using the orthogonality condition, we obtain

$$\phi_j(x) - \lambda_j \int_a^b K(x, t) \phi_j(t) dt = 0, \quad \dots (10)$$

Hence $\phi_j(x)$ and λ_j for $j \geq n+1$ are eigenfunctions and eigenvalues of the kernel $K^{(n+1)}(x, t)$.

(ii) Let λ and $\phi(x)$ be an eigenvalue and eigenfunction of the kernel $K^{(n+1)}(x, t)$ so that

$$\phi(x) - \lambda K \phi(x) + \lambda \sum_{m=1}^n \frac{\phi_m(x)}{\lambda_m} (\phi, \phi_m) = 0. \quad \dots (11)$$

Taking the scalar product of (11) with $\phi_j(x)$, $j \leq n$ and using the orthonormality of the $\phi_j(x)$, we have

$$(\phi, \phi_j) - \lambda (K\phi, \phi_j) + (\lambda / \lambda_j) \times (\phi, \phi_j) = 0. \quad \dots (12)$$

Now,

$$(K\phi, \phi_j) = (\phi, K\phi_j) = (1 / \lambda_j) \times (\phi, \phi_j). \quad \dots (13)$$

Using (13), (12) reduces to

$$(\phi, \phi_j) + (\lambda / \lambda_j) \times \{(\phi, \phi_j) - (\phi, \phi_j)\} \Rightarrow (\phi, \phi_j) = 0. \quad \dots (14)$$

In view of (14), we find that the last term in the L.H.S. of (11) vanishes and hence (11)

reduces to

$$\phi(x) - \lambda \int_a^b K(x, t) \phi(t) dt = 0, \quad \dots (15)$$

showing that λ and $\phi(x)$ are eigenvalue and eigenfunction of the kernel $K(x, t)$ and that $\phi \neq \phi_j$, $j \leq n$. In fact, we see that ϕ is orthogonal to all ϕ_j , $j \leq n$, and $\phi(x)$ and λ are always contained in the sequences $\{\phi_k(x)\}$ and $\{\lambda_k\}$, $k \geq n+1$, respectively.

Remark. Combining the results of the above two theorems I and II, we easily find that, if the symmetric kernel K has only a finite number of eigenvalues, then it must be separable. The proof follows by noting that $K^{(n+1)}(x, t)$ then has no eigenvalues and therefore it must be null. Hence,

we must have

$$K(x, t) = \sum_{m=1}^n \frac{\phi_m(x) \bar{\phi}_m(t)}{\lambda_m}$$

In chapter 4, we have already proved that every separable kernel has only a finite number of eigenvalues. Combining these two results, we have the following theorem.

Theorem III. A necessary and sufficient condition for a symmetric $\mathcal{L}_2$ kernel to be separable is that it must possess a finite number of eigenvalues.

7.4. HILBERT-SCHMIDT THEOREM. [Kanpur 2008, 09, 10; Meerut 2000, 01, 02, 07]

If $f(x)$ can be written in the form

$$f(x) = \int_a^b K(x, t) h(t) dt \quad \dots (1)$$

where $K(x, t)$ is a symmetric $\mathcal{L}_2$ -kernel and $h(t)$ is an $\mathcal{L}_2$ -function, then $f(x)$ can be expanded in an absolutely and uniformly convergent Fourier series with respect to the orthonormal system of eigenfunctions $\phi_1(x), \phi_2(x), \dots, \phi_n(x), \dots$ of the kernel $K(x, t)$:

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x), \quad \dots (2)$$

where

$$f_n = (f, \phi_n). \quad \dots (3)$$

The Fourier coefficients of the function $f(x)$ are related to the Fourier coefficients h_n of the function $h(x)$ by the relations

$$f_n = h_n / \lambda_n \quad \dots (4)$$

and

$$h_n = (h, \phi_n), \quad \dots (5)$$

where λ_n are the eigenvalues of the kernel $K(x, t)$.

Proof. Let $K(x, t)$ be a nonnull, symmetric kernel which has a finite or an infinite number of eigenvalues (always real and non-zero). We order them in the sequence

$$\lambda_1, \lambda_2, \dots, \lambda_n, \dots \quad \dots (6)$$

in such a way that each eigenvalue is repeated as many times as its multiplicity. We further agree to denumerate these eigenvalues in the order that corresponds to their absolute values, i.e.,

$$0 < |\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_n| \leq |\lambda_{n+1}| \leq \dots$$

$$\text{Let } \phi_1(x), \phi_2(x), \dots, \phi_n(x), \dots \quad \dots (7)$$

be the sequence of eigenfunctions corresponding to the eigenvalues given by the sequence (6) and arranged in such a way that they are no longer repeated and are linearly independent in each group corresponding to the same eigenvalue. Thus, to each eigenvalue λ_k in (6) there corresponds just one eigenfunction $\phi_k(x)$ in (2). Further, suppose that eigenfunctions $\phi_k(x)$ in (2) have been orthonormalized. (refer theorem VI of Art 7.2)

Now, the Fourier coefficients of the function $f(x)$ with respect to the orthonormal system $\{\phi_k(x)\}$ are given by

$$f_n = (f, \phi_n) = (Kh, \phi_n) = (h, K\phi_n) = (1/\lambda_n) \times (\lambda, \phi_n) = h_n / \lambda_n,$$

by using the self-adjoint property of the operator and the relation $\lambda_n K\phi_n = \phi_n$. Hence, the Fourier series for $f(x)$ is given by

$$f(x) \sim \sum_{n=1}^{\infty} f_n \phi_n(x) = \sum_{n=1}^{\infty} \frac{h_n}{\lambda_n} \phi_n(x). \quad \dots (8)$$

We now estimate the remainder term of the series (8) as shown below.

$$\left| \sum_{k=n+1}^{n+p} h_k \frac{\phi_k(x)}{\lambda_k} \right|^2 \leq \sum_{k=n+1}^{n+p} h_k^2 \sum_{k=n+1}^{n+p} \frac{|\phi_k(x)|^2}{\lambda_k^2} \leq \sum_{k=n+1}^{n+p} h_k^2 \sum_{k=1}^{\infty} \frac{\phi_k^2(x)}{\lambda_k^2}. \quad \dots (9)$$

Using the relation (7) of Art. 7.3, we see that the above series is bounded. Moreover, since

$h(x)$ is an $\mathcal{L}_2$ -function, it follows that the series $\sum_{k=1}^{\infty} h_k^2$ is convergent and the partial sum $\sum_{k=n+1}^{n+p} h_k^2$ can be made arbitrarily small. Hence, the series (8) converges absolutely and uniformly.

We now proceed to show that the series (8) converges to $f(x)$ in the mean. For this purpose,

$$\text{we denote its partial sum as } \psi_n(x) = \sum_{m=1}^n \frac{h_m}{\lambda_m} \phi_m(x) \quad \dots (10)$$

and estimate the value of $\|f(x) - \psi_n(x)\|$.

$$\text{Now, } f(x) - \psi_n(x) = Kh - \sum_{m=1}^n \frac{h_m}{\lambda_m} \phi_m(x) = Kh - \sum_{m=1}^n \frac{(h, \phi_m)}{\lambda_m} \phi_m(x) = K^{(n+1)}h, \quad \dots (11)$$

where $K^{(n+1)}$ is the truncated kernel. From (11), we have

$$\begin{aligned} \|f(x) - \psi_n(x)\|^2 &= \|K^{(n+1)}h\|^2 = (K^{(n+1)}h, K^{(n+1)}h) \\ &= (h, K^{(n+1)}K^{(n+1)}h) = (h, K_2^{(n+1)}h), \end{aligned} \quad \dots (12)$$

where we have used the self-adjoint property of the kernel $K^{(n+1)}$ and also the relation

$$K^{(n+1)}K^{(n+1)} = K_2^{(n+1)}.$$

We know that the set of eigenvalues of the second iterated kernel coincide with the set of squares of the eigenvalues of the given kernel (refer theorem VIII of Art. 7.2). Using this property and theorem II of Art. 7.3., we see that the least eigenvalue of the kernel $K_2^{(n+1)}$ is equal to λ_{n+1}^2 . Again, using the theorem IX of Art 7.2., we obtain

$$\frac{1}{\lambda_{n+1}^2} = \max \left[\frac{(h, K_2^{(n+1)}h)}{(h, h)} \right], \quad \dots (13)$$

where we have omitted the modulus sign from the scalar product $(h, K_2^{(n+1)}h)$, because it is a positive quantity.

Combining (12) and (13), we obtain

$$\|f(x) - \psi_n(x)\|^2 = (h, K_2^{(n+1)}h) \leq (h, h) / \lambda_{n+1}^2. \quad \dots (14)$$

$$\text{Since } \lambda_{n+1} \rightarrow \infty, \quad (14) \Rightarrow \|f(x) - \psi_n(x)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad \dots (15)$$

Now, we have the relation

$$\|f(x) - \psi(x)\| \leq \|f(x) - \psi_n(x)\| + \|\psi_n(x) - \psi(x)\|, \quad \dots (16)$$

where $\psi(x)$ is the limit of the series with partial sum ψ_n .

As shown above, the first terms on the R.H.S. of (16) tends to zero. To show that the second term of R.H.S. of (16) also tends to zero, we proceed as follows :

Since the series (8) converges uniformly, we have, for an arbitrarily small and positive ϵ ,

$$|\psi_n(x) - \psi(x)| < \epsilon, \quad \text{when } n \text{ is sufficiently large.}$$

$$\text{Therefore, } \|\psi_n(x) - \psi(x)\| < \epsilon (b-a)^{1/2} \Rightarrow \|\psi_n(x) - \psi(x)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, we see that both the terms on R.H.S. of (16) tend to zero as $n \rightarrow \infty$. Therefore, from (16), we have $f(x) = \psi(x)$ as required.

Remark. It is to be noted that we assumed neither the convergence of the Fourier series $h(x)$ nor the completeness of the orthonormal system while proving Hilbert-Schmidt theorem.

We now show that the Hilbert-Schmidt theorem easily leads to the bilinear form of the type (5) given in Art. 7.3. By definition, we have

$$K_m(x, t) = \int_a^b K(x, z) K_{m-1}(z, t) dz, \quad m = 2, 3, \dots, \quad \dots (17)$$

which is of the form (1) with $h(x) = K_m(x, t)$; t fixed. The Fourier coefficient $a_k(t)$ of $K_m(x, t)$ with respect to the system of eigenfunctions $\{\phi_k(x)\}$ of $K(x, t)$ is given by

$$a_k(t) = \int_a^b K_m(x, t) \bar{\phi}_k(x) dx = \lambda_k^{-m} \bar{\phi}_k(t)$$

Hence, with help of Hilbert-Schmidt theorem, we find that all the iterated kernels $K_m(x, t)$; $m \geq 2$, of a symmetric $\mathcal{L}_2$ -kernel $K(x, t)$ can be represented by the absolutely and uniformly convergent series

$$K_m(x, t) = \sum_{k=1}^{\infty} \lambda_k^{-m} \phi_k(x) \bar{\phi}_k(t) \quad \dots (18)$$

Replacing t by x in (18) and then integrating with respect to x from a to b , we have

$$\int_a^b K_m(x, x) dx = \sum_{k=1}^{\infty} \lambda_k^{-m} \left\{ \int_a^b \phi_k(x) \bar{\phi}_k(x) dx \right\} \quad \dots (19)$$

$$\text{But} \quad \left(\int_a^b \phi_k(x) \bar{\phi}_k(x) dx \right)^{1/2} = \|\phi_k(x)\| = 1, \quad \dots (20)$$

because $\phi_k(x)$ is a normalized function

Using (20), (19) reduces to

$$\sum_{k=1}^{\infty} \lambda_k^{-m} = \int_a^b K_m(x, x) dx = A_m, \quad \dots (21)$$

where A_m is the *trace of the iterated kernel* K_m .

Using the Riesz-Fischer theorem and relation (21) with $m = 2$, it follows that the series

$$\sum_{k=1}^{\infty} \frac{\phi_k(x) \bar{\phi}_k(t)}{\lambda_k} \quad \dots (22)$$

converges in the mean to a symmetric $\mathcal{L}_2$ -kernel $K(x, t)$ which, treated as a Fredholm kernel, possesses exactly the sequence of numbers $\{\lambda_k\}$ as eigenvalues.

7.5. DEFINITE KERNELS AND MERCER'S THEOREM

Nonnegative-definite kernel. Definition. A symmetric $\mathcal{L}_2$ -kernel $K(x, t)$ is said to be *nonnegative-definite* if $(K\phi, \phi) \geq 0$ for every $\mathcal{L}_2$ -function ϕ .

Positive-definite kernel. Definition. A symmetric $\mathcal{L}_2$ -kernel $K(x, t)$ is said to be *positive definite* if $(K\phi, \phi) \geq 0$ and $(K\phi, \phi) = 0 \Rightarrow \phi$ is null.

Nonpositive-definite kernel. Definition. A symmetric $\mathcal{L}_2$ -kernel $K(x, t)$ is said to be *nonpositive-definite* if $(K\phi, \phi) \leq 0$ for every $\mathcal{L}_2$ -function ϕ .

Negative-definite kernel. Definition. A symmetric $\mathcal{L}_2$ -kernel $K(x, t)$ is said to be *negative-definite* if $(K\phi, \phi) \leq 0$ and $(K\phi, \phi) = 0 \Rightarrow \phi$ is null.

Indefinite kernel. A symmetric kernel that does not fall into any of the above mentioned four types of kernels, is known as *indefinite kernel*.

The following theorem is an immediate consequence of the Hilbert-Schmidt theorem.

Theorem (i) *A nonnull, symmetric $\mathcal{L}_2$ -kernel $K(x, t)$ is nonnegative if and only if all its eigenvalues are positive*

(ii) It is positive definite if and only if the above condition is satisfied and, in addition, some (and therefore every) full orthonormal system of eigen functions of $K(x, t)$ is complete.

Proof (i) Using the Hilbert-Schmidt theorem, we have

$$f(x) = K\phi = \sum_{n=1}^{\infty} \frac{(\phi, \phi_n)}{\lambda_n} \phi_n(x) \quad \dots (1)$$

Taking the inner product of both sides of (1) with ϕ , we have

$$(K\phi, \phi) = \sum_{n=1}^{\infty} \frac{|\phi, \phi_n|^2}{\lambda_n} \quad \dots (2)$$

If $\lambda_n > 0$ for each n , then $(2) \Rightarrow (K\phi, \phi) \geq 0$ for all ϕ .

In addition, if any $\lambda_n < 0$, then $(K\phi_n, \phi_n) = \lambda_n^{-1} < 0$.

Thus, the part (i) of the theorem is proved.

(ii) Let $K(x, t)$ be nonnegative-definite. From (2), we have

$$(K\phi, \phi) = 0 \Leftrightarrow (\phi, \phi_n) = 0 \text{ for all } n,$$

showing that the kernel $K(x, t)$ will be positive-definite if and only if

$$(\phi, \phi_n) = 0 \text{ for all } n \Rightarrow \phi \equiv 0.$$

Now using the condition (refer condition (iv), Art. 7.1 (h)) for the completeness of an orthonormal system, part (ii) of the theorem easily follows.

We now state without proof, the following theorem.

Mercer's theorem. *If a nonnull, symmetric $\mathcal{L}_2$ -kernel is quasi-definite (that is, when all but a finite number of eigenvalues are of one sign) and continuous, then the series $\sum_{n=1}^{\infty} \lambda_n^{-1}$ is convergent*

and

$$K(x, t) = \sum_{n=1}^{\infty} \frac{\phi_n(x) \bar{\phi}_n(t)}{\lambda_n},$$

the series being uniformly and absolutely convergent.

Remark. The result of the above theorem gives the exact conditions for the bilinear form (5) given in Art. 7.3 to be extended to an infinite series.

We have to note that the continuity of the kernel is an important condition for the theorem to be true.

7.6. SCHMIDT'S SOLUTION OF NON-HOMOGENEOUS FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

[Meerut 2001, 02, 04, 06]

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt, \quad \dots (1)$$

where $K(x, t)$ is continuous, real and symmetric and λ is not an eigenvalue.

Statement of Hilbert-Schmidt theorem :

Let $F(x)$ be generated from a continuous function $y(x)$ by the operator

$$\lambda \int_a^b K(x, t) y(t) dt,$$

where $K(x, t)$ is continuous, real and symmetric, so that

$$F(x) = \lambda \int_a^b K(x, t) y(t) dt. \quad \dots (2)$$

Then $F(x)$ can be represented over interval (a, b) by a linear combination of the normalized eigenfunctions of homogeneous integral equation

$$y(x) = \lambda \int_a^b K(x, t) y(t) dt, \quad \dots (3)$$

having $K(x, t)$ as its kernel.

Procedure of solution of (1). Re-writing (1), we have

$$y(x) - f(x) = \lambda \int_a^b K(x, t) y(t) dt. \quad \dots (4)$$

Since (4) is of the form (2), it follows by Hilbert-Schmidt theorem

$$y(x) - f(x) = \sum_{m=1}^{\infty} a_m \phi_m(x), \quad a \leq x \leq b. \quad \dots (4)'$$

where $\phi_m(x)$ ($m = 1, 2, 3, \dots$) are the normalized eigenfunctions of homogeneous integral equation

$$y(x) = \lambda \int_a^b K(x, t) y(t) dt \quad \dots (5)$$

Let λ_m ($m = 1, 2, 3, \dots$) be the corresponding eigenvalues of (5)

$$\text{Let} \quad \lambda \neq \lambda_m, \quad \forall m = 1, 2, 3, \dots \quad \dots (6)$$

Since $\phi_m(x)$ ($m = 1, 2, \dots$) are normalized, we have

$$\int_a^b \phi_m(x) \phi_n(x) dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases} \quad \dots (7)$$

Multiplying both sides of (4)' by $\phi_m(x)$ and then integrating w.r.t. 'x' from a to b , we get

$$\begin{aligned} \int_a^b y(x) \phi_m(x) dx - \int_a^b f(x) \phi_m(x) dx \\ = a_1 \int_a^b \phi_1(x) \phi_m(x) dx + a^2 \int_a^b \phi_2(x) \phi_m(x) dx + \dots + a_m \int_a^b \phi_m(x) \phi_m(x) dx + \dots \end{aligned} \quad \dots (8)$$

$$\text{Let} \quad C_m = \int_a^b y(x) \phi_m(x) dx \quad \dots (9)$$

and

$$f_m = \int_a^b f(x) \phi_m(x) dx. \quad \dots (10)$$

$$\text{Making use of (7), (9) and (10) in (8), we obtain} \quad C_m - f_m = a_m \quad \dots (11)$$

Now, multiplying both sides of (1) by $\phi_m(x)$ and then integrating w.r.t. 'x' from a to b , we have

$$\int_a^b y(x) \phi_m(x) dx = \int_a^b f(x) \phi_m(x) dx + \lambda \int_a^b \left\{ \int_a^b K(x, t) y(t) dt \right\} \phi_m(x) dx$$

$$\text{or} \quad C_m = f_m + \lambda \int_a^b y(t) \left\{ \int_a^b K(x, t) \phi_m(x) dx \right\} dt \quad [\text{using (9) and (10) and also} \\ \text{changing the order of integration in double integral on R.H.S}]$$

$$\text{or} \quad C_m = f_m + \lambda \int_a^b y(t) \left\{ \int_a^b K(t, x) \phi_m(x) dx \right\} dt \quad \dots (12)$$

[since $K(x, t)$ is symmetric, so $K(x, t) = K(t, x)$]

Since $\phi_m(x)$ is eigenfunction corresponding to the eigenvalue λ_m of (5), by definition, we get

$$\begin{aligned}
& \phi_m(x) = \lambda_m \int_a^b K(x, t) \phi_m(t) dt \quad \text{or} \quad \phi_m(x) = \lambda_m \int_a^b K(x, z) \phi_m(z) dz. \\
\text{or} \quad & \phi_m(t) = \lambda_m \int_a^b K(t, z) \phi_m(z) dz \quad \text{or} \quad \phi_m(t) = \lambda_m \int_a^b K(t, x) \phi_m(x) dx \\
\text{or} \quad & \int_a^b K(t, x) \phi_m(x) dx = \frac{\phi_m(t)}{\lambda_m}. \quad \dots (13)
\end{aligned}$$

Using (13), (12) reduces to

$$\begin{aligned}
C_m &= f_m + \lambda \int_a^b \frac{y(t) \phi_m(t)}{\lambda_m} dt \quad \text{or} \quad C_m = f_m + \frac{\lambda}{\lambda_m} \int_a^b y(x) \phi_m(x) dx \\
\text{or} \quad & C_m = f_m + (\lambda / \lambda_m) \times C_m. \quad \text{using (9)} \quad \dots (14)
\end{aligned}$$

$$\text{From (11),} \quad C_m = a_m + f_m. \quad \dots (15)$$

Eliminating C_m from (14) and (15), we get

$$a_m + f_m = f_m + \frac{\lambda}{\lambda_m} (a_m + f_m) \quad \text{or} \quad a_m \left(1 - \frac{\lambda}{\lambda_m} \right) = \frac{\lambda}{\lambda_m} f_m$$

$$\therefore a_m = \frac{\lambda}{\lambda_m - \lambda} f_m, \quad \dots (16)$$

where $\lambda \neq \lambda_m$ and so a_m is well-defined.

Substituting the above value of a_m in (4)', the required solution of (1) is given by

$$\begin{aligned}
y(x) - f(x) &= \sum_{m=1} \frac{\lambda f_m}{\lambda_m - \lambda} \phi_m(x) \\
\text{or} \quad y(x) &= f(x) + \lambda \sum_m \frac{f_m}{\lambda_m - \lambda} \phi_m(x). \quad \dots (17)
\end{aligned}$$

$$\text{From (10),} \quad f_m = \int_a^b f(t) \phi_m(t) dt. \quad \dots (18)$$

Using (18), (17) may be re-written as

$$\begin{aligned}
y(x) &= f(x) + \lambda \sum_m \frac{\phi_m(x)}{\lambda_m - \lambda} \int_a^b f(t) \phi_m(t) dt \\
\text{or} \quad y(x) &= f(x) + \lambda \int_a^b \left[\sum_m \frac{\phi_m(x) \phi_m(t)}{\lambda_m - \lambda} \right] f(t) dt \\
\text{or} \quad y(x) &= f(x) + \lambda \int_a^b R(x, t; \lambda) f(t) dt \quad \dots (19)
\end{aligned}$$

where the resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = \sum_m \frac{\phi_m(x) \phi_m(t)}{\lambda_m - \lambda} \quad \dots (20)$$

Three important cases arise :

Case I. Unique Solutin. If condition (6) is satisfied, (16) gives well defined value of a_m for substituting in (4)'. Thus, solution (17) exists uniquely if and only if λ does not take on an eigenvalue.

Case II. No Solution. Let $\lambda = \lambda_k$, where λ_k is the k th eigenvalue and also let

$$f_k \neq 0, \text{ that is, } \int_a^b f(x) \phi_k(x) dx \neq 0,$$

i.e., eigenfunction $\phi_k(x)$ is not orthogonal to $f(x)$. Then, because of the presence of the term

$$\frac{f_k \phi_k(x)}{\lambda_k - \lambda} \quad \dots (21)$$

in (17), we find that no solution exists since the term (21) is undefined.

Case III. Infinitely many solutions exist. Let $\lambda = \lambda_k$, where λ_k is the k th eigenvalue and

$$\text{also let } f_k = 0, \text{ that is, } \int_a^b f(x) \phi_k(x) dx = 0,$$

i.e., eigenfunction $\phi_k(x)$ is orthogonal to $f(x)$. Then, (14) reduces to (for $m = k$)

$$C_k = 0 + (\lambda / \lambda_k) \times C_k \quad \text{or} \quad C_k = C_k,$$

which is a trivial identity and hence imposes no restriction on C_k . From (16) it then follows that the coefficient a_k of $\phi_k(x)$ in (17), which formally assumes the form $0/0$, is truly arbitrary. Hence, we re-write solution (17) as follows :

$$y(x) = f(x) + A \phi_k(x) + \lambda \sum'_m \frac{f_m}{\lambda_m - \lambda} \phi_m(x), \quad \dots (22)$$

where dash implies that we should neglect $m = k$ in the summation and A is an arbitrary constant. (22) shows that we arrive at infinitely many solutions of (1).

7.7. SOLVED EXAMPLES BASED ON ART. 7.6.

Ex. 1. Solve the symmetric integral equation $y(x) = (x+1)^2 + \int_{-1}^1 (xt + x^2 t^2) y(t) dt$,

by using Hilbert-Schmidt theorem.

[Kanpur 2010; Meerut 2009, 2012]

$$\text{Sol. Given } y(x) = (x+1)^2 + \int_{-1}^1 (xt + x^2 t^2) y(t) dt, \quad \dots (1)$$

$$\text{Comparing (1) with } y(x) = f(x) + \lambda \int_{-1}^1 (xt + x^2 t^2) y(t) dt, \quad \dots (2)$$

$$\text{here } f(x) = (x+1)^2 \quad \text{and} \quad \lambda = 1. \quad \dots (3)$$

We begin with determining eigenvalues and the corresponding normalized eigenfunctions of

$$y(x) = \lambda \int_{-1}^1 (xt + x^2 t^2) y(t) dt \quad \dots (4)$$

$$\text{Re-writing (4), } y(x) = \lambda x \int_{-1}^1 t y(t) dt + \lambda x^2 \int_{-1}^1 t^2 y(t) dt. \quad \dots (5)$$

$$\text{Let } C_1 = \int_{-1}^1 t y(t) dt \quad \dots (6)$$

$$\text{and } C_2 = \int_{-1}^1 t^2 y(t) dt. \quad \dots (7)$$

Then (5) reduces to $y(x) = \lambda C_1 x + \lambda C_2 x^2$ (8)

From (8), $y(t) = \lambda C_1 t + \lambda C_2 t^2$ (9)

Using (9), (6) becomes

$$C_1 = \int_{-1}^1 t (\lambda C_1 t + \lambda C_2 t^2) dt \quad \text{or} \quad C_1 = C_1 \lambda \left[\frac{t^3}{3} \right]_{-1}^1 + C_2 \lambda \left[\frac{t^4}{4} \right]_{-1}^1$$

or $C_1 = \frac{2C_1\lambda}{3} + 0$ or $C_1 \left(1 - \frac{2\lambda}{3} \right) + 0.C_2 = 0$ (10)

Again, using (9), (7) becomes

$$C_2 = \int_{-1}^1 t^2 (\lambda C_1 t + \lambda C_2 t^2) dt \quad \text{or} \quad C_2 = C_1 \lambda \left[\frac{t^4}{4} \right]_{-1}^1 + C_2 \lambda \left[\frac{t^5}{5} \right]_{-1}^1$$

or $C_2 = 0 + \frac{2C_2\lambda}{5}$ or $0.C_1 + \left(1 - \frac{2\lambda}{5} \right) C_2 = 0$ (11)

Equations (10) and (11) have a nontrivial solution only if

$$D(\lambda) = \begin{vmatrix} 1 - (2\lambda/3) & 0 \\ 0 & 1 - (2\lambda/5) \end{vmatrix} = 0$$

or $\{1 - (2\lambda/3)\} \{1 - (2\lambda/5)\} = 0$ giving $\lambda = 3/2$ or $5/2$.

Hence the required eigenvalues are $\lambda_1 = 3/2$ and $\lambda_2 = 5/2$... (12)

Determination of eigenfunction corresponding to $\lambda_1 = 3/2$.

Putting $\lambda = \lambda_1 = 3/2$ in (10) and (11), we obtain

$$C_1.0 + 0.C_2 = 0 \quad \text{and} \quad 0.C_1 + \left[1 - \left(\frac{2}{5} \times \frac{3}{2} \right) \right] C_2 = 0,$$

Hence $C_2 = 0$ and C_1 is arbitrary. Putting these values in (8) and noting that $\lambda = 3/2$, we have the required eigenfunction $y_1(x)$ given by

$$y_1(x) = (3/2) \times C_1 x.$$

Setting $(3/2) \times C_1 = 1$, we may take $y_1(x) = x$.

Now, the corresponding nonnormalized eigenfunction $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{y_1(x)}{\left[\int_{-1}^1 \{y_1(x)\}^2 dx \right]^{1/2}} = \frac{x}{\left[\int_{-1}^1 x^2 dx \right]^{1/2}} = \frac{x}{\left\{ \left[x^3/3 \right]_{-1}^1 \right\}^{1/2}}$$

Thus,
$$\phi_1(x) = \frac{x}{\sqrt{(2/3)}} = x \times \left(\frac{3}{2} \right)^{1/2} = \frac{x\sqrt{6}}{2}. \quad \dots (13)$$

Determination of eigenfunction corresponding to $\lambda_2 = 5/2$.

Putting $\lambda = \lambda_2 = 5/2$ in (10) and (11), we obtain

$$\left[1 - \left(\frac{2}{3} \times \frac{5}{2} \right) \right] C_1 + 0.C_2 = 0 \quad \text{and} \quad 0.C_1 + 0.C_2 = 0,$$

Hence $C_1 = 0$ and C_2 is arbitrary. Putting these values in (8) and noting that $\lambda = 5/2$, we have the required eigenfunction $y_2(x)$ given by

$$y_2(x) = (5/2) \times C_2 x^2.$$

Setting $(5/2) \times C_2 = 1$, we may take $y_2(x) = x^2$.

Now, the corresponding normalized eigenfunction $\phi_2(x)$ is given by

$$\phi_2(x) = \frac{y_2(x)}{\left[\int_{-1}^1 \{y_2(x)\}^2 dx \right]^{1/2}} = \frac{x^2}{\left[\int_{-1}^1 x^4 dx \right]^{1/2}} = \frac{\sqrt{10}}{2} x^2. \quad \dots (14)$$

Also, $f_1 = \int_{-1}^1 f(x) \phi_1(x) dx = \int_{-1}^1 (x+1)^2 \left(\frac{\sqrt{6}}{2} x \right) dx$, by (3) and (13)

$$= \frac{\sqrt{6}}{2} \int_{-1}^1 (x^2 + 2x + 1)x dx = \frac{\sqrt{6}}{2} \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} \right]_{-1}^1 = \frac{2\sqrt{6}}{3}. \quad \dots (15)$$

and

$$f_2 = \int_{-1}^1 f(x) \phi_2(x) dx = \int_{-1}^1 (x+1)^2 \left(\frac{\sqrt{10}}{2} x^2 \right) dx, \text{ by (3) and (14)}$$

$$= \frac{\sqrt{10}}{2} \left[\frac{x^5}{5} + \frac{2x^4}{4} + \frac{x^3}{3} \right]_{-1}^1 = \frac{8}{15} \sqrt{10}. \quad \dots (16)$$

From (3), $\lambda = 1$. Also $\lambda_1 = 3/2$ and $\lambda_2 = 5/2$. Hence $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$.

Therefore, (I) will possess a unique solution given by (refer case I, Art. 7.6)

$$y(x) = f(x) + \lambda \sum_{m=1}^2 \frac{f_m}{\lambda_m - \lambda} \phi_m(x) \quad \text{or} \quad y(x) = (x+1)^2 + \sum_{m=1}^2 \frac{f_m}{\lambda_m - 1} \phi_m(x), \text{ using (3)}$$

or

$$y(x) = (x+1)^2 + \frac{f_1 \phi_1(x)}{\lambda_1 - 1} + \frac{f_2 \phi_2(x)}{\lambda_2 - 1}$$

or

$$y(x) = (x+1)^2 + \frac{(2\sqrt{6}/3) \times (x\sqrt{6}/2)}{(3/2) - 1} + \frac{(8\sqrt{10}/15) \times (x^2\sqrt{10}/2)}{(5/2) - 1}$$

or

$$y(x) = (x+1)^2 + 4x + (16/9) \times x^2 = x^2 + 2x + 1 + 4x + (16/9) \times x^2$$

or

$$y(x) = (25/9) \times x^2 + 6x + 1.$$

Ex. 2. Using Hilbert-Schmidt theorem, find the solution of the symmetric integral equation

$$y(x) = x^2 + 1 + \frac{3}{2} \int_{-1}^1 (xt + x^2 t^2) y(t) dt. \quad [\text{Kanpur 2009, 11; Meerut 2000}]$$

Sol. Given

$$y(x) = x^2 + 1 + \frac{3}{2} \int_{-1}^1 (xt + x^2 t^2) y(t) dt. \quad \dots (1)$$

Comparing (1) with

$$y(x) = f(x) + \lambda \int_{-1}^1 (xt + x^2 t^2) y(t) dt, \quad \dots (2)$$

here

$$f(x) = x^2 + 1$$

and

$$\lambda = 3/2. \quad \dots (3)$$

We begin with determining eigenvalues and the corresponding normalized eigenfunctions of

$$y(x) = \lambda \int_{-1}^1 (xt + x^2 t^2) y(t) dt. \quad \dots (4)$$

Equation (4) is same as equation (4) of solved Ex. 1. So proceed as before and obtain

$$\lambda_1 = 3/2 \quad \lambda_2 = 5/2, \quad \phi_1(x) = (x\sqrt{6})/2 \quad \text{and} \quad \phi_2(x) = (x^2\sqrt{10})/2. \quad \dots (6)$$

$$\text{Also,} \quad f_1 = \int_{-1}^1 f(x) \phi_1(x) dx = \int_{-1}^1 (x^2 + 1) \cdot \frac{x\sqrt{6}}{2} dx = \frac{\sqrt{6}}{2} \left[\frac{x^4}{4} + \frac{x^2}{2} \right]_{-1}^1 = 0. \quad \dots (7)$$

$$\text{and} \quad f_2 = \int_{-1}^1 f(x) \phi_2(x) dx = \int_{-1}^1 (x^2 + 1) \frac{x^2\sqrt{10}}{2} dx = \frac{\sqrt{10}}{2} \left[\frac{x^5}{5} + \frac{x^3}{3} \right]_{-1}^1 = \frac{8\sqrt{10}}{15}. \quad \dots (8)$$

$$\text{Here} \quad \lambda = 3/2 = \lambda_1 \quad \text{and} \quad \lambda \neq \lambda_2.$$

Since $\lambda = \lambda_1$ and $f_1 = 0$, hence infinitely many solutions of (1) exist and are given by (refer case III, Art. 7.6)

$$y(x) = f(x) + A\phi_1(x) + \lambda \sum_{m=1}^{\infty} \frac{f_m}{\lambda_m - \lambda} \phi_m(x), \quad \dots (9)$$

where dash in the above sum means that the term with $m = 1$ must be neglected.

$$\therefore (9) \text{ takes the form} \quad y(x) = f(x) + A\phi_1(x) + \lambda \frac{f_2}{\lambda_2 - \lambda} \phi_2(x)$$

$$\text{or} \quad y(x) = x^2 + 1 + A \left(\frac{x\sqrt{6}}{2} \right) + \frac{3}{2} \times \frac{(8\sqrt{10}/15)}{(5/2) - (3/2)} \times \frac{x^2\sqrt{10}}{2} \quad \text{using (3), (5), (6) and (8)}$$

$$\text{or} \quad y(x) = x^2 + 1 + Cx + 4x^2 \quad \text{or} \quad y(x) = 5x^2 + Cx + 1,$$

where $C = (A\sqrt{6})/2$ is an arbitrary constant.

Ex. 3. Solve the following symmetric integral equation with the help of Hilbert-Schmidt

$$\text{theorem : } y(x) = 1 + \lambda \int_0^\pi \cos(x+t) y(t) dt. \quad [Meerut, 2010, 11]$$

$$\text{Sol. Given} \quad y(x) = 1 + \lambda \int_0^\pi \cos(x+t) y(t) dt. \quad \dots (1)$$

$$\text{Comparing (1) with} \quad y(x) = f(x) + \lambda \int_0^\pi \cos(x+t) y(t) dt, \quad \dots (2)$$

$$\text{here} \quad f(x) = 1, \quad \text{and} \quad \lambda = \lambda. \quad \dots (3)$$

We begin with determining eigenvalues and the corresponding normalized eigenfunctions of

$$y(x) = \lambda \int_0^\pi \cos(x+t) y(t) dt \quad \dots (4)$$

$$\text{Re-writing (4),} \quad y(x) = \lambda \int_0^\pi (\cos x \cos t - \sin x \sin t) dt$$

$$\text{or} \quad y(x) = \lambda \cos x \int_0^\pi \cos t y(t) dt - \lambda \sin x \int_0^\pi \sin t y(t) dt. \quad \dots (5)$$

$$\text{Let} \quad C_1 = \int_0^\pi \cos t y(t) dt \quad \dots (6)$$

$$\text{and} \quad C_2 = \int_0^\pi \sin t y(t) dt. \quad \dots (7)$$

Then (5) reduces to $y(x) = \lambda C_1 \cos x - \lambda C_2 \sin x$... (8)

From (8), $y(t) = \lambda C_1 \cos t - \lambda C_2 \sin t$ (9)

Using (9), (6) becomes $C_1 = \int_0^\pi \cos t (\lambda C_1 \cos t - \lambda C_2 \sin t) dt$

or $C_1 = \frac{\lambda C_1}{2} \int_0^\pi (1 + \cos 2t) dt - \frac{\lambda C_2}{2} \int_0^\pi \sin 2t dt = \frac{\lambda C_1}{2} \left[t + \frac{\sin 2t}{2} \right]_0^\pi + \frac{\lambda C_2}{4} [\cos 2t]_0^\pi$

or $C_1 = (\lambda C_1 \pi) / 2$ or $C_1 (2 - \lambda \pi) + 0 \cdot C_2 = 0$ (10)

Next, using (9), (7) becomes $C_2 = \int_0^\pi \sin t (\lambda C_1 \cos t - \lambda C_2 \sin t) dt$

or $C_2 = \frac{\lambda C_1}{2} \int_0^\pi \sin 2t dt - \frac{\lambda C_2}{2} \int_0^\pi (1 - \cos 2t) dt = -\frac{\lambda C_1}{2} [\cos 2t]_0^\pi - \frac{\lambda C_2}{2} \left[t - \frac{\sin 2t}{2} \right]_0^\pi$

or $C_2 = -(\lambda C_2 \pi) / 2$ or $0 \cdot C_1 + (2 + \lambda \pi) C_2 = 0$ (11)

Equations (10) and (11) have a nontrivial solution only if

$$D(\lambda) = \begin{vmatrix} 2 - \lambda \pi & 0 \\ 0 & 2 + \lambda \pi \end{vmatrix} = 0$$

or $(2 - \lambda \pi)(2 + \lambda \pi) = 0$, so that $\lambda = 2/\pi$ or $-2/\pi$.

Hence the required eigenvalues are $\lambda_1 = 2/\pi$ and $\lambda_2 = -2/\pi$ (12)

Determination of eigenfunction corresponding to $\lambda_1 = 2/\pi$.

Putting $\lambda = \lambda_1 = 2/\pi$ in (10) and (11), we obtain

$$0 \cdot C_1 + 0 \cdot C_2 = 0 \quad \text{and} \quad 0 \cdot C_1 + 4C_2 = 0,$$

Hence $C_2 = 0$ and C_1 is arbitrary. Putting these values in (8) and noting that $\lambda = 2/\pi$, we have the eigenfunction $y_1(x)$ given by $y_1(x) = (2/\pi) \times C_1 \cos x$.

Setting $(2/\pi) \times C_1 = 1$, we may take $y_1(x) = \cos x$.

The corresponding normalized eigenfunction $\phi_1(x)$ is given by

$$\begin{aligned} \phi_1(x) &= \frac{y_1(x)}{\left[\int_0^\pi \{y_1(x)\}^2 dx \right]^{1/2}} = \frac{\cos x}{\left[\int_0^\pi \cos^2 x dx \right]^{1/2}} = \frac{\cos x}{\left[\int_0^\pi \frac{1 + \cos 2x}{2} dx \right]^{1/2}} \\ &= \frac{\cos x}{\left\{ \left[\frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) \right]_0^\pi \right\}^{1/2}} = \frac{\cos x}{\sqrt{\pi/2}} = \left(\frac{2}{\pi} \right)^{1/2} \cos x. \end{aligned} \quad \dots (13)$$

Determination of eigenfunction corresponding to $\lambda_2 = -2/\pi$.

Putting $\lambda = \lambda_2 = -2/\pi$ in (10) and (11), we get

$$4C_1 + 0 \cdot C_2 = 0 \quad \text{and} \quad 0 \cdot C_1 + 0 \cdot C_2 = 0,$$

Hence $C_1 = 0$ and C_2 is arbitrary. Putting these values in (8) and noting that $\lambda = -2/\pi$, we have the eigenfunction $y_2(x)$ given by

$$y_2(x) = -(-2/\pi) \times C_2 \sin x.$$

Setting $(2/\pi) \times C_2 = 1$, we can take

$$y_2(x) = \sin x.$$

The corresponding normalized eigenfunction $\phi_2(x)$ is given by

$$\begin{aligned} \phi_2(x) &= \frac{y_2(x)}{\left[\int_0^\pi \{y_2(x)\}^2 dx \right]^{1/2}} = \frac{\sin x}{\left[\int_0^\pi \sin^2 x dx \right]^{1/2}} = \frac{\sin x}{\left[\int_0^\pi \frac{1 - \cos 2x}{2} dx \right]^{1/2}} \\ &= \frac{\sin x}{\left\{ \left[\frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) \right]_0^\pi \right\}^{1/2}} = \frac{\sin x}{\sqrt{\pi/2}} = \left(\frac{2}{\pi} \right)^{1/2} \sin x. \end{aligned} \quad \dots (14)$$

Also, $f_1 = \int_0^\pi f(x) \phi_1(x) dx = \int_0^\pi \cos x \left(\frac{2}{\pi} \right)^{1/2} dx$, by (3) and (13)

$$= (2/\pi)^{1/2} \times [\sin x]_0^\pi = 0 \quad \dots (15)$$

and $f_2 = \int_0^\pi f(x) \phi_2(x) dx = \int_0^\pi \sin x \left(\frac{2}{\pi} \right)^{1/2} dx$, by (3) and (14)

$$= (2/\pi)^{1/2} \times [-\cos x]_0^\pi = 2 \times (2/\pi)^{1/2}. \quad \dots (16)$$

Three cases arise :

Case I : Let $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$. Then (1) will possess unique solution given by

$$y(x) = f(x) + \lambda \sum_{m=1}^2 \frac{f_m}{\lambda_m - \lambda} \phi_m(x), \text{ by case I, Art. 7.6}$$

or $y(x) = 1 + \frac{\lambda}{\lambda_1 - \lambda} f_1 \phi_1(x) + \frac{\lambda}{\lambda_2 - \lambda} f_2 \phi_2(x)$, by (3)

or $y(x) = 1 + \left[\frac{\lambda \phi_1(x)}{(2/\pi) - \lambda} \right] \times 0 + \frac{\lambda}{-(2/\pi) - \lambda} \times 2 \left(\frac{2}{\pi} \right)^{1/2} \times \left(\frac{2}{\pi} \right)^{1/2} \sin x$

or $y(x) = 1 - \frac{4\lambda \sin x}{2 + \lambda\pi}. \quad \dots (17)$

Case II. Let $\lambda = \lambda_2 = -2/\pi$. Since $f_2 \neq 0$, so (1) possesses no solution. (refer case II, Art. 7.6)

Case III. Let $\lambda = \lambda_1 = 2/\pi$. Since $f_1 = 0$, there exist infinitely many solutions given by (refer case III, Art. 7.6)

$$y(x) = f(x) + A \phi_1(x) + \lambda \sum'_{m=1}^2 \frac{f_m}{\lambda_m - \lambda} \phi_m(x), \quad \dots (18)$$

where dash is employed to indicate that the term corresponding to $m = 1$ in the sum is to be omitted. Accordingly, (18) reduces to

$$y(x) = f(x) + A \phi_1(x) + \frac{\lambda}{\lambda_2 - \lambda} f_2 \phi_2(x)$$

or
$$y(x) = 1 + A \left(\frac{2}{\pi} \right)^{1/2} \cos x + \frac{(2/\pi)}{-(2/\pi) - (2/\pi)} \times 2 \left(\frac{2}{\pi} \right)^{1/2} \times \left(\frac{2}{\pi} \right)^{1/2} \sin x$$

or
$$y(x) = 1 + C \cos x - (2/\pi) \times \sin x, \text{ when } C (= 4\sqrt{2}/\sqrt{\pi}) \text{ is an arbitrary constant.}$$

Ex. 4. Solve the symmetric integral equation $y(x) = f(x) + \lambda \int_a^b k(x) k(t) y(t) dt.$

Sol. Given
$$y(x) = f(x) + \lambda \int_a^b k(x) k(t) y(t) dt. \quad \dots (1)$$

We begin with determining eigenvalues and the corresponding normalized eigenfunctions of

$$y(x) = \lambda \int_a^b k(x) k(t) y(t) dt. \quad \dots (2)$$

Re-writing (2),
$$y(x) = \lambda k(x) \int_a^b k(t) y(t) dt. \quad \dots (3)$$

Let
$$C = \int_a^b k(t) y(t) dt. \quad \dots (4)$$

Then (3) reduces to
$$y(x) = \lambda C k(x). \quad \dots (5)$$

From (5),
$$y(t) = \lambda C k(t). \quad \dots (6)$$

Using (6), (4) becomes

$$C = \int_a^b [k(t) \cdot \lambda C k(t) dt] \quad \text{or} \quad C \left[1 - \lambda \int_a^b \{k(t)\}^2 dt \right] = 0 \quad \dots (7)$$

For eigenfunction of (2), clearly $C \neq 0$. Hence (7) gives

$$1 - \lambda \int_a^b \{k(t)\}^2 dt = 0 \quad \text{or} \quad \lambda = \lambda_1 = 1 / \left[\int_a^b \{k(x)\}^2 dx \right], \quad \dots (8)$$

which is the only eigenvalue of (2). Putting this value of λ in (5), the corresponding eigen function $y_1(x)$ is given by

$$y_1(x) = C k(x) / \left[\int_a^b \{k(x)\}^2 dx \right].$$

Setting $C / \left[\int_a^b \{k(x)\}^2 dx \right] = 1$, we can take
$$y_1(x) = k(x). \quad \dots (9)$$

Hence the corresponding normalized eigenfunction $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{y_1(x)}{\left[\int_a^b \{y_1(x)\}^2 dx \right]^{1/2}} = \frac{k(x)}{\left[\int_a^b \{k(x)\}^2 dx \right]^{1/2}}. \quad \dots (10)$$

Also,
$$f_1 = \int_a^b f(x) \phi_1(x) dx = \int_a^b \left[f(x) k(x) / \left[\int_a^b \{k(x)\}^2 dx \right]^{1/2} \right] dx, \text{ by (10)}$$

Thus,
$$f_1 = \left[\int_a^b f(x) k(x) dx \right] / \left[\int_a^b \{k(x)\}^2 dx \right]^{1/2}. \quad \dots (11)$$

Three cases arise :

Case I. Let $\lambda \neq \lambda_1$. Then (1) will possess unique solution given by

$$y(x) = f(x) + \lambda \sum_{m=1}^{\infty} \frac{f_m}{\lambda_m - \lambda} \phi_m(x) \quad \text{or} \quad y(x) = f(x) + \frac{\lambda}{\lambda_1 - \lambda} f_1 \phi_1(x)$$

or
$$y(x) = f(x) + \frac{\lambda}{\left[\int_a^b \{k(x)\}^2 dx\right]^{-1} - \lambda} \times \frac{\int_a^b f(x) k(x) dx}{\left[\int_a^b \{k(x)\}^2 dx\right]^{1/2}} \times \frac{k(x)}{\left[\int_0^b \{k(x)\}^2 dx\right]^{1/2}}$$
 [using (8), (10) and (11)]

or
$$y(x) = f(x) + \frac{\lambda k(x) \int_a^b f(x) k(x) dx}{1 - \lambda \int_a^b \{k(x)\}^2 dx} \dots (12)$$

Case II. Let $\lambda = \lambda_1$. Suppose that $f(x)$ be not orthogonal to $\phi_1(x)$, this is,

$$f_1 = \int_a^b f(x) \phi_1(x) dx \neq 0.$$

Then (1) possesses no solution.

Case III. Let $\lambda = \lambda_1$. Suppose that $f(x)$ orthogonal to $\phi_1(x)$, that is,

$$f_1 = \int_a^b f(x) \phi_1(x) dx = 0$$

Then (1) possesses infinitely many solutions given by

$$y(x) = f(x) + A\phi_1(x), \text{ A being an arbitrary constant.}$$

or
$$y(x) = f(x) + Ak(x) \cdot \left[\int_a^b \{k(x)\}^2 dx\right]^{-1/2}, \text{ by (10)}$$

or
$$y(x) = f(x) + Ck(x), \quad \text{where} \quad C = A \left[\int_a^b \{k(x)\}^2 dx\right]^{-1/2} = \text{new arbitrary constant.}$$

Ex. 5. Determine the eigenvalues and the corresponding eigenfunctions of the equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} \sin(x+t) y(t) dt,$$

where $f(x) = x$. Obtain the solution of this equation when λ is not an eigenvalue.

Sol. Given
$$y(x) = x + \lambda \int_0^{2\pi} \sin(x+t) y(t) dt. \dots (1)$$

Comparing (1) with
$$y(x) = f(x) + \lambda \int_0^{2\pi} \sin(x+t) y(t) dt, \dots (2)$$

here
$$f(x) = x \quad \text{and} \quad \lambda = \lambda \dots (3)$$

We begin with determining eigenvalues and the corresponding normalized eigenfunctions of

$$y(x) = \lambda \int_0^{2\pi} \sin(x+t) y(t) dt. \dots (4)$$

Re-writing (4),
$$y(x) = \lambda \int_0^{2\pi} (\sin x \cos t \cos x \sin t) y(t) dt.$$

or
$$y(x) = \lambda \sin x \int_0^{2\pi} \cos t y(t) dt + \lambda \cos x \int_0^{2\pi} \sin t y(t) dt \quad \dots (5)$$

Let
$$C_1 = \int_0^{2\pi} \cos t y(t) dt \quad \dots (6)$$

and
$$C_2 = \int_0^{2\pi} \sin t y(t) dt. \quad \dots (7)$$

Then (5) reduces to
$$y(x) = \lambda C_1 \sin x + \lambda C_2 \cos x. \quad \dots (8)$$

From (8),
$$y(t) = \lambda C_1 \sin t + \lambda C_2 \cos t. \quad \dots (9)$$

Using (9), (6) becomes
$$C_1 = \int_0^{2\pi} \cos t (\lambda C_1 \sin t + \lambda C_2 \cos t) dt$$

or
$$C_1 = \frac{\lambda C_1}{2} \int_0^{2\pi} \sin 2t dt + \frac{\lambda C_2}{2} \int_0^{2\pi} (1 + \cos 2t) dt = -\frac{\lambda C_1}{2} [\cos 2t]_0^{2\pi} + \frac{\lambda C_2}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi}$$

or
$$C_1 = \lambda \pi C_2 \quad \text{or} \quad C_1 - \lambda \pi C_2 = 0. \quad \dots (10)$$

Using (9), (7) becomes
$$C_2 = \int_0^{2\pi} \sin t (\lambda C_1 \sin t + \lambda C_2 \cos t) dt$$

or
$$C_2 = \frac{\lambda C_1}{2} \int_0^{2\pi} (1 - \cos 2t) dt + \frac{\lambda C_2}{2} \int_0^{2\pi} \sin 2t dt$$

or
$$C_2 = \lambda \pi C_1 \quad \text{or} \quad \lambda \pi C_1 - C_2 = 0. \quad \dots (11)$$

Equations (10) and (11) have a non-trivial solution only if

$$D(\lambda) = \begin{vmatrix} 1 & -\lambda\pi \\ \lambda\pi & -1 \end{vmatrix} = 0 \quad \text{or} \quad -1 + \lambda^2 \pi^2 = 0 \quad \text{so that} \quad \lambda = 1/\pi \quad \text{or} \quad -1/\pi.$$

Hence the required eigenvalue are $\lambda_1 = 1/\pi$ and $\lambda_2 = -1/\pi$.

Determination of eigenfunction corresponding to $\lambda_1 = 1/\pi$.

Putting $\lambda = \lambda_1 = 1/\pi$ in (10) and (11), we get

$$C_1 - C_2 = 0 \quad \text{and} \quad C_1 - C_2 = 0,$$

giving $C_2 = C_1$. Putting $C_2 = C_1$ in (8), the required eigenfunction $y_1(x)$ is given by

$$y_1(x) = (C_1/\pi) \times (\sin x + \cos x).$$

Setting $(C_1/\pi) = 1$, we may take $y_1(x) = \sin x + \cos x$.

Hence the corresponding normalized eigenfunction $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{y_1(x)}{\left[\int_0^{2\pi} \{y_1(x)\}^2 dx \right]^{1/2}} = \frac{\sin x + \cos x}{\left[\int_0^{2\pi} (\sin x + \cos x)^2 dx \right]^{1/2}}$$

$$= \frac{\sin x + \cos x}{\left[\int_0^{2\pi} (1 + \sin 2x) dx \right]^{1/2}} = \frac{\sin x + \cos x}{\left\{ \left[x - \frac{\cos 2x}{2} \right]_0^{2\pi} \right\}^{1/2}} = \frac{\sin x + \cos x}{\sqrt{2\pi}} \quad \dots(13)$$

Determination of eigenfunction corresponding to $\lambda_2 = -1/\pi$.

Putting $\lambda = \lambda_2 = -1/\pi$ in (10) and (11), we get

$$C_1 + C_2 = 0 \quad \text{and} \quad -C_1 - C_2 = 0,$$

giving $C_2 = -C_1$. Putting $C_2 = -C_1$ in (8), the required eigenfunction $y_2(x)$ is given by

$$y_2(x) = (C_1 / \pi) \times (\sin x - \cos x).$$

Setting $(C_1 / \pi) = 1$, we may take $y_2(x) = \sin x - \cos x$.

Hence the corresponding normalized eigenfunction $\phi_2(x)$ is given by

$$\begin{aligned} \phi_2(x) &= \frac{y_2(x)}{\left[\int_0^{2\pi} \{y_2(x)\}^2 dx \right]^{1/2}} = \frac{\sin x - \cos x}{\left[\int_0^{2\pi} (\sin x - \cos x)^2 dx \right]^{1/2}} \\ &= \frac{\sin x - \cos x}{\left[\int_0^{2\pi} (1 - \sin 2x) dx \right]^{1/2}} = \frac{\sin x - \cos x}{\left\{ \left[x + \frac{\cos 2x}{2} \right]_0^{2\pi} \right\}^{1/2}} = \frac{\sin x - \cos x}{\sqrt{2\pi}} \quad \dots (14) \end{aligned}$$

Also, $f_1 = \int_0^{2\pi} f(x) \phi_1(x) dx = \int_0^{2\pi} \frac{x(\sin x + \cos x)}{\sqrt{2\pi}} dx$, by (3) and (13)

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}} \left[\{x(\sin x - \cos x)\}_0^{2\pi} - \int_0^{2\pi} (\sin x - \cos x) dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left\{ -2\pi - [-\cos x - \sin x]_0^{2\pi} \right\} = -\sqrt{2\pi}, \end{aligned}$$

and

$$\begin{aligned} f_2 &= \int_0^{2\pi} f(x) \phi_2(x) dx = \int_0^{2\pi} \frac{x(\sin x - \cos x)}{\sqrt{2\pi}} dx, \text{ by (3) and (14)} \\ &= \frac{1}{\sqrt{2\pi}} \left[\{x(-\cos x - \sin x)\}_0^{2\pi} - \int_0^{2\pi} (-\cos x - \sin x) dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left\{ -2\pi - [-\sin x + \cos x]_0^{2\pi} \right\} = -\sqrt{2\pi}. \end{aligned}$$

Given that $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$. Hence (1) will possess unique solution given by

$$y(x) = f(x) + \lambda \sum_{m=1}^2 \frac{f_m}{\lambda_m - \lambda} \phi_m(x) \quad \text{or} \quad y(x) = x + \lambda \sum_{m=1}^2 \frac{f_m}{\lambda_m - \lambda} \phi_m(x), \text{ using (3)}$$

or

$$y(x) = x + \frac{\lambda f_1 \phi_1(x)}{\lambda_1 - \lambda} + \frac{\lambda f_2 \phi_2(x)}{\lambda_2 - \lambda}$$

or
$$y(x) = x + \frac{\lambda(-\sqrt{2\pi})(\sin x + \cos x)}{\{(1/\pi) - \lambda\}\sqrt{2\pi}} + \frac{\lambda(-\sqrt{2\pi})(\sin x - \cos x)}{\{(-1/\pi) - \lambda\}\sqrt{2\pi}}$$

or
$$y(x) = x - \frac{\lambda\pi(\sin x + \cos x)}{(1 - \lambda\pi)} + \frac{\lambda\pi(\sin x - \cos x)}{(1 + \lambda\pi)}$$

or
$$y(x) = x - \lambda\pi \sin x \left(\frac{1}{1 - \lambda\pi} - \frac{1}{1 + \lambda\pi} \right) - \lambda\pi \cos x \left(\frac{1}{1 - \lambda\pi} + \frac{1}{1 + \lambda\pi} \right)$$

or
$$y(x) = x - \frac{2\lambda^2\pi^2 \sin x}{1 - \lambda^2\pi^2} - \frac{2\lambda\pi \cos x}{1 - \lambda^2\pi^2}$$

Ex. 6. Using Hilbert-Schmidt method, solve $y(x) = x + \lambda \int_0^1 K(x, t) y(t) dt$, where

$$K(x, t) = \begin{cases} x(t-1), & 0 \leq x \leq t, \\ t(x-1), & t \leq x \leq 1. \end{cases} \quad [\text{Kanpur 2005; Meerut 2001, 02, 05}]$$

Sol. Given
$$y(x) = x + \lambda \int_0^1 K(x, t) y(t) dt, \quad \dots (1)$$

where
$$K(x, t) = \begin{cases} x(t-1), & 0 \leq x \leq t, \\ t(x-1), & t \leq x \leq 1. \end{cases} \quad \dots (2)$$

Comparing (1) with
$$y(x) = f(x) + \lambda \int_0^1 K(x, t) y(t) dt.$$

here, $f(x) = x.$ and $\lambda = \lambda.$... (3)

We begin with determining eigenvalues and the corresponding normalized eigenfunctions of

$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt \quad \dots (4)$$

Re-writing (4), we have
$$y(x) = \lambda \left[\int_0^x K(x, t) y(t) dt + \int_x^1 K(x, t) y(t) dt \right]$$

or
$$y(x) = \int_0^x \lambda t(x-1) y(t) dt + \int_x^1 \lambda x(t-1) y(t) dt, \text{ using (2)} \quad \dots (5)$$

Differentiating both sides of (5) w.r.t. 'x' and using Leibnitz's rule (see Art. 1.13), we get

$$y'(x) = \frac{d}{dx} \int_0^x \lambda t(x-1) y(t) dt + \frac{d}{dx} \int_x^1 \lambda x(t-1) y(t) dt$$

or
$$y'(x) = \int_0^x \frac{\partial}{\partial x} \{ \lambda t(x-1) y(t) \} dt + \lambda x(x-1) y(x) \frac{dx}{dx} - \lambda \cdot 0 \cdot (x-1) y(0) \frac{d0}{dx} \\ + \int_x^1 \frac{\partial}{\partial x} \{ \lambda x(t-1) y(t) \} dt + \lambda x(1-1) y(1) \frac{d1}{dx} - \lambda x(x-1) y(x) \frac{dx}{dx}$$

or
$$y'(x) = \int_0^x \lambda t y(t) dt + \lambda x(x-1) y(x) + \int_x^1 \lambda(t-1) y(t) dt - \lambda x(x-1) y(x)$$

or
$$y'(x) = \int_0^x \lambda t y(t) dt + \int_x^1 \lambda(t-1) y(t) dt. \quad \dots (6)$$

Differentiating both sides of (6) w.r.t. 'x', we get

$$y''(x) = \frac{d}{dx} \int_0^x \lambda t y(t) dt + \frac{d}{dx} \int_x^1 \lambda(t-1) y(t) dt$$

$$\begin{aligned} \text{or } y''(x) &= \int_0^x \frac{\partial}{\partial x} \{\lambda t y(t)\} dt + \lambda x y(x) \frac{dx}{dx} - \lambda \cdot 0 \cdot y(0) \frac{d0}{dx} + \int_x^1 \frac{\partial}{\partial x} \{\lambda(t-1) y(t)\} dt \\ &\quad + \lambda(1-1)y(1) \frac{d1}{dx} - \lambda(x-1) y(x) \frac{dx}{dx}, \text{ using Leibnitz-rule again} \end{aligned}$$

$$\text{or } y''(x) = \lambda x y(x) - \lambda(x-1) y(x).$$

$$\text{or } y''(x) = \lambda y(x) \quad \text{or } y''(x) - \lambda y(x) = 0. \quad \dots (7)$$

$$\text{Putting } x = 0 \text{ in (5), we get } y(0) = 0. \quad \dots (8)$$

$$\text{Putting } x = 1 \text{ in (5) we get } y(1) = 0. \quad \dots (9)$$

Now, we shall solve Sturm-Liouville problem given by differential equation (7) together with boundary conditions (8) and (9) by the usual procedure to get eigenvalues and the corresponding eigenfunctions.

Three cases arise :

Case I. Let $\lambda = 0$. Then (7) reduces to $y''(x) = 0$ whose general solution is

$$y(x) = Ax + B. \quad \dots (10)$$

$$\text{Putting } x = 0 \text{ in (10) and using (8), we get } B = 0. \quad \dots (11)$$

$$\text{Putting } x = 1 \text{ in (10) and using (9), we get } 0 = A + B. \quad \dots (12)$$

Solving (11) and (12), $A = B = 0$. Hence (10) gives $y(x) = 0$, which is not an eigenfunction and so $\lambda = 0$ is not an eigenvalue.

Case II. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (7) reduces to $y''(x) - \mu^2 y(x) = 0$, whose

$$\text{general solution is } y(x) = Ae^{\mu x} + Be^{-\mu x}. \quad \dots (13)$$

$$\text{Putting } x = 0 \text{ in (13) and using (8), we get } 0 = A + B. \quad \dots (14)$$

$$\text{Putting } x = 1 \text{ in (13) and using (9), we get } 0 = Ae^{\mu} + Be^{-\mu}. \quad \dots (15)$$

Solving (14) and (15), $A = B = 0$. Hence (13) gives $y(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (7) reduces to $y''(x) + \mu^2 y(x) = 0$ whose general solution is

$$y(x) = A \cos \mu x + B \sin \mu x. \quad \dots (16)$$

$$\text{Putting } x = 0 \text{ in (16) and using (8), we get } 0 = A. \quad \dots (17)$$

$$\text{Putting } x = 1 \text{ in (16) and using (9), we get } 0 = A \cos \mu + B \sin \mu. \quad \dots (18)$$

$$\text{Using (17), (18) gives } B \sin \mu = 0 \quad \dots (19)$$

Now, we must take $B \neq 0$, otherwise $A = 0$ and $B = 0$ will give $y(x) = 0$ as before and hence we shall not get eigenfunction. Since $B \neq 0$, (19) reduces to $\sin \mu = 0$.

$$\therefore \mu = n\pi, \text{ when } n \text{ is any integer. Hence } \lambda = -\mu^2 = -n^2 \pi^2$$

$$\text{Hence the required eigenvalue } \lambda_n \text{ are given } \lambda_n = -n^2 \pi^2, n = 1, 2, 3, \dots \quad \dots (21)$$

$$\text{Putting } A = 0 \text{ and } \mu = n\pi \text{ in (16), we get } y(x) = B \sin n\pi x.$$

Setting $B = 1$, required eigenfunctions $y_n(x)$ are given by

$$y_n(x) = \sin n\pi x, \quad n = 1, 2, 3, \dots \quad \dots (22)$$

The normalized eigenfunctions $\phi_n(x)$ are given by

$$\begin{aligned} \phi_n(x) &= \frac{y_n(x)}{\left[\int_0^1 \{y_n(x)\}^2 dx \right]^{1/2}} = \frac{\sin n\pi x}{\left[\int_0^1 \sin^2 n\pi x dx \right]^{1/2}} = \frac{\sin n\pi x}{\left[\int_0^1 \frac{1 - \cos 2n\pi x}{2} dx \right]^{1/2}} \\ &= \frac{\sin n\pi x}{\left\{ \left[\frac{1}{2} \left(x - \frac{\sin 2n\pi x}{2n\pi} \right) \right]_0^1 \right\}^{1/2}} = \frac{\sin n\pi x}{1/\sqrt{2}} = \sqrt{2} \sin n\pi x. \end{aligned} \quad \dots (23)$$

Now, $f_n = \int_0^1 f(x) \phi_n(x) dx = \int_0^1 (x) \times (\sqrt{2} \sin n\pi x) dx$, by (3) and (23)

$$\begin{aligned} &= \sqrt{2} \left\{ \left[x \left(-\frac{\cos n\pi x}{n\pi} \right) \right]_0^1 - \int_0^1 \left(-\frac{\cos n\pi x}{n\pi} \right) dx \right\} = \sqrt{2} \left\{ -\frac{\cos n\pi}{n\pi} + \frac{1}{n\pi} \int_0^1 \cos n\pi x dx \right\} \\ &= \sqrt{2} \left\{ -\frac{(-1)^n}{n\pi} + \frac{1}{n^2 \pi^2} [\sin n\pi x]_0^1 \right\} = \frac{(-1)^{n+1} \sqrt{2}}{n\pi}. \end{aligned} \quad \dots (24)$$

Now, two cases arise.

Case (i). Let λ be not an eigenvalue, that is, $\lambda \neq \lambda_n$ for $n = 1, 2, 3, \dots$

Then (1) will possess unique solution given by

$$y(x) = f(x) + \lambda \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \lambda} \phi_n(x) \quad \text{or} \quad y(x) = x + \lambda \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{2}}{n\pi} \times \frac{\sqrt{2} \sin n\pi x}{-n^2 \pi^2 - \lambda}$$

or

$$y(x) = x + \frac{2\lambda}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin n\pi x}{n(n^2 \pi^2 + \lambda)}.$$

Case (ii) Let $\lambda = \lambda_n = -n^2 \pi^2$, $n = 1, 2, 3, \dots$ Then since from (24), $f_n \neq 0$ for $n = 1, 2, 3, \dots$

Hence (1) will possess no solution.

Ex. 7. Solve the symmetric integral equation $y(x) = e^x + \lambda \int_0^1 K(x, t) y(t) dt$, where

$$K(x, t) = \begin{cases} \frac{\sinh x \sinh(t-1)}{\sinh 1}, & 0 \leq x \leq t, \\ \frac{\sinh t \sinh(x-1)}{\sinh 1}, & t \leq x \leq 1. \end{cases} \quad [\text{Kanpur 2005}]$$

Sol. Given

$$y(x) = e^x + \lambda \int_0^1 K(x, t) y(t) dt, \quad \dots (1)$$

where

$$K(x, t) = \begin{cases} \frac{\sinh x \sinh (t-1)}{\sinh 1}, & 0 \leq x \leq t, \\ \frac{\sinh t \sinh (x-1)}{\sinh 1}, & t \leq x \leq 1. \end{cases} \quad \dots (2)$$

Comparing (1) with $y(x) = f(x) + \lambda \int_0^1 K(x, t) y(t) dt$, ... (3)

here $f(x) = e^x$ and $\lambda = \lambda$... (4)

We begin with determination of eigenvalues and the corresponding normalized eigenfunctions of the homogeneous integral equation

$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt. \quad \dots (5)$$

Re-writing (5), we have $y(x) = \lambda \left[\int_0^x K(x, t) y(t) dt + \int_x^1 K(x, t) y(t) dt \right]$

or $y(x) = \int_0^x \frac{\lambda \sinh t \sinh (x-1) y(t)}{\sinh 1} dt + \int_x^1 \frac{\lambda \sinh x \sinh (t-1) y(t)}{\sinh 1} dt$, by (2) ... (6)

Differentiating both sides of (6) w.r.t. 'x' and using Leibnitz's-rule (see Art. 1.13), we have

$$y'(x) = \int_0^x \frac{\lambda \sinh t \cosh (x-1) y(t)}{\sinh 1} dt + \frac{\lambda \sinh x \sinh (x-1) y(x)}{\sinh 1} \\ + \int_x^1 \frac{\lambda \cosh x \sinh (t-1) y(t)}{\sinh 1} dt - \frac{\lambda \sinh x \sinh (x-1) y(x)}{\sinh 1}$$

or $y'(x) = \int_0^x \frac{\lambda \sinh t \cosh (x-1) y(t)}{\sinh 1} dt + \int_x^1 \frac{\lambda \cosh x \sinh (t-1) y(t)}{\sinh 1} dt$... (7)

Differentiating both sides of (7) w.r.t. 'x' and using Leibnitz-rule (Art. 1.13), we have

$$y''(x) = \int_0^x \frac{\lambda \sinh t \sinh (x-1) y(t)}{\sinh 1} dt + \frac{\lambda \sinh x \cosh (x-1) y(x)}{\sinh 1} \\ + \int_x^1 \frac{\lambda \sinh x \sinh (t-1) y(t)}{\sinh 1} dt - \frac{\lambda \cosh x \sinh (x-1) y(x)}{\sinh 1}$$

or $y''(x) = y(x) + \frac{\lambda y(x)}{\sinh 1} [\sinh x \cosh (x-1) - \cosh x \sinh (x-1)]$, using (6)

or $y''(x) = y(x) + \frac{\lambda y(x)}{\sinh 1} \sinh \{x - (x-1)\}$

or $y'' = y(x) + \lambda y(x)$ or $y''(x) - (1 + \lambda)y(x) = 0$... (8)

Putting $x = 0$ in (6), we get $y(0) = 0$ (9)

Putting $x = 1$ in (6), we get $y(1) = 0$ (10)

Three cases arise :

Case I. Let $1 + \lambda = 0$, that is, $\lambda = -1$. Then (8) reduces to $y''(x) = 0$ whose general solution is

$$y(x) = Ax + B. \quad \dots (11)$$

Putting $x = 0$ in (11) and using (9), we get $0 = B$ (12)

Putting $x = 1$ in (11) and using (10), we get $0 = A + B$ (13)

Solving (12) and (13), $A = B = 0$. Hence (11) gives $y(x) = 0$, which is not an eigenfunction and so $\lambda = -1$ is not an eigenvalue.

Case II. Let $\lambda + 1 = \mu^2$, where $\mu \neq 0$. Then (8) reduces to $y'' - \mu^2 y(x) = 0$ whose general solution is

$$y(x) = A e^{\mu x} + B e^{-\mu x} \quad \dots (14)$$

Putting $x = 0$ in (14) and using (9), we get $0 = A + B$... (15)

Putting $x = 1$ in (14) and using (10), we get $0 = A e^{\mu} + B e^{-\mu}$ (16)

Solving (15) and (16), $A = B = 0$. Hence (14) gives $y(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda + 1 = -\mu^2$, where $\mu \neq 0$. Then (8) reduces to $y''(x) + \mu^2 y(x) = 0$ whose general solution is

$$y(x) = A \cos \mu x + B \sin \mu x. \quad \dots (17)$$

Putting $x = 0$ in (17) and using (9), we get $0 = A$ (18)

Putting $x = 1$ in (17) and using (10), we get $0 = A \cos \mu + B \sin \mu$ (19)

Using (18), (19) gives $B \sin \mu = 0$ (20)

Now, we must take $B \neq 0$, otherwise $A = 0$ and $B = 0$ will give $y(x) = 0$ as before and hence we shall not get eigenfunction. Since $B \neq 0$, (20) reduces to $\sin \mu = 0$.

$\therefore \mu = n\pi$, where n is any integer. ... (21)

$$\therefore 1 + \lambda = -\mu^2 = -n^2 \pi^2 \quad \text{or} \quad \lambda = -(1 + n^2 \pi^2).$$

Hence the required eigenvalues λ_n are given by

$$\lambda_n = -(1 + n^2 \pi^2), \quad n = 1, 2, 3, \dots \quad \dots (22)$$

Putting $A = 0$ and $\mu = n\pi$ in (17), $y(x) = B \sin n\pi x$.

Setting $B = 1$, the required eigenfunctions $y_n(x)$ are given by

$$y_n(x) = \sin n\pi x, \quad n = 1, 2, 3, \dots \quad \dots (23)$$

The normalized eigenfunctions $\phi_n(x)$ are given by

$$\begin{aligned} \phi_n(x) &= \frac{y_n(x)}{\left[\int_0^1 \{y_n(x)\}^2 dx \right]^{1/2}} = \frac{\sin n\pi x}{\left[\int_0^1 \sin^2 n\pi x dx \right]^{1/2}} = \frac{\sin n\pi x}{\left[\int_0^1 \frac{1 - \cos 2n\pi x}{2} dx \right]^{1/2}} \\ &= \frac{\sin n\pi x}{\left\{ \left[\frac{1}{2} \left(x - \frac{\sin 2n\pi x}{2n\pi} \right) \right]_0^1 \right\}^{1/2}} = \frac{\sin n\pi x}{(1/\sqrt{2})} = \sqrt{2} \sin nx. \end{aligned} \quad \dots (24)$$

Now, $f_n = \int_0^1 f(x) \phi_n(x) dx = \int_0^1 e^x (\sqrt{2} \sin n\pi x) dx$, by (4) and (24)

$$= \sqrt{2} \left[\frac{e^x}{1 + n^2 \pi^2} (\sin n\pi x - n\pi \cos n\pi x) \right]_0^1, \text{ as } \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$= \frac{\sqrt{2}}{1+n^2\pi^2} [-e n \pi \cos n\pi - (-n\pi)] = \frac{n \pi \sqrt{2} \{1 - e(-1)^n\}}{1+n^2\pi^2}, n = 1, 2, 3, \dots \quad \dots (25)$$

Now, two cases arise

Case (i). Let λ be not an eigenvalue, that is, $\lambda \neq \lambda_n$, for $n = 1, 2, 3, \dots$

Then (1) will possess unique solution given by

$$y(x) = f(x) + \lambda \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \lambda} \phi_n(x) = e^x + \lambda \sum_{n=1}^{\infty} \frac{n\pi\sqrt{2} \{1 - e(-1)^n\}}{1+n^2\pi^2} \times \frac{\sqrt{2} \sin n\pi x}{-n^2\pi^2 - 1 - \lambda}$$

or

$$y(x) = e^x - 2\lambda\pi \sum_{n=1}^{\infty} \frac{n\{1 - e(-1)^n\} \sin n\pi x}{(1+n^2\pi^2)(1+n^2\pi^2 + \lambda)}$$

Case (ii). Let $\lambda = \lambda_n = -1 - n^2\pi^2$, $n = 1, 2, 3, \dots$. Then since from (25), $f_n \neq 0$ for $n = 1, 2, 3, \dots$, hence (1) will possess no solution.

Ex. 8. Given that the eigenvalues of the integral equation $y(x) = \lambda \int_0^{2\pi} \cos(x+t) y(t) dt$ are $1/\pi$ and $-1/\pi$ with respective eigenfunctions $\cos x$ and $\sin x$. Then the integral equation

$$y(x) = \sin x + \cos x + \lambda \int_0^{2\pi} \cos(x+t) y(t) dt \text{ has}$$

- (a) unique solution for $\lambda = 1/\pi$ (b) unique solution for $\lambda = -1/\pi$
 (c) unique solution for $\lambda = \pi$ (d) no solution for $\lambda = -\pi$. [GATE 2004]

Sol. Ans (c). Refer cases I, II and III of Art. 7.6 on pages 7.23 and 7.24. If $\lambda = 1/\pi$ or $-1/\pi$ which are eigenvalues, then by cases II and III either there is no solution or there exist infinitely many solutions. Thus, results (a) and (b) are wrong. If $\lambda = -\pi$, which is not an eigenvalue, then by case I there exists unique solution. So result (d) is wrong. But if $\lambda = \pi$, which is not an eigenvalue, then by case I, there exist a unique solution. Thus result (c) is correct

Ex. 9. Let $K(x, t) = \begin{cases} x+t, & 0 \leq t \leq x \\ 0, & \text{Otherwise} \end{cases}$. Then the integral equation $y(x) = 1 + \lambda \int_0^1 K(x, t) y(t) dt$ has

- (a) unique solution for every value of λ (b) no solution for any value of λ
 (c) a unique solution for finitely many values of λ only
 (d) infinitely many solutions for finitely many values of λ [GATE 2003]

Sol. Ans (d). Refer cases I, II and III of Art. 7.6 on pages 7.23 and 7.24 carefully. It follows that results (a), (b) and (c) are incorrect while result (d) is correct.

EXERCISE

1. State and prove Hilbert-Schmidt theorem. [Merrut 2000, 01, 02, 07]
2. State and prove Riesz-Fischer theorem. [Merrut 2004]
3. State Hilbert-Schmidt theorem. Derive Schmidt's solution of the integral equation

$$y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt, \text{ where } K(x, t) \text{ is symmetric and } \lambda \text{ is not an eigenvalue.}$$

4. Using Hilbert-Schmidt theorem, solve the following symmetric integral equations :

$$(i) \quad y(x) = x + \int_0^1 (x+1) y(t) dt, \lambda \neq \lambda_1, \lambda_2. \quad \text{Ans. } y(x) = \frac{(6\lambda - 12)x - 4\lambda}{\lambda^2 + 12\lambda - 12}.$$

$$(ii) \quad y(x) = (1 - x\sqrt{3}) + (-6 + 4\sqrt{3}) \int_0^1 (x+t) y(t) dt.$$

Ans. $y(x) = (1 - x\sqrt{3}) + C(1 + \sqrt{3}) - (1 + 3x/2)$, where C is an arbitrary constant.

$$(iii) \quad y(x) = (1 + x\sqrt{3}) + (-6 - 4\sqrt{2}) \int_0^1 (x+t) y(t) dt.$$

Ans. $y(x) = (1 + x\sqrt{3}) + C(1 - x\sqrt{3}) - (1 + 3x/2)$, where C is an arbitrary constant.

$$(iv) \quad y(x) = x + \lambda \int_0^1 y(t) dt, (\lambda \neq 1). \quad \text{Ans. } y(x) = x + \lambda / \{2(1 - \lambda)\}$$

$$(v) \quad y(x) = \frac{1}{2} - x + \int_0^1 y(t) dt. \quad \text{Ans. } y(x) = \frac{1}{2} - x + C$$

5. Determine the eigenvalues and the corresponding eigenfunctions of the equation

$$y(x) = F(x) + \lambda \int_0^{2\pi} \cos(x+t) y(t) dt, \text{ where } \lambda \text{ is not an eigenvalue.}$$

Ans. Eigenvalues are $\lambda_1 = 1/\pi, \lambda_2 = -1/\pi$ and corresponding eigenfunctions are

$$\phi_1(x) = (\cos x) / \sqrt{\pi}, \phi_2(x) = (\sin x) / \sqrt{x}. \quad R(x, t; \lambda) = \frac{\cos x \cos t}{1 - \lambda\pi} - \frac{\sin x \sin t}{1 + \lambda\pi}$$

6. State and prove Hilber-Schmidt theorem and use it for finding the solution of the symmetric integral equation $\phi(x) = \cos \pi x + \lambda \int_0^1 K(x, t) \phi(t) dt$, where $K(x, t) = \begin{cases} (x+1)t, & 0 \leq x \leq t \\ (t+1)x, & t \leq x \leq 1. \end{cases}$

7. Show that solution of the integral equation $y(x) = e^x + \lambda \int_0^1 K(x, t) y(t) dt$, where

$$K(x, t) = \begin{cases} (1-t)x, & 0 \leq t \leq x \\ (1-x)t, & x \leq t \leq 1 \end{cases} \quad \text{is} \quad y(x) = e^x - 2\lambda \sum_{n=1}^{\infty} \frac{n\pi \{1 - (-1)^n e\} \sin n\pi x}{(1 + n^2 \pi^2)(\lambda - n^2 \pi^2)}$$

8. Show that the solution of the integral equation

$$y(x) = \sin m\pi x + n^2 \pi^2 \int_0^1 K(x, t) y(t) dt, m \neq n \quad \text{where} \quad K(x, t) = \begin{cases} (1-t)x, & 0 \leq t \leq x \\ (1-x)t, & x \leq t \leq 1 \end{cases}$$

is $y(x) = \sin m\pi x + C \sin n\pi t - (n^2 \sin m\pi x) / (n^2 - m^2)$, where C is an arbitrary constant.

7.8. SOLUTION OF THE FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND WITH SYMMETRIC KERNEL.

[Meerut 2005]

Consider the Fredholm integral equation of the first kind

$$f(x) = \int_a^b K(x, t) y(t) dt, \quad \dots (1)$$

where the kernel $K(x, t)$ is a known symmetric $\mathcal{L}_2$ -kernel and $y(t)$ is unknown function while $f(x)$ is a known function. Suppose that the sequence of eigenvalues $\{\lambda_n\}$ of its kernel $K(x, t)$ and corresponding eigenfunctions $\{\phi_n(x)\}$ are known and arranged as in relations (6) and (7) of Art. 7.4.

From the relation (4) of Art. 7.4, we have

$$f_n = y_n / \lambda_n \quad \text{so that} \quad y_n = \lambda_n f_n \quad \dots (2)$$

In view of the Reisz-Fischer theorem (refer Art. 7.1 (g), there exist the following two situations :

$$(i) \text{ If the infinite series } \sum_{n=1}^{\infty} f_n^2 \lambda_n^2 \quad \dots (3)$$

diverges, then (1) has no solution

(ii) If the infinite series (3) converges, then there exists a unique $\mathcal{L}_2$ -function $y(x)$ which is the solution of (1). Then the solution of (1) can be determined by taking the limit in mean

$$y(x) = \lim_{m \rightarrow \infty} \sum_{n=1}^m \lambda_n f_n \phi_n(x) \quad \dots (4)$$

7.9. SOLVED EXAMPLES BASED ON ART. 7.8

Ex. 1. Solve the symmetric Fredholm integral equation of the first kind

$$\int_0^1 K(x, t) y(t) dt = f(x), \quad \text{where} \quad K(x, t) = \begin{cases} x(1-t), & x < t \\ t(1-x), & x > t \end{cases}$$

$$\text{Sol. Given} \quad \int_0^1 K(x, t) y(t) dt = f(x), \quad \dots (1)$$

$$\text{where} \quad K(x, t) = \begin{cases} x(1-t), & x < t \\ t(1-x), & x > t \end{cases} \quad \dots (2)$$

We shall first find the eigenvalues and the corresponding eigenfunction of the kernel $K(x, t)$. To this end we solve the associated homogeneous Fredholm integral equation

$$y(x) = \lambda \int_0^1 K(x, t) y(t) dt \quad \dots (3)$$

$$\text{From (3),} \quad y(x) = \lambda \left[\int_0^x K(x, t) y(t) dt + \int_x^1 K(x, t) y(t) dt \right]$$

$$\text{or} \quad y(x) = \lambda \int_0^x t(1-x) y(t) dt + \lambda \int_x^1 x(1-t) y(t) dt, \text{ by (2)} \quad \dots (4)$$

Differentiating both sides of (4) w.r.t. 'x' and using Leibnitz's rule on the R.H.S., we have

$$y'(x) = -\lambda \int_0^x t y(t) dt + \lambda x(1-x) y(x) + \lambda \int_x^1 (1-t) y(t) dt - \lambda x(1-x) y(x)$$

$$\text{or} \quad y'(x) = -\lambda \int_0^x t y(t) dt + \lambda \int_x^1 (1-t) y(t) dt. \quad \dots (5)$$

Differentiating both sides of (5) w.r.t. 'x' and using Leibnitz's rule on the R.H.S., we have

$$y''(x) = -\lambda x y(x) - \lambda(1-x) y(x) \quad \text{or} \quad y''(x) + \lambda y(x) = 0$$

$$\text{or} \quad (D^2 + \lambda)y = 0, \quad \text{where} \quad D \equiv d/dx \quad \dots (6)$$

$$\text{Putting } x = 0 \text{ in (4), we get} \quad y(0) = 0 \quad \dots (7)$$

$$\text{Putting } x = 1 \text{ in (4), we get} \quad y(1) = 0 \quad \dots (8)$$

Now, we shall solve Sturm-Liouville problem given by (6) together with boundary conditions (7) and (8) by the usual procedure to find the required eigenvalues and the corresponding normalized eigenfunctions.

Three cases arise :

Case I. Let $\lambda = 0$. Then (6) reduces to $y'' = 0$ whose solution is

$$y(x) = Ax + B \quad \dots (3)$$

Putting $x = 0$ and $x = 1$ by turn in (9) and using (7) and (8), we get

$$B = 0 \quad \text{and} \quad A + B = 0 \quad \text{so that} \quad A = B = 0.$$

Hence (9) gives $y(x) = 0$ which is not an eigenfunction.

Case II. Let $\lambda = -\mu^2, \mu \neq 0$. Then (6) reduces to $(D^2 - \mu^2)y = 0$ whose solution is

$$y(x) = Ae^{\mu x} + Be^{-\mu x} \quad \dots (10)$$

Putting $x = 0$ and $x = 1$ by turn in (10) and using (7) and (8), we have

$$A + B = 0 \quad \text{and} \quad Ae^{\mu} + Be^{-\mu} = 0 \quad \text{so that} \quad A = B = 0.$$

Hence (10) gives $y(x) = 0$ which is not an eigenfunction.

Case III. Let $\lambda = \mu^2, \mu \neq 0$. Then (6) reduces to $(D^2 + \mu^2)y = 0$ whose solution is

$$y(x) = A \cos \mu x + B \sin \mu x. \quad \dots (11)$$

$$\text{Putting } x = 0 \text{ in (11) and using (7), we get} \quad A = 0 \quad \dots (12)$$

$$\text{Putting } x = 1 \text{ in (11) and using (8), we get} \quad B \sin \mu = 0 \quad \dots (13)$$

Now, we must take $B \neq 0$, otherwise $A = B = 0$ will give $y(x) = 0$ and so eigenfunction will not exist. So for the existence of an eigenfunction, we take $B \neq 0$. Then (13) reduces to

$$\sin \mu = 0 \quad \text{so that} \quad \mu = n\pi, n = 1, 2, 3, \dots$$

Hence the required eigenvalues λ_n are given by

$$\lambda_n = n^2 \pi^2, n = 1, 2, 3, \dots \quad \dots (14)$$

Then from (11), setting $B = 1$, the corresponding eigenfunctions $y_n(x)$ are given by

$$y_n(x) = \sin n\pi x, \quad n = 1, 2, 3, \dots$$

The normalized eigenfunctions $\phi_n(x)$ are given by

$$\phi_n(x) = \frac{y_n(x)}{\|y_n(x)\|} = \frac{\sin n\pi x}{\left[\int_0^1 \sin^2 n\pi x \, dx\right]^{1/2}} = \sqrt{2} \sin n\pi x, \text{ on simplifications } \dots (5)$$

From result (3) of Art. 7.4, we have

$$f_n = (f, \phi_n) = \int_0^1 f(x) \phi_n(x) \, dx = \sqrt{2} \int_0^1 f(x) \sin n\pi x \, dx \quad \dots (16)$$

Then, the given integral equation (1) has a solution of class $\mathcal{L}_2$ if and only if the infinite series $\sum_{n=1}^{\infty} f_n^2 \lambda_n^2$ i.e., $\pi^4 \sum_{n=1}^{\infty} (n^4 f_n^2)$ converges, where f_n is given by (16).

Ex. 2. Solve the Poisson's integral equation

$$f(\theta) = \frac{1-\rho^2}{2\pi} \int_0^{2\pi} \frac{y(\psi) \, d\psi}{1-2\rho \cos(\theta-\psi)+\rho^2}, \quad 0 \leq \theta \leq 2\pi, \quad 0 < \rho < 1$$

Solution. Comparing the given integral with

$$f(\theta) = \int_0^{2\pi} K(\theta, \psi) y(\psi) \, d\psi,$$

we get $K(\theta, \psi) = \{(1 - \rho^2) / 2\pi\} \times \{1 - 2\rho \cos(\theta - \psi) + \rho^2\}^{-1}$,
which is clearly a symmetric kernel. Expanding it, we get

$$K(\theta, \psi) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \rho^n \cos \{n(\theta - \psi)\} \quad \dots (1)$$

We can easily show that (1) leads to

$$\begin{aligned} \int_0^{2\pi} K(\theta, \psi) d\psi = 1 \quad \text{so that} \quad \int_0^{2\pi} (2\pi)^{-1/2} K(\theta, \psi) d\psi = (2\pi)^{-1/2}. \quad \dots (2) \\ \Rightarrow \quad \lambda_0 = 1 \quad \text{and} \quad \phi_0(x) = (2\pi)^{-1/2}, \end{aligned}$$

where λ_0 is an eigenvalue of $K(\theta, \psi)$ and $\phi_0(x)$ is the corresponding normalized eigenfunction.

Similarly for $n = 1, 2, 3, \dots$, using the well known formulas

$$\int_0^{2\pi} K(\theta, \psi) \cos n\psi d\psi = \rho^n \cos n\theta \quad \text{and} \quad \int_0^{2\pi} K(\theta, \psi) \sin n\psi d\psi = \rho^n \sin n\theta,$$

we can easily show that

$$\lambda_{2n-1} = \lambda_{2n} = \rho^{-n}; \quad \phi_{2n-1}(x) = \pi^{-1/2} \cos nx; \quad \phi_{2n}(x) = \pi^{-1/2} \sin nx, \quad n = 1, 2, 3, \dots \dots (3)$$

where λ_n is n th eigenvalue and $\phi_n(x)$ is the corresponding normalized eigenfunction of $K(\theta, \psi)$.

We know that the given Poisson's integral equation has unique $\mathcal{L}_2$ -solution provided the series $\sum_{n=1}^{\infty} f_n^2 \lambda_n^2$ converges, where f_n can be evaluated with help of result (3) of Art. 7.4, namely $f_n = (f, \phi_n)$.

Then we can show that the given integral equation has an $\mathcal{L}_2$ -solution if and only if the infinite series $\sum_{n=1}^{\infty} (a_n^2 + b_n^2) / \rho^{2n}$, where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta \quad \text{and} \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta, \quad \text{converges.}$$

Ex. 3. Find a Fourier series solution for the integral equation

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \rho^2}{1 - 2\rho \cos(x - t) + \rho^2} y(t) dt, \quad 0 < \rho < 1, \quad -\pi \leq x \leq \pi \quad \dots (1)$$

Sol. We can show that

$$\frac{1}{2\pi} \times \frac{1 - \rho^2}{1 - 2\rho \cos(x - t) + \rho^2} = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \rho^n (\cos nx \cos nt + \sin nx \sin nt) \quad \dots (2)$$

and the series is absolutely convergent.

We know that the Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \dots (3)$$

$$\text{where} \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n \geq 0 \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n > 0 \quad \dots (4)$$

$$\text{Let} \quad y(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos nt + d_n \sin nt) \quad \dots (5)$$

Then, we have

$$\int_{-\pi}^{\pi} \left\{ \frac{\pi}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n (\cos nx \cos nt + \sin nx \sin nt) \right\} \times \left\{ \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos nt + d_n \sin nt) \right\} dt$$

$$= \frac{c_0}{2} + \sum_{n=1}^{\infty} \alpha^n (c_n \cos nx + d_n \sin nx)$$

$$\Rightarrow a_n = c_n \rho^n \quad \text{and} \quad b_n = d_n \rho^n$$

Hence the solution of the given integral equation (1) is given by the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^{-n} (a_n \cos nt + b_n \sin nt) \quad \dots (6)$$

provided (6) converges.

7.10. APPROXIMATION OF A GENERAL $\mathcal{L}_2$ -KERNEL (NOT NECESSARILY SYMMETRIC) BY A SEPARABLE KERNEL.

In Art. 4.5, we approximated an analytic kernel $x(e^{xt} - 1)$ by a separable kernel. In the present article, we propose to prove that it is possible to approximate every $\mathcal{L}_2$ -kernel in the mean by a separable kernel.

In what follows we shall use the result of Art. 7.1 (k), namely, the availability of a two-dimensional complete orthonormal set.

Let $K(x, t)$ be an $\mathcal{L}_2$ -kernel and let $\{\psi_i(x)\}$ be an arbitrary, complete, orthonormal set over $a \leq x \leq b$. Then, the set $\{\Psi_i(x) \bar{\Psi}_j(t)\}$ is a complete orthonormal set over the square $a \leq x \leq b, a \leq t \leq b$. Here the bar denotes the complex conjugate. The Fourier expansion of the kernel in this set is given by

$$K(x, t) = \sum_{i,j=1}^n K_{ij} \psi_i(x) \bar{\psi}_j(t), \quad \dots (1)$$

where the K_{ij} are the Fourier coefficients given by

$$K_{ij} = \int_a^b \int_a^b K(x, t) \bar{\psi}_i(x) \psi_j(t) dx dt \quad \dots (2)$$

$$\text{Parseval's identity gives} \quad \int_a^b \int_a^b |K(x, t)|^2 dx dt = \sum_{i,j=1}^{\infty} |K_{ij}|^2 \quad \dots (3)$$

Define a separable kernel $k(x, t)$ as follows :

$$k(x, t) = \sum_{i,j=1}^n K_{ij} \psi_i(x) \bar{\psi}_j(t) \quad \dots (4)$$

Then, we can easily show that

$$\int_a^b \int_a^b |K(x, t) - k(x, t)|^2 dx dt = \sum_{i,j=n+1}^{\infty} |K_{ij}|^2 \quad \dots (5)$$

Since the series (3) is convergent by hypothesis, the sum in (5) can be made as small as we require by choosing a sufficiently large n .

This prove what we wished to prove.

7.11. THE OPERATOR METHOD IN THE THEORY OF INTEGRAL EQUATIONS.

In the present article we propose to deal with a Fredholm integral equation by using the concepts of functional analysis. We have already dealt with the properties of a function space in Art. 7.1. (e). Recall the transformation or Fredholm operator K given by

$$K\phi = \int_a^b K(x, t) \phi(t) dt \quad \dots (1)$$

Let α be a constant. Then, we can show that

$$K(\phi_1 + \phi_2) = K\phi_1 + K\phi_2 \quad \text{and} \quad K(\alpha\phi) = \alpha K(\phi),$$

showing that K is a *linear operator*.

The operator K is said to be *bounded* if $\|K\phi\| \leq M \|\phi\|$ for an $\mathcal{L}_2$ -kernel $K(x, t)$, an $\mathcal{L}_2$ -function ϕ and a constant M .

The *norm* $\|K\|$ of K is defined as

$$\|K\| = l.u.b. (\|Ky\| / \|y\|) \quad \text{or} \quad \|K\| = l.u.b. \|Ky\|, \|y\| = 1, \quad \dots (2)$$

where *l.u.b.* stands for least upper bound. The two criterions given in (2) are equivalent.

A transformation K is said to be *continuous* in an $\mathcal{L}_2$ -space if, whenever $\{\phi_n\}$ is a sequence in the domain of K with limit ϕ , then $K\phi_n \rightarrow K\phi$. A transformation is said to be *continuous in the entire domain* of K if it is continuous at every point therein. It is to be noted that *a linear transformation is continuous if it is bounded*.

7.11 (a) To show that the operator K given by (1) is bounded

Proof. We begin with the relation

$$\psi(x) = K\phi = \int_a^b K(x, t) \phi(t) dt$$

$$\begin{aligned} \text{Then,} \quad |\psi(x)|^2 &= \left| \int_a^b K(x, t) \phi(t) dt \right|^2 \leq \left[\int_a^b |K(x, t)|^2 dt \right] \times \left[\int_a^b |\phi(t)|^2 dt \right] \\ &\quad \text{[using Schwarz inequality]} \end{aligned}$$

$$\begin{aligned} \text{or} \quad |\psi(x)|^2 &\leq \|\phi\|^2 \int_a^b |K(x, t)|^2 dt \\ \Rightarrow \int_a^b |\psi(x)|^2 dx &\leq \|\phi\|^2 \int_a^b \int_a^b |K(x, t)|^2 dx dt \\ \Rightarrow \|\psi\|^2 &\leq \|\phi\|^2 \int_a^b \int_a^b |K(x, t)|^2 dx dt \\ \Rightarrow \|\psi\| = \|K\phi\| &\leq \|\phi\| \left\{ \int_a^b \int_a^b |K(x, t)|^2 dx dt \right\}^{1/2} \\ \Rightarrow \|K\| &\leq \left\{ \int_a^b \int_a^b |K(x, t)|^2 dx dt \right\}^{1/2}, \text{ as } \|K\phi\| = \|K\| \|\phi\| \quad \dots (3) \end{aligned}$$

showing that the operator K is bounded.

7.11 (b) The concept of complete continuity.

Recall that a set S of elements ϕ is said to be *compact* if a subsequence having a limit can be extracted from any sequence of elements of S . Now, an operator is said to be a *completely continuous* if it transforms a bounded set into a compact set. Clearly, a completely continuous operator is continuous (and therefore bounded), but the converse is not true. Again any bounded operator K whose range is finite dimensional is completely continuous because it transforms a bounded set in $\mathcal{L}_2[a, b]$ into a bounded finite-dimensional set which is always compact. Fortunately, number of the integral operators, that arise in applications are completely continuous. We shall now establish this fact.

7.11 (c) To show that a separable kernel $K(x, t)$ given by

$$K(x, t) = \sum_{i=1}^n f_i(x) g_i(t), \quad \dots (4)$$

where $f_i(x)$ and $g_i(t)$ are $\mathcal{L}_2$ -functions, is completely continuous.

Proof. For each $\mathcal{L}_2$ -function $y(x)$

$$Ky = \int_a^b \left[\sum_{i=1}^n f_i(x) g_i(t) y(t) \right] dt = \sum_{i=1}^n C_i f_i(x), \quad \dots (5)$$

where

$$C_i = \int_a^b g_i(t) y(t) dt \quad \dots (6)$$

(5) shows that the range of K is a finite dimensional subspace of $\mathcal{L}_2(a, b)$.

$$\|Ky\| = \left\| \sum_{i=1}^n C_i f_i(x) \right\| \leq \sum_{i=1}^n |C_i| \|f_i\|$$

or

$$\|Ky\| \leq \sum_{i=1}^n \|f_i\| \int_a^b |g_i(t)| |y(t)| dt, \text{ by (6)} \quad \dots (7)$$

Using the Schwarz inequality (see Art. 7.1 (d)), (7) gives

$$\|Ky\| \leq M \|y\|, \quad \dots (8)$$

where

$$M = \sum_{i=1}^n \|f_i\| \|g_i\|$$

It follows that K is a bounded operator with finite-dimensional range and hence is completely continuous.

7.11. (d). To show that an $\mathcal{L}_2$ -kernel $K(x, t)$ is completely continuous

Proof. To this end we shall use the result of an important theorem, namely, *if K can be approximated in norm by a completely continuous operator, then K is completely continuous.* By using this theorem the required result follows because an $\mathcal{L}_2$ -kernel can always be approximated by a separable kernel (see Art. 7.1 (a)) and separable kernel is completely continuous as just proved in 7.11 (c).

7.11 (e). To show that the norms of K and of its adjoint $\bar{K}$ are equal

Proof. We know that (refer result (13) in Art. 7.1 (e))

$$(K\phi, \psi) = (\phi, \bar{K}\psi), \quad \dots (9)$$

which is valid for each pair of $\mathcal{L}_2$ -functions ϕ, ψ .

Replacing ψ by $K\phi$ in (9) and then using the Schwarz inequality, we get

$$(K\phi, K\phi) = (\phi, \bar{K}K\phi) \leq \|\phi\| \|\bar{K}K\phi\|,$$

remembering that $(K\phi, K\phi)$ is a nonnegative real number.

$$\begin{aligned} \therefore \quad & \|K\phi\|^2 \leq \|\phi\| \|\bar{K}\| \|K\phi\| \quad \text{or} \quad \|K\phi\| \leq \|\bar{K}\| \|\phi\| \\ \Rightarrow \quad & \|K\| \|\phi\| \leq \|\bar{K}\| \|\phi\| \Rightarrow \|K\| \leq \|\bar{K}\| \end{aligned} \quad \dots (10)$$

Next replacing ϕ by $\bar{K}\psi$ in (9) and proceeding as before, we obtain

$$\|\bar{K}\| \leq \|K\| \quad \dots (11)$$

Then, (10) and (11) $\Rightarrow \|K\| = \|\bar{K}\|$, as required ... (12)

7.11. (f). Theorem. *The reciprocal of the modulus of the eigenvalue with the smallest modulus for a symmetric $\mathcal{L}_2$ -kernel K is equal to the maximum value of $|(K\phi, \phi)|$ with $\|\phi\| = 1$.*

[Refer theorem IX of Art. 7.2.. The result of this theorem will now be proved in this Art. 7.11 (f)]

Proof. We can easily show that an upper bound for the reciprocal of the eigenvalue can be easily obtained because, for the eigenvalue problem $\lambda K\phi = \phi$, we have

$$\begin{aligned} (K\phi, \phi) &= (1/\lambda) (\phi, \phi) = (1/\lambda) \|\phi\|^2 \\ \Rightarrow (1/\lambda) \times \|\phi\|^2 &= (K\phi, \phi) \leq \|K\phi\| \|\phi\| \leq \|K\| \|\phi\|^2 \end{aligned} \quad \dots (13)$$

$$\text{From (3) and (13), } |1/\lambda| \leq \left[\int_a^b \int_a^b |K(x, t)|^2 dx dt \right]^{1/2}, \quad \dots (14)$$

from which an upper bound of the reciprocal of the modulus of the eigenvalue λ can be obtained.

When the $\mathcal{L}_2$ -kernel is also symmetric, the following result from the theory of operators can be used. *"If K is a symmetric and completely continuous operator; at least one of the numbers $\|K\|$ or $-\|K\|$ is the reciprocal of an eigenvalue of K and no other eigenvalue of K has smaller absolute value"*

Using the definition of $\|K\|$ and the fact that a symmetric $\mathcal{L}_2$ -kernel generates completely continuous operator, we have proved the required result stated in theorem 7.11 (f).

Corollary. *Every symmetric kernel with a norm not equal to zero has at least one eigenvalue.*

Proof. The result follows from theorem of Art. 7.11 (f) [Merrut 2000, 01, 02, 06]

Remark. We have already proved a special case of the above result for real symmetric kernel is theorem II of Art. 7.2.

7.11 (g) Procedure for getting the eigenvalues and eigenfunctions arranged in the sequences (1) and (2) as given in Art. 7.3.

Suppose the first eigenvalue λ_1 and the corresponding eigenfunction are known. Then, to evaluate λ_2 and ϕ_2 , we shorten the kernel K by subtracting the factor $(\phi_1 \bar{\phi}_1)/\lambda_1$ from it. Then, from theorem II of Art. 7.3, we know that the kernel

$$K^{(2)} = K - (\phi_1 \bar{\phi}_1)/\lambda_1$$

satisfies all the requirements of a symmetric $\mathcal{L}_2$ -kernel. Proceeding as explained earlier, we see that at least one of the numbers $\|K_2\|$ or $-\|K_2\|$ is the reciprocal of λ_2 .

Proceeding likewise, the process can be continued until all the eigenvalues and eigenfunctions are obtained.

Remarks. The only draw back in the above process is that, to find the n th eigenvalues, we have to obtain the first $(n-1)$ eigenvalues.

MISCELLANEOUS EXERCISES ON CHAPTER 7

1. Compute the iterated kernels for symmetric kernel $K(x, t) = \sum_{n=1}^{\infty} n^{-1} \sin n\pi x \sin n\pi t$.

2. Consider the eigenvalue problem $y(x) = \lambda \int_{-1}^1 (1 - |x-t|) y(t) dt$

Differentiate under the integral sign to obtain the corresponding differential equation and the boundary conditions. Show that the kernel of this integral equation is positive.

3. Compute the eigenvalues and eigenfunctions of the symmetric kernel $K(x, t) = \min(x, t)$ in the basic interval $0 < x < 1, 0 < t < 1$.

4. Show that the kernel $K(x, t), 0 < x < 1, 0 < t < 1$,

$$K(x, t) = \begin{cases} x(1-t), & x < t \\ t(1-x), & x > t \end{cases} \text{ has the bilinear form} \quad K(x, t) = 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi t}{(n\pi)^2}$$

$$\text{Hence deduce that } \sum_{n=1}^{\infty} (1/n^2) = \pi^2/6$$

5. Use the Gram-Schmidt process to orthogonalize $1, x, x^2, x^3$ in the interval $-1 \leq x \leq 1$ and thus compute eigenvalues and eigenfunctions of the symmetric kernel $1 + xt + x^2 t^2 + x^3 t^3$.

6. Consider the kernel $K(x, t) = \log [1 - \cos(x - t)], 0 \leq x \leq 2\pi, 0 \leq t \leq 2\pi$. Prove that

(i) it is a symmetric $\mathcal{L}_2$ -kernel

(ii) $K(x, t) = -\log 2 + 2 \log [1 - e^{i(x-t)}]$

$$= -\log 2 - 2 \sum_{n=1}^{\infty} \frac{\cos nx \cos nt}{n} - 2 \sum_{n=1}^{\infty} \frac{\sin nx \sin nt}{n}$$

(iii) its eigenvalues are $\lambda_0 = -1/(2\pi \log 2), \lambda_n = -n/2\pi, n = 1, 2, \dots$ with eigenfunctions

$$\phi_0(x) = C, \phi_n(x) = A \cos nx + B \sin nx, A, B \text{ and } C \text{ are constants.}$$

7. Which one of the following sets of functions is not orthogonal (with respect to the L^2 -inner product) over the given interval

(a) $\{\sin nx : n \in \mathbb{N}, -\pi < x < \pi\}$

(b) $\{\cos nx : n \in \mathbb{N}, -\pi < x < \pi\}$

(c) $\{x^{2n+1/2} : n \in \mathbb{N}, -1 < x < 1\}$

(d) $\{x^{2n+1} : n \in \mathbb{N}, -1 < x < 1\}$. [GATE 2010]

Solution. Ans. (d). Use definition of orthogonality given in Art. 7.1 (c), page 7.2. Here,

$$\text{for } n \neq m, \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \{\cos(n-m)x - \cos(n+m)x\} \, dx = \frac{1}{2} \left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m} \right]_{-\pi}^{\pi} = 0$$

$$\text{Similarly, } \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \quad \text{and} \quad \int_{-1}^1 x^{2n+1/2} x^{2m+1/2} \, dx = \left[\frac{x^{2n+2m+2}}{2n+2m+2} \right]_{-1}^1 = 0$$

$$\text{But, we have} \quad \int_{-1}^1 x^{2m+1} x^{2n+1} \, dx = \left[\frac{x^{2m+2n+3}}{2m+2n+3} \right]_{-1}^1 = \frac{2}{2m+2n+3} \neq 0.$$

Hence, the sets (A), (B) and (C) are orthogonal while the set (D) is not orthogonal.

CHAPTER 8

Singular Integral Equations

8.1. SINGULAR INTEGRAL EQUATION. DEFINITION.

[Meerut 2001, 06, 07]

An integral equation in which the range of integration is infinite, or in which the kernel is discontinuous, is known as a *singular integral equation*. Thus, for example, the equations

$$f(x) = \int_0^\infty \sin(xt) y(t) dt, \quad f(x) = \int_0^\infty e^{-xt} y(t) dt, \quad \text{and} \quad f(x) = \int_0^x \frac{y(t)}{\sqrt{x-t}} dt$$

are all *singular integral equations of the first kind*.

Remark. Singular integral equations possess very unusual properties.

8.2. THE SOLUTION OF THE ABEL INTEGRAL EQUATION, NAMELY,

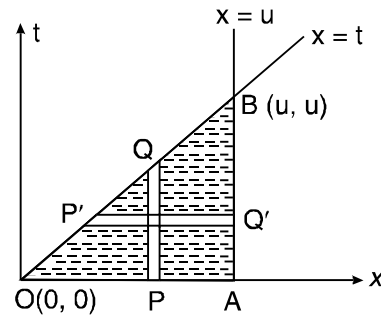
$$f(x) = \int_0^x \frac{y(t)}{(x-t)^\alpha} dt, \quad 0 < \alpha < 1. \quad \dots (1)$$

in which $f(x)$ is a known function while $y(t)$ is to be determined. [Kanpur 2006, Meerut 2004]

Multiplying both sides of (1) by $1/(u-x)^{1-\alpha}$ and then integrating w.r.t. 'x' from 0 to u, we

$$\text{obtain} \quad \int_0^u \frac{f(x) dx}{(u-x)^{1-\alpha}} = \int_{x=0}^{x=u} \frac{1}{(u-x)^{1-\alpha}} \left\{ \int_{t=0}^{t=x} \frac{y(t) dt}{(x-t)^\alpha} \right\} dx. \quad \dots (2)$$

Consider the double integral on right side of (2). This integral is to be first integrated in the t -direction from $t = 0$ to $t = x$ and then the resulting integral is to be integrated in the x -direction from $x = 0$ to $x = u$. The region of integration is the triangular area OAB . In the integral under consideration, the area OAB is divided in strips parallel to t -axis (for example strip PQ). To reverse the order of integration, we have to first integrate with respect to x regarding t as constant and then with respect to t . This is done by dividing the above mentioned area OAB in strips parallel to the x -axis (for example, $P'Q'$).



Thus, we note that first we must integrate from $x=t$ to $x=u$ in x -direction and afterwards in the t -direction from $t = 0$ to $t = u$. Thus, changing the order of integration on right side of (2), we obtain

$$\int_0^u \frac{f(x) dx}{(u-x)^{1-\alpha}} = \int_{t=0}^{t=u} y(t) \left\{ \int_{x=t}^{x=u} \frac{dx}{(u-x)^{1-\alpha} (x-t)^\alpha} \right\} dt. \quad \dots (3)$$

$$\text{Let} \quad I = \int_{x=t}^{x=u} \frac{dx}{(u-x)^{1-\alpha} (x-t)^\alpha}. \quad \dots (4)$$

$$\text{Put} \quad (u-x)/(u-t) = y \quad \dots (5A)$$

$$\text{that is,} \quad x = u - (u-t)y. \quad \dots (5B)$$

From (5B) $dx = -(u-t)dy$ (6)

Using (5A), (5B) and (6) in (4), we have

$$\begin{aligned} I &= - \int_{y=1}^{y=0} \frac{(u-t) dy}{(u-t)^{1-\alpha} y^{1-\alpha} \{u-(u-t)y-t\}^\alpha} = \int_0^1 \frac{(u-t) dy}{(u-t)^{1-\alpha} y^{1-\alpha} (u-t)^\alpha (1-y)^\alpha} \\ &= \int_0^1 y^{\alpha-1} (1-y)^{-\alpha} dy = \int_0^1 y^{\alpha-1} (1-y)^{(1-\alpha)-1} dy \\ &= B(\alpha, 1-\alpha), \text{ by the definition of the Beta function} \\ &= \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)} = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \alpha\pi}, \end{aligned} \quad \dots (7)$$

where $\Gamma(\alpha)$ is the usual gamma function.

From (4) and (7),
$$\int_{x=t}^{x=u} \frac{dx}{(u-x)^{1-\alpha} (x-t)^\alpha} = \frac{\pi}{\sin \alpha\pi}. \quad \dots (8)$$

Substituting (8) in (3), we have
$$\int_0^u \frac{f(x) dx}{(u-x)^{1-\alpha}} = \int_0^u y(t) \left\{ \frac{\pi}{\sin \alpha\pi} \right\} dt.$$

or
$$\int_0^u y(t) dt = \frac{\sin \alpha\pi}{\pi} \int_0^u \frac{f(x) dx}{(u-x)^{1-\alpha}}. \quad \dots (9)$$

Differentiating both sides of (9) w.r.t. u and using Leibnitz's rule of differentiation under the sign of integration (refer Art. 1.13), we have

$$y(u) = \frac{\sin \alpha\pi}{\pi} \frac{d}{du} \left[\int_0^u \frac{f(x)}{(u-x)^{1-\alpha}} dx \right] \quad \dots (10)$$

Replacing u by t on both sides of (10), we get

$$y(t) = \frac{\sin \alpha\pi}{\pi} \frac{d}{du} \left[\int_0^t \frac{f(x)}{(t-x)^{1-\alpha}} dx \right]. \quad \dots (11)$$

which is required solution of (1).

Example. Solve the singular integral equation $x = \int_0^x \frac{y(t) dt}{(x-t)^{1/2}}.$

[Kanpur 2006, 10; Meerut 2006, 09, 12]

Sol. Given $x = \int_0^x \frac{y(t) dt}{(x-t)^{1/2}}. \quad \dots (1)$

Comparing (1) with the Abel singular integral equation

$$f(x) = \int_0^x \frac{y(t) dt}{(x-t)^\alpha} \quad \dots (2)$$

here $f(x) = x$ and $\alpha = 1/2. \quad \dots (3)$

We know that the solution of (2) is given by

$$y(t) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dt} \left[\int_0^t \frac{f(x)}{(t-x)^{1-\alpha}} dx \right]. \quad \dots (4)$$

Putting values of $f(x)$ and α from (3) in (4), the required solution is given by

$$y(t) = \frac{\sin(\pi/2)}{\pi} \frac{d}{dt} \left[\int_0^t \frac{x}{(t-x)^{1/2}} dx \right] \quad \text{or} \quad y(t) = \frac{1}{\pi} \frac{dI}{dt}, \quad \dots (5)$$

where

$$I = \int_0^t \frac{x}{(t-x)^{1/2}} dx. \quad \dots (6)$$

Put $t-x = z^2$ so that $dx = -2z dz$ (7)

$$\begin{aligned} \therefore \text{From (6),} \quad I &= \int_{\sqrt{t}}^0 \frac{(t-z^2)(-2z dz)}{z} = -2 \int_{\sqrt{t}}^0 (t-z^2) dz \\ &= -2 \left[tz - z^3/3 \right]_{\sqrt{t}}^0 = -2[-t\sqrt{t} + (1/3) \times t\sqrt{t}] = (4/3) \times t^{3/2}. \end{aligned}$$

Putting the above value of I in (5), we get

$$y(t) = \frac{1}{\pi} \frac{d}{dt} \left(\frac{4}{3} t^{3/2} \right) = \frac{1}{\pi} \times \frac{4}{3} \times \frac{3}{2} \times t^{1/2} = \frac{2\sqrt{t}}{\pi}.$$

8.3. GENERAL FORM OF THE ABEL SINGULAR INTEGRAL EQUATION.

It is given by
$$f(x) = \int_a^x \frac{y(t) dt}{[h(x) - h(t)]^\alpha}, \quad 0 < \alpha < 1, \quad \dots (1)$$

where $h(t)$ is a strictly monotonically increasing and differentiable in (a, b) and $h'(t) \neq 0$.

Determination of solution of (1). To solve (1), we consider an integral I given by

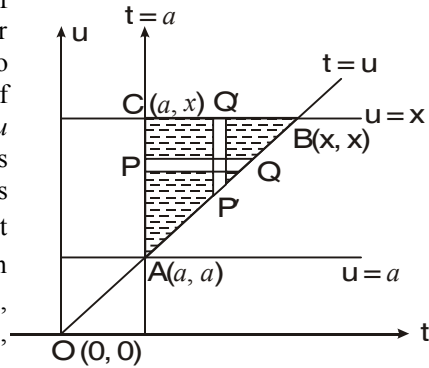
$$I = \int_a^x \frac{h'(u) f(u) du}{[h(x) - h(u)]^{1-\alpha}}. \quad \dots (2)$$

From (1),
$$f(u) = \int_a^u \frac{y(t) dt}{[h(u) - h(t)]^\alpha}. \quad \dots (3)$$

Substituting the above value of $f(u)$ in (2), we obtain

$$I = \int_{u=a}^{u=x} \frac{h'(u)}{[h(x) - h(u)]^{1-\alpha}} \left\{ \int_{t=a}^{t=u} \frac{y(t) dt}{[h(u) - h(t)]^\alpha} \right\} du. \quad \dots (4)$$

Clearly, the double integral on the right hand side of (4) is to be first integrated in the t -direction from $t=a$ to $t=u$ and then the resulting integral is to be integrated in the u -direction from $u=a$ to $u=x$. The region of integration is the triangular area ABC . In the integral under consideration, the area OAB is divided in strips parallel to t -axis (for example, strip PQ). To reverse the order of integration, we have to first integrate with respect to u regarding t as constant and then with respect to t . This is done by dividing the above mentioned area ABC in strips parallel to u -axis (for example, $P'Q'$). Thus, we note that first we must integrate from $u=t$ to $u=x$ in u -direction and afterwards in the t -direction from $t=a$ to $t=x$. Thus, changing the order of integration on right hand side of (4), we obtain.



$$I = \int_{t=a}^{t=x} y(t) \left\{ \int_{u=t}^{u=x} \frac{h'(u) du}{[h(x) - h(u)]^{1-\alpha} [h(u) - h(t)]^\alpha} \right\} dt. \quad \dots (5)$$

Re-writing (5), we have
$$I = \int_{t=a}^{t=x} J y(t) dt, \quad \dots (6)$$

where
$$J = \int_{u=t}^{u=x} \frac{h'(u) du}{[h(x) - h(u)]^{1-\alpha} [h(u) - h(t)]^\alpha}. \quad \dots (7)$$

We shall now simplify integral J .

Put $\{h(x) - h(u)\} / \{h(x) - h(t)\} = y, \quad \dots (8)$

so that $h(u) = h(x) - [h(x) - h(t)] y. \quad \dots (9)$

From (9) $h'(u) du = -[h(x) - h(t)] dy. \quad \dots (10)$

Using (8), (9) and (10) in (7), we have

$$\begin{aligned} J &= - \int_1^0 \frac{[h(x) - h(t)] dy}{[h(x) - h(t)]^{1-\alpha} y^{1-\alpha} [h(x) - \{h(x) - h(t)\} y - h(t)]^\alpha} \\ &= \int_0^1 \frac{[h(x) - h(t)] dy}{[h(x) - h(t)]^{1-\alpha} y^{1-\alpha} [h(x) - h(t)]^\alpha (1-y)^\alpha} \\ &= \int_0^1 y^{\alpha-1} (1-y)^{-\alpha} dy = \int_0^1 y^{\alpha-1} (1-y)^{(1-\alpha)-1} dy \\ &= B(\alpha, 1-\alpha), \text{ by the definition of the Beta function} \\ &= \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)} = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \alpha\pi}. \quad \dots (11) \end{aligned}$$

Substituting the above value of J in (6), we have

$$I = \int_a^x \frac{\pi}{\sin \alpha\pi} y(t) dt = \frac{\pi}{\sin \alpha\pi} \int_a^x y(t) dt. \quad \dots (12)$$

Equating the two values of I from (2) and (12), we have

$$\begin{aligned} \frac{\pi}{\sin \alpha\pi} \int_a^x y(t) dt &= \int_a^x \frac{h'(u) f(u) du}{[h(x) - h(u)]^{1-\alpha}} \\ \text{or } \int_a^x y(t) dt &= \frac{\sin \alpha\pi}{\pi} \int_a^x \frac{h'(u) f(u) du}{[h(x) - h(u)]^{1-\alpha}}. \quad \dots (13) \end{aligned}$$

Differentiating both sides of (13) w.r.t. 'x' and using Leibnitz's rule of differentiation under the sign of integration (refer Art. 1.13), we have

$$y(x) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dx} \left[\int_a^x \frac{h'(u) f(u) du}{[h(x) - h(u)]^{1-\alpha}} \right]. \quad \dots (14)$$

Replacing x by t on both sides of (14), we have

$$y(t) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dt} \left[\int_a^t \frac{h'(u) f(u) du}{[h(t) - h(u)]^{1-\alpha}} \right], \quad \dots (15)$$

which is the required solution of (1).

8.4. ANOTHER GENERAL FORM OF THE ABEL SINGULAR INTEGRAL EQUATION.

It is given by
$$f(x) = \int_x^b \frac{y(t) dt}{[h(t) - h(x)]^\alpha}, 0 < \alpha < 1 \quad \dots (1)$$

where $a < x < b$ and $h(t)$ is a monotonically increasing function.

Determination of solution of (1). To solve (1), we consider an integral I given by

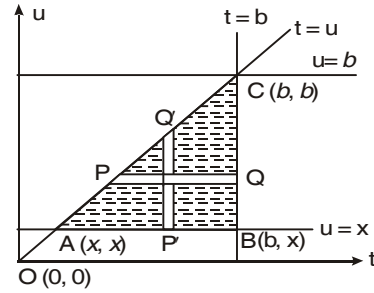
$$I = \int_x^b \frac{h'(u)f(u) du}{[h(t) - h(x)]^{1-\alpha}}. \quad \dots (2)$$

From (1),
$$f(u) = \int_u^b \frac{y(t) dt}{[h(t) - h(u)]^\alpha}. \quad \dots (3)$$

Substituting the above values of $f(u)$ in (2), we obtain

$$I = \int_{u=x}^{u=b} \frac{h'(u)}{[h(u) - h(x)]^{1-\alpha}} \left\{ \int_{t=u}^{t=b} \frac{y(t) dt}{[h(t) - h(u)]^\alpha} \right\} du. \quad \dots (4)$$

Clearly, the double integral on the right hand side of (4) is to be integrated in the t -direction from $t=u$ to $t=b$ and then the resulting integral is to be integrated in the u -direction from $u=x$ to $u=b$. The region of integration is the triangular area ABC . In the integral under consideration, the area ABC is divided in strips parallel to t -axis (for example, strip PQ). To reverse the order of integration, we have to first integrate with respect to u regarding t as constant and then with respect to t . This is done by dividing the above mentioned area ABC in strips parallel to u -axis (for example, strip $P'Q'$). Thus, we note that first we must integrate from $u=x$ to $u=t$ in u -direction and afterwards in the t -direction from $t=x$ to $t=b$. Thus, changing the order of integration on right hand side of (4), we obtain



$$I = \int_{t=x}^{t=b} y(t) \left\{ \int_{u=x}^{u=t} \frac{h'(u) du}{[h(u) - h(x)]^{1-\alpha} [h(t) - h(u)]^\alpha} \right\} dt. \quad \dots (5)$$

Re-writing (5), we have
$$I = \int_{t=x}^{t=b} J y(t) dt, \quad \dots (6)$$

where
$$J = \int_{u=x}^{u=t} \frac{h'(u) du}{[h(u) - h(x)]^{1-\alpha} [h(t) - h(u)]^\alpha}. \quad \dots (7)$$

We shall now simplify J .

Put $\{h(u) - h(x)\} / \{h(t) - h(x)\} = y \quad \dots (8)$

so that $h(u) = h(x) + [h(t) - h(x)] y. \quad \dots (9)$

From (9), $h'(u) du = \{h(t) - h(x)\} dy \quad \dots (10)$

Using (8), (9) and (10) in (7), we have

$$J = \int_0^1 \frac{[h(t) - h(x)] dy}{[h(t) - h(x)]^{1-\alpha} y^{1-\alpha} [h(t) - h(x) - \{h(t) - h(x)\} y]^\alpha}$$

$$\begin{aligned}
&= \int_0^1 \frac{[h(t) - h(x)] dy}{[h(t) - h(x)]^{1-\alpha} y^{1-\alpha} [h(t) - h(x)]^\alpha (1-y)^\alpha} \\
&= \int_0^1 y^{\alpha-1} (1-y)^{-\alpha} dy = \int_0^1 y^{\alpha-1} (1-y)^{(1-\alpha)-1} dy \\
&= B(\alpha, 1-\alpha), \text{ by the definition of the Beta function} \\
&= \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)} = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \alpha\pi}. \quad \dots (11)
\end{aligned}$$

Substituting the above value of J in (6), we have

$$I = \int_{t=x}^{t=b} \frac{\pi}{\sin \alpha\pi} y(t) dt = \frac{\pi}{\sin \alpha\pi} \int_x^b y(t) dt. \quad \dots (12)$$

Equating the two values of I from (2) and (12), we have

$$\begin{aligned}
&\frac{\pi}{\sin \alpha\pi} \int_x^b y(t) dt = \int_x^b \frac{h'(u) f(u) du}{[h(u) - h(x)]^{1-\alpha}} \\
\text{or} \quad &\int_x^b y(t) dt = \frac{\sin \alpha\pi}{\pi} \int_x^b \frac{h'(u) f(u) du}{[h(u) - h(x)]^{1-\alpha}}. \quad \dots (13)
\end{aligned}$$

Differentiating both sides of (13) w.r.t. 'x' and using Leibnitz's rule of differentiation under the sign of integration (refer Art. 1.13), we have

$$-y(x) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dx} \left[\int_x^b \frac{h'(u) f(u) du}{[h(u) - h(x)]^{1-\alpha}} \right]. \quad \dots (14)$$

Replacing x by t in (14) and multiplying both sides by (-1) , we have

$$y(t) = -\frac{\sin \alpha\pi}{\pi} \frac{d}{dt} \left[\int_t^b \frac{h'(u) f(u) du}{[h(u) - h(t)]^{1-\alpha}} \right]. \quad \dots (15)$$

Weakly singular kernel. Definition. Consider a Fredholm integral equation with the kernel

$$\text{of the form} \quad K(x, t) = \frac{H(x, t)}{|t - x|^\alpha}, \quad 0 < \alpha < 1 \quad \dots (1)$$

where $H(x, t)$ is a bounded function. Then the kernel (1) is known as *weakly singular*.

A weakly singular kernel can be transformed to a kernel which is bounded. This is done by means of iterated kernels. It has been established that if the singular kernel has the form (1), then there always exist a positive number p_0 , dependent in α , such that, for $p > p_0$, the iterated kernel $K_p(x, t)$ is bounded.

8.5. SOLVED EXAMPLES.

Ex. 1. Solve the integral equation $f(x) = \int_a^x \frac{y(t) dt}{(\cos t - \cos x)^{1/2}}, \quad 0 \leq a < x < b \leq \pi.$

[Meerut 2003, 10, 11]

Sol. Given that $f(x) = \int_a^x \frac{y(t) dt}{(\cos t - \cos x)^{1/2}}, \quad \dots (1)$

Re-writing (1), we have $f(x) = \int_a^x \frac{y(t) dt}{[(1 - \cos x) - (1 - \cos t)]^{1/2}}. \quad \dots (2)$

Comparing (2) with general form of Abel integral equation (refer Art. 8.3)

$$f(x) = \int_a^x \frac{y(t) dt}{[h(x) - h(t)]^\alpha}, 0 < \alpha < 1 \quad \dots (3)$$

where $h(t)$ is a strictly monotonically increasing and differentiable function in (a, b) and $h'(t) \neq 0$ in this interval, we have $\alpha = 1/2$, $h(t) = 1 - \cos t$.

Clearly $h(t)$ is a strictly monotonically increasing function in $(0, \pi)$.

We know that (refer equation (15) Art. 8.3) the solution of (3) is given by

$$y(t) = \frac{\sin \alpha \pi}{\pi} \frac{d}{dt} \left[\int_a^t \frac{h'(u) f(u) du}{[h(t) - h(u)]^{1-\alpha}} \right]. \quad \dots (4)$$

Putting $\alpha = 1/2$, $h(t) = 1 - \cos t$, $h(u) = 1 - \cos u$ and $h'(u) = \sin u du$ in (4), the required solution of (1) is

$$y(t) = \frac{\sin(\pi/2)}{\pi} \frac{d}{dt} \left[\int_a^t \frac{\sin u f(u) du}{[(1 - \cos t) - (1 - \cos u)]^{1/2}} \right]$$

or

$$y(t) = \frac{1}{\pi} \frac{d}{dt} \left[\int_a^t \frac{\sin u f(u) du}{(\cos u - \cos t)^{1/2}} \right], a < t < b.$$

Ex. 2. Solve $f(x) = \int_x^b \frac{y(t) dt}{(\cos x - \cos t)^{1/2}}, 0 \leq a < x < b \leq \pi$.

Sol. Given that $f(x) = \int_x^b \frac{y(t) dt}{(\cos x - \cos t)^{1/2}}, 0 \leq a < x < b \leq \pi \quad \dots (1)$

Re-writing (1), we have $f(x) = \int_x^b \frac{y(t) dt}{[(1 - \cos t) - (1 - \cos x)]^{1/2}}. \quad \dots (2)$

Comparing (2) with general form of Abel integral equation (refer Art. 8.4)

$$f(x) = \int_x^b \frac{y(t) dt}{[h(t) - h(x)]^\alpha}, 0 < \alpha < 1. \quad \dots (3)$$

where $h(t)$ is a monotonically increasing function, we have $\alpha = 1/2$, $h(t) = 1 - \cos t$.

Clearly $h(t)$ is a monotonically increasing function,

We known that (refer equation (15) in Art. 8.4) the solution of (3) is given by

$$y(t) = -\frac{\sin \alpha \pi}{\pi} \frac{d}{dt} \left[\int_t^b \frac{h'(u) f(u) du}{[h(u) - h(t)]^{1-\alpha}} \right]. \quad \dots (4)$$

Putting $\alpha = 1/2$, $h(t) = 1 - \cos t$, $h(u) = 1 - \cos u$ and $h'(u) = \sin u du$ in (4), the required solution of (1) is

$$y(t) = -\frac{\sin(\pi/2)}{\pi} \frac{d}{dt} \left[\int_t^b \frac{\sin u f(u) du}{[1 - \cos u - (1 - \cos t)]^{1/2}} \right]$$

or

$$y(t) = -\frac{1}{\pi} \frac{d}{dt} \left[\int_t^b \frac{\sin u f(u) du}{(\cos t - \cos u)^{1/2}} \right], a < t < b.$$

Ex. 3. Solve the integral equation $f(x) = \int_a^x \frac{y(t) dt}{(x^2 - t^2)^\alpha}, 0 < \alpha < 1; a < x < b$.

Sol. Given that
$$f(x) = \int_a^x \frac{y(t) dt}{(x^2 - t^2)^\alpha}, 0 < \alpha < 1; a < x < b \quad \dots (1)$$

Comparing (2) with general form of Abel integral equation (refer Art. 8.3)

$$f(x) = \int_a^x \frac{y(t) dt}{[h(x) - h(t)]^\alpha}, 0 < \alpha < 1$$

where $h(t)$ is a strictly monotonically increasing and differentiable function in (a, b) and $h'(t) \neq 0$ in this interval, we have

$$h(t) = t^2. \quad \dots (3)$$

Clearly $h(t)$ is a strictly monotonically increasing and differentiable function.

We know that (refer equation (15) in Art. 8.3) the solution of (2) is given by

$$y(t) = \frac{\sin \alpha \pi}{\pi} \frac{d}{dt} \left[\int_a^t \frac{h'(u) f(u) du}{[h(t) - h(u)]^{1-\alpha}} \right]. \quad \dots (4)$$

Putting $h(t) = t^2$, $h(u) = u^2$ and $h'(u) = 2u du$ in (4), the required solution is given by

$$y(t) = \frac{\sin \alpha \pi}{\pi} \frac{d}{dt} \left[\int_a^t \frac{2u f(u) du}{(t^2 - u^2)^{1-\alpha}} \right], a < t < b$$

or

$$y(t) = \frac{2 \sin \alpha \pi}{\pi} \frac{d}{dt} \left[\int_a^t \frac{u f(u) du}{(t^2 - u^2)^{1-\alpha}} \right], a < t < b. \quad \dots (5)$$

Remark. The result (5) remains valid when $a \rightarrow 0$. Hence, the solution of the integral equation

$$f(x) = \int_0^x \frac{y(t) dt}{(x^2 - t^2)^\alpha}, 0 < \alpha < 1 \quad \dots (6)$$

is

$$y(t) = \frac{2 \sin \alpha \pi}{\pi} \frac{d}{dt} \left[\int_0^t \frac{u f(u) du}{(t^2 - u^2)^{1-\alpha}} \right]. \quad \dots (7)$$

Ex. 4. Solve the integral equation $f(x) = \int_x^b \frac{y(t) dt}{(t^2 - x^2)^\alpha}, 0 < \alpha < 1; a < x < b$. [Meerut 2004, 07]

Sol. Given that
$$f(x) = \int_x^b \frac{y(t) dt}{(t^2 - x^2)^\alpha}, 0 < \alpha < 1; a < x < b. \quad \dots (1)$$

Comparing (2) with general form of Abel integral equation (refer Art. 8.4)

$$f(x) = \int_x^b \frac{y(t) dt}{[h(t) - h(x)]^\alpha}, 0 < \alpha < 1 \quad \dots (2)$$

where $h(t)$ is a monotonically increasing function, we have

$$h(t) = t^2. \quad \dots (3)$$

Clearly, $h(t)$ is a strictly monotonically increasing function.

We known that (refer equation (15) in Art. 8.4) the solution of (2) is given by

$$y(t) = -\frac{\sin \alpha \pi}{\pi} \frac{d}{dt} \left[\int_t^b \frac{h'(u) f(u) du}{[h(u) - h(t)]^{1-\alpha}} \right]. \quad \dots (4)$$

Putting $h(t) = t^2$, $h(u) = u^2$ and $h'(u) = 2u du$ in (4), the required solution is given by

$$y(t) = -\frac{\sin \alpha \pi}{\pi} \frac{d}{dt} \left[\int_t^b \frac{2u f(u) du}{(u^2 - t^2)^{1-\alpha}} \right]$$

or
$$y(t) = -\frac{2 \sin \alpha \pi}{\pi} \frac{d}{dt} \left[\int_t^b \frac{u f(u) du}{(u^2 - t^2)^{1-\alpha}} \right]. \quad \dots (5)$$

Remark. The result (5) remains valid when $b \rightarrow \infty$. Hence from (1) and (5), it follows that the solution of the integral equation

$$f(x) = \int_x^\infty \frac{y(t) dt}{(t^2 - x^2)^\alpha}, \quad 0 < \alpha < 1, \quad \dots (6)$$

is
$$y(t) = -\frac{2 \sin \alpha \pi}{\pi} \frac{d}{dt} \left[\int_t^\infty \frac{u f(u) du}{(u^2 - t^2)^{1-\alpha}} \right]. \quad \dots (7)$$

8.6. CAUCHY PRINCIPAL VALUE FOR INTEGRALS.

[Meerut 2001]

According to Riemann, the theory of the definite integrals $\int_a^b f(x) dx$ is based on two assumptions, namely, (i) the integrand $f(x)$ is bounded and (ii) The range of integration (a, b) is finite. These are called *ordinary integrals*.

Cauchy extended the theory of Riemann integration to include the following exceptional cases :

(i) When $f(x)$ becomes infinite at one of the limits of integration.

(ii) When $f(x)$ becomes infinite for one or more values of x between the range of integration.

(iii) When one or both the limits of integration are infinite.

Such integrals are called *improper integrals*. Integrals of these forms may have a finite value, it may be infinite or indeterminate depending upon the function $f(x)$ and the limits a and b .

Cauchy's general and principal values. Singular integrals. In case of an improper integral $\int_a^b f(x) dx$, where $f(x)$ is unbounded at $x = c$, but is bounded in each of the intervals $(a, c - \epsilon_1)$ and $(c + \epsilon_2, b)$ where ϵ_1 and ϵ_2 are arbitrary small positive numbers. Then, the limit

$$\int_a^b f(x) dx = \lim_{\substack{\epsilon_1 \rightarrow 0 \\ \epsilon_2 \rightarrow 0}} \left[\int_a^{c-\epsilon_1} f(x) dx + \int_{c+\epsilon_2}^b f(x) dx \right], \quad \dots (1)$$

if it exists, is called the *general value* of the improper integral. Here it is understood that ϵ_1 and ϵ_2 tend to zero independently. But it may happen that the limit (1) does not exist when ϵ_1 and ϵ_2 tend to zero independently of each other, but it exists when $\epsilon_1 = \epsilon_2 = \epsilon$, say. Such a limit is known as the *principal value* of the integral and is usually denoted by

$$P \int_a^b f(x) dx \quad \text{or} \quad \int_a^b f(x) dx. \\ \therefore P \int_a^b f(x) dx = \int_a^b f(x) dx = \lim_{\epsilon \rightarrow 0} \left[\int_a^{c-\epsilon} f(x) dx + \int_{c+\epsilon}^b f(x) dx \right]. \quad \dots (2)$$

Similarly, the general value of $\int_{-\infty}^\infty f(x) dx$ is defined by the limit,

$$\int_{-\infty}^\infty f(x) dx = \lim_{\substack{\epsilon_1 \rightarrow 0 \\ \epsilon_2 \rightarrow 0}} \int_{-1/\epsilon_2}^{1/\epsilon_1} f(x) dx \quad \dots (3)$$

and the corresponding principal value is given by

$$P \int_{-\infty}^{\infty} f(x) dx = \lim_{\epsilon \rightarrow 0} \int_{-1/\epsilon}^{1/\epsilon} f(x) dx. \quad \dots (4)$$

The limits (2) and (4) are also known as *singular integrals*. Such singular integrals exist when the integrand $f(x)$ satisfies the following regularity condition.

Hölder Condition. A function $f(x)$ is said to satisfy the Hölder condition if there exist constants k and α , $0 < \alpha \leq 1$, such that, for every pair of points x_1, x_2 lying in the range $a \leq x \leq b$, we have

$$|f(x_1) - f(x_2)| < k |x_1 - x_2|^\alpha. \quad \dots (5)$$

A function satisfying the Hölder condition is known as *Hölder continuous*.

In particular, when $\alpha = 1$, then condition (5) is known as *Lipschitz condition*.

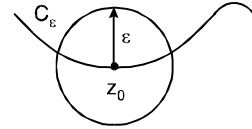
The Hölder condition can be extended to functions of more than one variable. Thus, the kernel $K(x, t)$ is Hölder continuous with respect to both variables if there exist constants k and α , $0 < \alpha < 1$, such that

$$|K(x_1, t_1) - K(x_2, t_2)| < k [|x_1 - x_2|^\alpha + |t_1 - t_2|^\alpha], \quad \dots (6)$$

where (x_1, t_1) and (x_2, t_2) lie within the range of definition.

The definition of Cauchy principal value for contour integrals.

We know that a contour integral of a complex-valued function with a pole z_0 on the contour does not exist. However it may have the Cauchy principal value which will be defined now. To this end, let C be a closed or open regular curve (refer adjoining figure). We enclose the point z_0 by a small circle of radius ϵ with centre at z_0 . Let C_ϵ denote the part of the contour outside this circle.



If a complex-valued function $f(z)$ is integrable along C_ϵ , however small the positive number ϵ , then the limit

$$\lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} f(z) dz, \quad \dots (7)$$

if it exists, is known as the Cauchy principal value and is denoted as

$$P \int_C f(z) dz \quad \text{or} \quad \int_C^* f(z) dz. \quad \dots (8)$$

In what follows, we shall study the contour integrals of the Cauchy type, that is,

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta. \quad \dots (9)$$

From the theory of functions of complex variable, we know that if $f(z)$ satisfies the Hölder condition

$$|f(z_1) - f(z_2)| < k |z_1 - z_2|^\alpha, \quad \dots (10)$$

where z_1, z_2 is any pair of points on the curve C , while k and α are constants such that $0 < \alpha \leq 1$, then the integral (9) exists for all points z on the curve C , except perhaps its end points. The function

$f_1(z)$ defined by

$$f_1(z) = \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad \dots (11)$$

is also Hölder continuous which possesses the similar properties as possessed by the corresponding real functions.

The definition (10) can be extended to complex-valued function of more than one variable as already done for real-valued functions.

The functions $f(\zeta)$ occurring in the integral (9) is known as the *density* of the Cauchy integral.

8.7. THE CAUCHY INTEGRALS.

The integral equation
$$f(z) = \frac{1}{2\pi i} \int_C \frac{y(\zeta)}{\zeta - z} d\zeta, \quad \dots (1)$$

where C is a regular curve, is known as a *Cauchy-type integral*. First of all, we shall discuss (1) when C is a closed contour.

Plemelj formulas. Let $y(\zeta)$ be a Hölder continuous function of a point on a regular closed contour C and let a point z tend, in an arbitrary manner, from inside or outside the contour C , to the point t on this contour; then the integral (1) tends to the limit

$$f^+(t) = \frac{1}{2} y(t) + \frac{1}{2\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta, \quad \dots (2)$$

or
$$f^-(t) = -\frac{1}{2} y(t) + \frac{1}{2\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta, \quad \dots (3)$$

respectively. The formulas (2) and (3) are known as *Plemelj formulas*. We adopt the standard convention of counterclockwise traversal of the closed contour C . It follows that the boundary value $f^+(t)$ relates to the values of the Cauchy integral inside the region bounded by C , while the second boundary value $f^-(t)$ relates to the value in the outside region.

Poincare-Bertrand transformation formula.

Let $y(t)$ be Hölder continuous function and let C be a closed contour. Then

$$\frac{1}{(2\pi i)^2} \int_C^* \frac{d\zeta_1}{\zeta_1 - t} \int_C^* \frac{y(\zeta)}{\zeta - \zeta_1} d\zeta = \frac{1}{4} y(t). \quad \dots (4)$$

Proof. Let
$$y_1(t) = \frac{1}{2\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta \quad \dots (5)$$

and
$$y_2(t) = \frac{1}{2\pi i} \int_C^* \frac{y_1(\zeta)}{\zeta - t} d\zeta \quad \dots (6)$$

be two singular integrals, where y_1 and y_2 are Hölder continuous functions.

Now, we shall obtain an iterated integral connecting y_2 and y . To this end, consider the Cauchy-type integrals
$$f(z) = \frac{1}{2\pi i} \int_C \frac{y(\zeta)}{\zeta - z} d\zeta \quad \dots (7)$$

and
$$f_1(z) = \frac{1}{2\pi i} \int_C \frac{y_1(\zeta)}{\zeta - z} d\zeta. \quad \dots (8)$$

Using the Plemelj formula (2) and integrals (7) and (8), we arrive at the limiting values

$$f^+(t) = \frac{1}{2} y(t) + \frac{1}{2\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta, \quad \dots (9)$$

and

$$f_1^+(t) = \frac{1}{2} y_1(t) + \frac{1}{2\pi i} \int_C^* \frac{y_1(\zeta)}{\zeta - t} d\zeta. \quad \dots (10)$$

Comparing (5) and (9), we have

$$y_1(t) = f^+(t) - (1/2) \times y(t). \quad \dots (11)$$

Again, comparing (6) and (10), we have

$$y_2(t) = f_1^+(t) - (1/2) \times y_1(t). \quad \dots (12)$$

From (11),

$$y_1(\zeta) = f^+(\zeta) - (1/2) \times y(\zeta). \quad \dots (13)$$

Putting the above value of $y_1(\zeta)$ in (8), we get

$$f_1(z) = \frac{1}{2\pi i} \int_C \frac{f^+(\zeta) - (1/2) \times y(\zeta)}{\zeta - z} d\zeta$$

or

$$f_1(z) = \frac{1}{2\pi i} \int_C \frac{f^+(\zeta)}{\zeta - z} d\zeta - \frac{1}{4\pi i} \int_C \frac{y(\zeta)}{\zeta - z} d\zeta. \quad \dots (14)$$

Consider the first integral on R.H.S. of (14). Since its density $f^+(\zeta)$ is the limiting value of $f(z)$, which is regular inside C , using the *Cauchy integral* formula, we have

$$\frac{1}{2\pi i} \int_C \frac{f^+(\zeta)}{\zeta - z} d\zeta = f(z). \quad \dots (15)$$

Comparing the second integral on R.H.S. of (14) with the integral in (7), we see that

$$\frac{1}{4\pi i} \int_C \frac{y(\zeta)}{\zeta - z} d\zeta = \frac{1}{2} f(z). \quad \dots (16)$$

Using (15) and (16), (14) becomes

$$f_1(z) = f(z) - (1/2) \times f(z) \quad \text{or} \quad f_1(z) = (1/2) \times f(z). \quad \dots (17)$$

From (17),

$$f_1^+(t) = (1/2) \times f^+(t). \quad \dots (18)$$

Now, from (12), we obtain

$$y_2(t) = f_1^+(t) - (1/2) \times y_1(t)$$

or

$$y_2(t) = (1/2) \times f^+(t) - (1/2) \times [f^+(t) - (1/2) \times y(t)], \text{ by (11) and (18)}$$

or

$$y_2(t) = (1/4) \times y(t). \quad \dots (19)$$

Now, from (6), we have

$$y_2(t) = \frac{1}{2\pi i} \int_C^* \frac{y_1(\zeta_1)}{\zeta_1 - t} d\zeta_1. \quad \dots (20)$$

Similarly, from (5), we have

$$y_1(\zeta_1) = \frac{1}{2\pi i} \int_C^* \frac{y(\zeta)}{\zeta - \zeta_1} d\zeta. \quad \dots (21)$$

Substituting the value of $y_1(\zeta_1)$ from (21) in (20), we have

$$y_2(t) = \frac{1}{2\pi i} \int_C^* \frac{1}{\zeta_1 - t} \left\{ \frac{1}{2\pi i} \int_C^* \frac{y(\zeta)}{\zeta - \zeta_1} d\zeta \right\} d\zeta_1$$

or
$$\frac{1}{(2\pi i)^2} \int_C^* \frac{d\zeta_1}{\zeta_1 - t} \int_C^* \frac{y(\zeta)}{\zeta - \zeta_1} d\zeta = \frac{1}{4} y(t), \text{ using (19)} \quad \dots (22)$$

which is the desired *Poincare-Bertrand transformation formula*.

Remark. The reader should note carefully that in the formula (22), it is not allowed to change the order of integration. Thus, while solving double integral, first integration is performed with respect to ζ and then the resulting integral is integrated with respect to ζ_1 .

8.8. SOLUTION OF THE CAUCHY-TYPE SINGULAR INTEGRAL EQUATION.

Two cases arise :

Case I. When there is closed contour C.

We are to solve the integral equation of the second kind.

$$ay(t) = f(t) - \frac{b}{\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta, \quad \dots (1)$$

where a and b are known complex constants, $y(\zeta)$ is a Hölder-continuous function, and C is a regular closed contour.

We introduce an operator L defined as
$$Ly = ay(t) + \frac{b}{2\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta. \quad \dots (2)$$

Re-writing (1),
$$ay(t) + \frac{b}{\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta = f(t)$$

or
$$Ly = f(t), \text{ using the definition of operator } L \quad \dots (3)$$

Now, we define an “adjoint” operator

$$Mg = ag(t) - \frac{b}{\pi i} \int_C^* \frac{g(\zeta_1)}{\zeta_1 - t} d\zeta_1. \quad \dots (4)$$

From (3), we have

$$M[Ly] = Mf$$

or
$$M \left[ay(t) + \frac{b}{\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta \right] = Mf, \text{ by (2)} \quad \dots (5)$$

Let
$$g(t) = ay(t) + \frac{b}{\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta \quad \dots (6)$$

so that
$$g(\zeta_1) = ay(\zeta_1) + \frac{b}{\pi i} \int_C^* \frac{y(\zeta)}{\zeta - \zeta_1} d\zeta. \quad \dots (7)$$

Using (6), (5) becomes

$$Mg = Mf$$

or
$$ag(t) - \frac{b}{\pi i} \int_C^* \frac{g(\zeta_1)}{\zeta_1 - t} d\zeta_1 = af(t) - \frac{b}{\pi i} \int_C^* \frac{f(\zeta_1)}{\zeta_1 - t} d\zeta_1, \text{ using the definition of operator } M$$

$$\begin{aligned} \text{or} \quad & a \left[ay(t) + \frac{b}{\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t_1} d\zeta \right] - \frac{b}{\pi i} \int_C^* \frac{1}{\zeta_1 - t} \left[ay(\zeta_1) + \frac{b}{\pi i} \int_C^* \frac{y(\zeta)}{\zeta - \zeta_1} d\zeta \right] d\zeta_1 \\ & = af(t) - \frac{b}{\pi i} \int_C^* \frac{f(\zeta_1)}{\zeta_1 - t} d\zeta_1 \end{aligned}$$

$$\begin{aligned} \text{or} \quad & a^2 y(t) + \frac{ab}{\pi i} \int_C^* \frac{y(\zeta) d\zeta}{\zeta - t} - \frac{ab}{\pi i} \int_C^* \frac{y(\zeta_1) d\zeta_1}{\zeta_1 - t} \\ & - b^2 \frac{1}{(\pi i)^2} \int_C^* \frac{d\zeta_1}{\zeta_1 - t} \int_C^* \frac{y(\zeta)}{\zeta - \zeta_1} d\zeta = af(t) - \frac{b}{\pi i} \int_C^* \frac{f(\zeta_1)}{\zeta_1 - t} d\zeta_1 \end{aligned}$$

$$\begin{aligned} \text{or} \quad & a^2 y(t) + \frac{ab}{\pi i} \int_C^* \frac{y(\zeta) d\zeta}{\zeta - t} - \frac{ab}{\pi i} \int_C^* \frac{y(\zeta) d\zeta}{\zeta - t} - 4b^2 \times \frac{1}{(2\pi i)^2} \int_C^* \frac{d\zeta_1}{\zeta_1 - t} \int_C^* \frac{y(\zeta)}{\zeta - \zeta_1} d\zeta \\ & = af(t) - \frac{b}{\pi i} \int_C^* \frac{f(\zeta_1)}{\zeta_1 - t} d\zeta_1 \end{aligned}$$

$$\text{or} \quad a^2 y(t) - 4b^2 \times \frac{1}{4} y(t) = af(t) - \frac{b}{\pi i} \int_C^* \frac{f(\zeta_1)}{\zeta_1 - t} d\zeta_1, \text{ using the Poincare-Bertrand formula}$$

$$\text{or} \quad (a^2 - b^2) y(t) = af(t) - \frac{b}{\pi i} \int_C^* \frac{f(\zeta)}{\zeta - t} d\zeta. \quad \dots (8)$$

Supposing that $(a^2 - b^2) \neq 0$, (8) gives

$$y(t) = \frac{a}{a^2 - b^2} f(t) - \frac{b}{(a^2 - b^2) \pi i} \int_C^* \frac{f(\zeta)}{\zeta - t} d\zeta. \quad \dots (9)$$

Particular Case. Putting $a = 0$ in (9), the solution of the Cauchy-type integral equation of the first kind

$$f(t) = \frac{b}{\pi i} \int_C^* \frac{y(t)}{\zeta - t} d\zeta \quad \dots (10)$$

$$\text{is} \quad y(t) = \frac{1}{b \pi i} \int_C^* \frac{f(\zeta)}{\zeta - t} d\zeta. \quad \dots (11)$$

Deduction. Putting $b = 1$ in (10) and (11), we see that the solution of

$$f(t) = \frac{1}{\pi i} \int_C^* \frac{y(t)}{\zeta - t} d\zeta \quad \dots (12)$$

$$\text{is} \quad y(t) = \frac{1}{\pi i} \int_C^* \frac{f(\zeta)}{\zeta - t} d\zeta. \quad \dots (13)$$

(12) and (13) exhibit the reciprocity of these relations.

Case II. When there is unclosed contour. The Riemann-Hilbert problem.

Plemelj formulas (2) and (3) of Art. 8.7 still hold for an arc also when we define the plus and minus directions as follows. To this end, we supplement the arc L with another arc L' so as to form a closed contour $L + L'$. Then, the interior and exterior of this closed contour stand for the plus and minus directions. Accordingly, for an arc C , we obtain

$$f^+(t) = \frac{1}{2}y(t) + \frac{1}{2\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta \quad \dots (14)$$

and
$$f^-(t) = -\frac{1}{2}y(t) + \frac{1}{2\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta. \quad \dots (15)$$

Re-writing (14) and (15), we have
$$y(t) = f^+(t) - f^-(t) \quad \dots (16)$$

and
$$\frac{1}{\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} dt = f^+(t) + f^-(t). \quad \dots (17)$$

Let a function $w(t)$ be prescribed on an arc L and let it satisfy the Hölder condition on L . Then, we wish to obtain a function $W(z)$ analytic for all points z on L such that it satisfies the boundary (or jump) condition

$$W^+(t) - W^-(t) = w(t), \quad t \in L. \quad \dots (18)$$

The problem posed in (18) is a special case of the so called Riemann-Hilbert problem, which requires the determination of a function $W(z)$ analytic for all point z not lying on L such that, for t on L ,

$$W^+(t) - Z(t) W^-(t) = w(t), \quad \dots (19)$$

where $w(t)$ and $Z(t)$ are given complex-valued functions.

By substituting the formulas (16) and (17) in the integral-equation

$$ay(t) = F(t) - \frac{b}{\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta, \quad \dots (20)$$

it follows that the solution of (20) is reduced to solving the Riemann-Hilbert problem

$$(a+b)f^+(t) - (a-b)f^-(t) = F(t). \quad \dots (21)$$

Let L be a regular unclosed curve. Then the solution of the singular integral equation (1) is

$$y(t) = \frac{a f(t)}{a^2 - b^2} + \frac{k}{(t-\alpha)^{1-m}(t-\beta)^m} - \frac{b}{(a^2 - b^2)\pi i} \left(\frac{t-\alpha}{t-\beta} \right)^m \int_C^* \left(\frac{\zeta-\beta}{\zeta-\alpha} \right)^m f(\zeta) \frac{d\zeta}{\zeta-t}, \quad \dots (22)$$

where α and β are the beginning and end points of the contour C and the number m is given by

$$m = \frac{1}{2\pi i} \log \frac{a+b}{a-b}, \quad \dots (23)$$

and the quantity k is an arbitrary constant and is suitably chosen so that $y(t)$ is bounded at α or at β .

Particular Case. The solution of the first kind (by setting $b = 1$ also without any loss of generality),

$$f(t) = \frac{1}{\pi i} \int_C^* \frac{y(\zeta)}{\zeta - t} d\zeta, \quad \dots (24)$$

is given by (22) with $a = 0$, $b = 1$. Putting $a = 0$, $b = 1$ in (23) yields $m = 1/2$ and so (22) gives the required solution of (24) as

$$y(t) = \frac{k}{[(t-\alpha)(t-\beta)]^{1/2}} + \frac{1}{\pi i} \left(\frac{t-\alpha}{t-\beta} \right)^{1/2} \int_C^* \left(\frac{\zeta-\beta}{\zeta-\alpha} \right)^{1/2} \frac{f(\zeta)}{\zeta-t} dt \quad \dots (25)$$

Example : Prove that the solution of the integral equation

$$f(t) = \frac{2t^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^t t^{2\eta+\zeta} (t^2 - \zeta^2)^{\alpha-1} y(\zeta) d\zeta, \quad 0 < \alpha < 1 \quad \dots (i)$$

is
$$y(t) = \frac{t^{-2\eta-1}}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \zeta^{2\alpha+2\eta+1} (t^2 - \zeta^2)^{-\alpha} f(\zeta) d\zeta \quad \dots (ii)$$

Hint : The required solution can be easily obtained by comparing (i) and (ii) with equations (24) and (25) respectively of Art. 8.8, and by setting $\alpha = 0, \beta = 1$.

8.9. THE HILBERT KERNEL. DEFINITION. [Kanpur 2011, Meerut 2001, 04, 07, 09]

A kernel of the form
$$K(x, t) = \cot\left(\frac{t-x}{2}\right),$$

where x and t are real variables, is known as the *Hilbert kernel*.

Consider the integral equation

$$y(x) = f(x) - \lambda \int_0^{2\pi} F(x, t) \cot\left(\frac{t-x}{2}\right) y(t) dt. \quad \dots (2)$$

where $f(x)$ and $F(x, t)$ are known continuous functions of period 2π .

Then the integral equation (2) is equivalent to the Cauchy-type integral equation

$$y(\zeta) = f(\zeta) - \lambda \int_C^* \frac{G(\zeta, \tau)}{\tau - \zeta} y(\tau) d\tau, \quad \dots (3)$$

where ζ and τ are complex variables and the contour C is the circumference of the unit disc with the centre at the point $z = 0$.

Let ζ and τ denotes the points on the boundary C corresponding to the arguments x and t , respectively. Then, we have

$$\zeta = e^{ix} \quad \text{and} \quad \tau = e^{it}. \quad \dots (4)$$

From (4), $d\tau = i e^{it} dt$ or $d\tau = i \tau dt$, ($\because \tau = e^{it}$)

so that $(1/\tau) \times d\tau = i dt. \quad \dots (5)$

Now, we have
$$\frac{d\tau}{\tau - \zeta} = \frac{i e^{it} dt}{e^{it} - e^{ix}} \quad \dots (6)$$

Also,
$$\begin{aligned} \frac{1}{2} \cot \frac{t-x}{2} + \frac{i}{2} &= \frac{i}{2} \frac{e^{i(t-x)/2} + e^{-i(t-x)/2}}{e^{i(t-x)/2} - e^{-i(t-x)/2}} + \frac{i}{2} = \frac{i}{2} \left[\frac{e^{i(t-x)/2} + e^{-i(t-x)/2}}{e^{i(t-x)/2} - e^{-i(t-x)/2}} + 1 \right] \\ &= \frac{i}{2} \frac{2e^{i(t-x)/2}}{e^{i(t-x)/2} - e^{-i(t-x)/2}} = i \frac{e^{it/2} e^{-ix/2}}{e^{it/2} e^{-ix/2} - e^{-it/2} e^{ix/2}} = \frac{i e^{it}}{e^{it} - e^{ix}}. \quad \dots (7) \end{aligned}$$

Using (6) and (7), we have

$$\frac{i e^{it} dt}{e^{it} - e^{ix}} = \frac{d\tau}{\tau - \zeta} = \left[\frac{1}{2} \cot \frac{t-x}{2} + \frac{i}{2} \right] dt \quad \text{or} \quad \frac{2 d\tau}{\tau - \zeta} = \cot \frac{t-x}{2} dt + i dt$$

or
$$\frac{2 d\tau}{\tau - \zeta} = \cot \frac{t-x}{2} dt + \frac{d\tau}{\tau}, \quad \text{using (5)}$$

or
$$\cot\left(\frac{t-x}{2}\right) dt = \frac{2 d\tau}{\tau - \zeta} - \frac{d\tau}{\tau} \quad \text{or} \quad \cot\left(\frac{t-x}{2}\right) dt = \frac{\tau + \zeta}{\tau - \zeta} \frac{d\tau}{\tau}. \quad \dots (8)$$

With help of (8), we find that (2) reduces to the form (3).

The Hilbert kernel is also related to the Poisson kernel in the integral representation formula for a harmonic function $U(r, x)$:

$$U(r, x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r \cos(t-x)} u(t) dt, \quad \dots (9)$$

inside the disc $r < 1$. The function $u(t) = U(1, t)$ is the prescribed value of the harmonic function on the circumference C of the disc.

$$\text{Putting } z = re^{ix} \quad \text{and} \quad \tau = e^{it} \quad \dots (10)$$

$$\text{in (9), we obtain as before} \quad U(r, x) = \text{Re} \left\{ \frac{1}{2\pi i} \int_C u(t) \frac{\tau+z}{\tau-z} \frac{d\tau}{\tau} \right\}, \quad \dots (11)$$

where Re denotes the real part of the expression that follows.

Again, let $V(r, x)$ be the function that is harmonic conjugate to $U(r, x)$. Then, we have

$$U(r, x) + i V(r, x) = \frac{1}{2\pi i} \int_C u(t) \frac{\tau+z}{\tau-z} \frac{d\tau}{\tau}, \quad \dots (12)$$

$$\text{such that } V(r, x) = 0 \text{ at the centre of the disc, that is, } [V(r, x)]_{r=0} = 0 \quad \dots (13)$$

Then, the function $V(r, x)$ is uniquely defined.

When $r \rightarrow 0$, so that z tends a point ζ of the circumference C from within the disc, we can apply the Plemelj formula (refer formula (4) in Art. 8.7) to the analytic function given by (12) and obtain (keeping (8) in view)

$$v(x) = -\frac{1}{2\pi} \int_0^{2\pi} u(t) \cot\left(\frac{t-x}{2}\right) dt, \quad \dots (14)$$

where $v(x) = V(1, x)$, is the limiting value of the harmonic function on C . Thus, (14) gives the relation between the limiting values of the conjugate harmonic functions $U(r, x)$ and $V(r, x)$ on the circumference.

We now state and prove an important formula, namely,

Hilbert formula. To prove that [Meerut 2001, 07, 09]

$$\frac{1}{4\pi^2} \int_0^{2\pi} \cot\left(\frac{\sigma-x}{2}\right) \left\{ \int_0^{2\pi} y(t) \cot\left(\frac{t-\sigma}{2}\right) dt \right\} d\sigma = -y(x) + \frac{1}{2\pi} \int_0^{2\pi} y(t) dt. \quad \dots (15)$$

Proof. Consider the following integral equations with Hilbert kernel :

$$y_1(x) = \frac{1}{2\pi} \int_0^{2\pi} y(t) \cot\left(\frac{x-t}{2}\right) dt. \quad \dots (16)$$

and

$$y_2(x) = \frac{1}{2\pi} \int_0^{2\pi} y_1(t) \cot\left(\frac{x-t}{2}\right) dt. \quad \dots (17)$$

Let $U(r, x)$, $U_1(r, x)$ and $U_2(r, x)$ be the functions which are harmonic inside the disc $r < 1$, and whose values on the circumference $r = 1$ are equal respectively to $y(x)$, $y_1(x)$ and $y_2(x)$. Then, from (9), it follows that $U_1(r, x)$ is harmonic conjugate to $U(r, x)$ and $U_2(r, x)$ is harmonic conjugate to $U_1(r, x)$. Using the well known Cauchy-Riemann equations of the theory of complex variables, we have

$$\partial U / \partial r = -(\partial U_2 / \partial r), \quad \text{and} \quad \partial U / \partial x = -(\partial U_2 / \partial x)$$

$$\text{giving} \quad U_2(r, x) = -U(r, x) + k. \quad \dots (18)$$

where k is an arbitrary constant. To determine k , we use (13) and obtain

$$k = [U(r, x)]_{r=0} = \frac{1}{2\pi} \int_0^{2\pi} U(1, t) dt = \frac{1}{2\pi} \int_0^{2\pi} y(t) dt, \quad \dots (19)$$

where we have used the mean-value property of the harmonic function.

$$\text{From (18) and (19),} \quad U_2(r, x) = -U(r, x) + \frac{1}{2\pi} \int_0^{2\pi} y(t) dt. \quad \dots (20)$$

Putting $r = 1$ in (20), we have

$$U_2(1, x) = -U(1, x) + \frac{1}{2\pi} \int_0^{2\pi} y(t) dt. \quad \dots (21)$$

$$\text{But} \quad U(1, x) = y(x) \quad \text{and} \quad U_2(1, x) = y_2(x). \quad \dots (22)$$

$$\therefore \text{From (21) and (22),} \quad y_2(x) = -y(x) + \frac{1}{2\pi} \int_0^{2\pi} y(t) dt. \quad \dots (23)$$

$$\begin{aligned} \text{From (17),} \quad y_2(x) &= -\frac{1}{2\pi} \int_0^{2\pi} y_1(t) \cot\left(\frac{t-x}{2}\right) dt \\ \Rightarrow \quad y_2(x) &= -\frac{1}{2\pi} \int_0^{2\pi} y_1(\sigma) \cot\left(\frac{\sigma-x}{2}\right) d\sigma. \quad \dots (24) \end{aligned}$$

$$\begin{aligned} \text{Now, from (16),} \quad y_1(x) &= -\frac{1}{2\pi} \int_0^{2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt \\ \Rightarrow \quad y_1(\sigma) &= -\frac{1}{2\pi} \int_0^{2\pi} y(t) \cot\left(\frac{t-\sigma}{2}\right) dt. \quad \dots (25) \end{aligned}$$

Substituting the values of $y_1(\sigma)$ given by (25) in (24), we have

$$y_2(x) = -\frac{1}{2\pi} \int_0^{2\pi} \cot\left(\frac{\sigma-x}{2}\right) \left\{ -\frac{1}{2\pi} \int_0^{2\pi} y(t) \cot\left(\frac{t-\sigma}{2}\right) dt \right\} d\sigma$$

$$\text{or} \quad y_2(x) = \frac{1}{4\pi^2} \int_0^{2\pi} \cot\left(\frac{\sigma-x}{2}\right) \left\{ \int_0^{2\pi} y(t) \cot\left(\frac{t-\sigma}{2}\right) dt \right\} d\sigma \quad \dots (26)$$

Equating the two values of $y_2(x)$ given by (23) and (26), we have

$$\frac{1}{4\pi^2} \int_0^{2\pi} \cot\left(\frac{\sigma-x}{2}\right) \left\{ \int_0^{2\pi} y(t) \cot\left(\frac{t-\sigma}{2}\right) dt \right\} d\sigma = -y(x) + \frac{1}{2\pi} \int_0^{2\pi} y(t) dt, \quad \dots (27)$$

which is the required *Hilbert formula*.

8.10. SOLUTION OF THE HILBERT-TYPE SINGULAR INTEGRAL EQUATION OF THE SECOND KIND, NAMELY,

$$ay(x) = f(x) - \frac{b}{2\pi} \int_0^{2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt, \quad \dots (1)$$

where a and b are complex constants.

[Kanpur 2005; Meerut 2002, 04, 06]

We introduce an operator L defined as

$$Ly = a y(x) + \frac{b}{2\pi} \int_0^{*2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt. \quad \dots (2)$$

Re-writing (1), we have $a y(x) + \frac{b}{2\pi} \int_0^{*2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt = f(x)$

or $Ly = f(x)$, using the definition of operator L ... (3)

Now, we define an “adjoint” operator

$$Mg = a g(x) - \frac{b}{2\pi} \int_0^{*2\pi} g(t) \cot\left(\frac{t-x}{2}\right) dt. \quad \dots (4)$$

From (3), we have

$$MLy = Mf$$

or $M \left[a y(x) + \frac{b}{2\pi} \int_0^{*2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt \right] = Mf$, using (2) ... (5)

Let $g(x) = a y(x) + \frac{b}{2\pi} \int_0^{*2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt.$... (6)

$$\Rightarrow g(x) = a y(x) + \frac{b}{2\pi} \int_0^{*2\pi} y(\sigma) \cot\left(\frac{\sigma-x}{2}\right) d\sigma.$$

$$\Rightarrow g(t) = a y(t) + \frac{b}{2\pi} \int_0^{*2\pi} y(\sigma) \cot\left(\frac{\sigma-t}{2}\right) d\sigma. \quad \dots (7)$$

Using (6), (5) reduces to

$$Mg = Mf$$

or $a g(x) - \frac{b}{2\pi} \int_0^{*2\pi} g(t) \cot\left(\frac{t-x}{2}\right) dt = a f(x) - \frac{b}{2\pi} \int_0^{*2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt$... (8)
[using the definition (4) of operator M]

Let $F(x) = a f(x) - \frac{b}{2\pi} \int_0^{*2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt.$... (9)

Using (9), (8) reduces to

$$a g(x) - \frac{b}{2\pi} \int_0^{*2\pi} g(t) \cot\left(\frac{t-x}{2}\right) dt = F(x). \quad \dots (10)$$

Substituting the values of $g(x)$ and $g(t)$ given by (6) and (7) in (10), we have

$$a \left[a y(x) + \frac{b}{2\pi} \int_0^{*2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt. \right] \\ - \frac{b}{2\pi} \int_0^{*2\pi} \cot\left(\frac{t-x}{2}\right) \left\{ a y(t) + \frac{b}{2\pi} \int_0^{*2\pi} y(\sigma) \cot\left(\frac{\sigma-t}{2}\right) d\sigma \right\} dt = F(x)$$

$$\begin{aligned} \text{or} \quad a^2 y(x) + \frac{ab}{2\pi} \int_0^{2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt - \frac{ab}{2\pi} \int_0^{2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt \\ - b^2 \times \frac{1}{4\pi^2} \int_0^{2\pi} \cot\left(\frac{t-x}{2}\right) \left\{ \int_0^{2\pi} y(\sigma) \cot\left(\frac{\sigma-t}{2}\right) d\sigma \right\} dt = F(x) \end{aligned}$$

$$\text{or} \quad a^2 y(x) - b^2 \left[-y(x) + \frac{1}{2\pi} \int_0^{2\pi} y(\sigma) d\sigma \right] = F(x), \text{ using the Hilbert formula of Art. 8.9.}$$

$$\text{or} \quad (a^2 + b^2) y(x) - \frac{b^2}{2\pi} \int_0^{2\pi} y(t) dt = F(x)$$

Assuming that $(a^2 + b^2) \neq 0$, we have

$$y(x) = \frac{1}{a^2 + b^2} F(x) + \frac{b^2}{2\pi(a^2 + b^2)} \int_0^{2\pi} y(t) dt, \quad \dots (11)$$

which is Fredholm integral equation of the second kind with separable kernel. To solve (11), we now proceed by usual method*.

$$\text{Let} \quad C = \int_0^{2\pi} y(t) dt. \quad \dots (12)$$

$$\therefore \text{ From (11),} \quad y(x) = \frac{1}{a^2 + b^2} F(x) + \frac{b^2 C}{2\pi(a^2 + b^2)}. \quad \dots (13)$$

$$\text{From (13),} \quad y(t) = \frac{1}{a^2 + b^2} F(t) + \frac{b^2 C}{2\pi(a^2 + b^2)}. \quad \dots (14)$$

Substituting the value of $y(t)$ as given by (14) in (12), we have

$$C = \int_0^{2\pi} \left[\frac{F(t)}{a^2 + b^2} + \frac{b^2 C}{2\pi(a^2 + b^2)} \right] dt = \frac{1}{a^2 + b^2} \int_0^{2\pi} F(t) dt + \frac{b^2 C}{2\pi(a^2 + b^2)} [t]_0^{2\pi}$$

$$\text{or} \quad C = \frac{1}{a^2 + b^2} \int_0^{2\pi} F(t) dt + \frac{b^2 C \times 2\pi}{2\pi(a^2 + b^2)} \quad \text{or} \quad C \left[1 - \frac{b^2}{a^2 + b^2} \right] = \frac{1}{a^2 + b^2} \int_0^{2\pi} F(t) dt$$

$$\text{or} \quad C = \frac{1}{a^2} \int_0^{2\pi} F(t) dt = \frac{1}{a} \int_0^{2\pi} f(t) dt \quad [\because \text{ from (9), } F(t) = a f(t)] \quad \dots (15)$$

Substituting this value of C in (13), we have

$$y(x) = \frac{F(x)}{a^2 + b^2} + \frac{b^2}{2\pi a(a^2 + b^2)} \int_0^{2\pi} f(t) dt$$

$$\text{or} \quad y(x) = \frac{1}{a^2 + b^2} \left[a f(x) - \frac{b}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \right] + \frac{b^2}{2\pi a(a^2 + b^2)} \int_0^{2\pi} f(t) dt, \text{ using (9)}$$

* We adopt here the usual method outlined in Art. 4.1 and Art. 4.2 of chapter 4.

or
$$y(x) = \frac{a}{a^2 + b^2} f(x) - \frac{b}{2\pi(a^2 + b^2)} \int_0^{*2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt$$

$$+ \frac{b^2}{2\pi a(a^2 + b^2)} \int_0^{2\pi} y(t) dt, \quad \dots (16)$$

which is the required solution of given integral equation (1).

8.11. SOLUTION OF THE HILBERT-TYPE SINGULAR INTEGRAL EQUATION OF THE FIRST KIND, NAMELY,

$$f(x) = \frac{b}{2\pi} \int_0^{*2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt. \quad \dots (1)$$

To solve (1), let us reconsider (1) with the constant b incorporated in $y(t)$. Thus, we shall

first solve
$$f(x) = \frac{1}{2\pi} \int_0^{*2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt. \quad \dots (2)$$

Re-writing (2), we have
$$f(x) = \frac{1}{2\pi} \int_0^{*2\pi} y(\sigma) \cot\left(\frac{\sigma-x}{2}\right) d\sigma. \quad \dots (3)$$

Replacing x by t in (3), we have
$$f(t) = \frac{1}{2\pi} \int_0^{*2\pi} y(\sigma) \cot\left(\frac{\sigma-t}{2}\right) d\sigma. \quad \dots (4)$$

Multiplying both sides of (4) by $\frac{1}{2\pi} \cot\left(\frac{t-x}{2}\right)$ and then integrating both sides w.r.t. ' t ' from 0 to 2π , we have

$$\frac{1}{2\pi} \int_0^{*2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt = \frac{1}{4\pi^2} \int_0^{*2\pi} \cot\left(\frac{t-x}{2}\right) \left\{ \int_0^{*2\pi} y(\sigma) \cot\left(\frac{\sigma-t}{2}\right) d\sigma \right\} dt$$

or
$$\frac{1}{2\pi} \int_0^{*2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt = -y(x) + \frac{1}{2\pi} \int_0^{2\pi} y(\sigma) d\sigma \quad \dots (5)$$

[using Hilbert formula of Art. 8.9.]

Let
$$F(x) = -\frac{1}{2\pi} \int_0^{*2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt. \quad \dots (6)$$

Using (6), (5) becomes
$$-F(x) = -y(x) + \frac{1}{2\pi} \int_0^{2\pi} y(\sigma) d\sigma$$

or
$$y(x) = F(x) + \frac{1}{2\pi} \int_0^{2\pi} y(t) dt, \quad \dots (7)$$

which is Fredholm integral-equation of the second kind with separable kernel. We now proceed to solve (7) by the usual method*.

Let
$$C = \frac{1}{2\pi} \int_0^{2\pi} y(t) dt. \quad \dots (8)$$

* We shall adopt the method outline in Art. 4.1 and 4.1 of chapter 4

Using (8), (7) reduces to $y(x) = F(x) + C$ (9)

From (9), $y(t) = F(t) + C$ (10)

Substituting the value of $y(t)$ given by (10) in (8), we have

$$C = \frac{1}{2\pi} \int_0^{2\pi} [F(t) + C] dt \quad \text{or} \quad C = \frac{1}{2\pi} \int_0^{2\pi} F(t) dt + \frac{C}{2\pi} \int_0^{2\pi} dt$$

$$\text{or} \quad C = \frac{1}{2\pi} \int_0^{2\pi} F(t) dt + \frac{C}{2\pi} \times 2\pi \quad \text{or} \quad \int_0^{2\pi} F(t) dt = 0 \quad \text{or} \quad \int_0^{2\pi} F(x) dx = 0 \quad \dots (11)$$

By virtue of relation (7), it follows that (11) holds for all values of the function $f(x)$. Hence, C must be an arbitrary constant and so we find that infinite number of solutions of (2) exist and are given by (9), that is,

$$y(x) = C + F(x) \quad \text{or} \quad y(x) = C - \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt, \text{ by (6)} \quad \dots (12)$$

By substituting (12) in (2), we find that $f(x)$ given by (12) satisfies (2), if and only if,

$$\int_0^{2\pi} f(x) dt = 0, \quad \dots (13)$$

showing that the necessary and sufficient condition for (2) to possess a solution is that condition (13) must hold good.

Deduction. To find solution of the integral equation (1)

$$\text{Re-writing (1), we have} \quad f(x) = \frac{1}{2\pi} \int_0^{2\pi} [by(t)] \cot\left(\frac{t-x}{2}\right) dt$$

$$\text{or} \quad f(x) = \frac{1}{2\pi} \int_0^{2\pi} Y(t) \cot\left(\frac{t-x}{2}\right) dt, \quad \dots (14)$$

$$\text{where} \quad Y(t) = b y(t) \quad \dots (15)$$

Now, proceed as above from (2) upto (12) after replacing $y(t)$ by $Y(t)$ and obtain

$$Y(x) = C - \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \quad \dots (16)$$

$$\text{But from (15),} \quad Y(x) = b y(x). \quad \dots (17)$$

$$\text{Using (17), (16) reduces to} \quad b y(x) = C - \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt$$

$$\text{or} \quad y(x) = \frac{C}{b} - \frac{1}{2\pi b} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt,$$

$$\text{or} \quad y(x) = C' - \frac{1}{2\pi b} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt, \quad \dots (18)$$

where $C' = (C/b)$ is another arbitrary constant. (18) gives the desired solution of (1).

Alternative method. Use the result of Art. 8.9, where we have connected the Hilbert kernel with the Cauchy kernel. To this end, write $y(e^{it}) = y(t)$, etc. and suppose that $y(t)$ and $f(t)$ are

periodic functions with period 2π . Also, replace $f(t)$ by $f(t)/i$. Then the formulas (12) and (13) of Art. 8.8 with help of the transformation (7) of Art. 8.9 give rise to the following reciprocal relations

$$\frac{1}{2\pi} \int_0^{2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt + \frac{i}{2\pi} \int_0^{2\pi} y(t) dt = f(x) \quad \dots (19)$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt + \frac{i}{2\pi} \int_0^{2\pi} f(t) dt = -y(x) \quad \dots (20)$$

Using the above pair of equations, the solution of (2) can be deduced.

From the pair (19) – (20), we can prove that for periodic functions $f(x)$ and $y(x)$, if the condition $\int_0^{2\pi} f(t) dt = 0$ is satisfied, then we must also have $\int_0^{2\pi} y(t) dt = 0$.

Example : Solve the integral equation

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2\pi} \int_0^{2\pi} y(t) \cos\left(\frac{t-x}{2}\right) dt$$

Hint. Here the function $f(x) = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is a periodic function with period

2π . Also, we have

$$\int_0^{2\pi} f(t) dt = 0$$

Hence, from the reciprocal pair (19) – (20), it follows that

$$y(x) = -\frac{1}{2\pi} \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \right\} \cos\left(\frac{t-x}{2}\right) dt$$

or

$$y(x) = \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx), \text{ on simplification.}$$

EXERCISE

1. Solve $x^2 = \int_2^x \frac{y(t) dt}{(x^2 - t^2)^{1/3}}, 2 < x < 4$.
2. Solve $x^2 = \int_x^4 \frac{y(t) dt}{(t^2 - x^2)^{1/3}}, 2 < x < 4$.
3. Solve the integral equation $y(x) = \sin x - \frac{1}{2\pi} \int_0^{2\pi} y(t) \cot\left(\frac{t-x}{2}\right) dt$. [Merrut 2008]
4. If a, b, c, d are real constants, solve the following integral equations :
 - (i) $ax + bx^2 = \int_0^x \frac{y(t) dt}{(x-t)^{1/2}}$.
 - (ii) $a + bx + cx^2 + dx^3 = \int_1^x \frac{y(t) dt}{(\cos t - \cos x)^{1/2}}, 1 < x < 2$.
 - (iii) $a + bx + cx^2 + dx^3 = \int_x^2 \frac{y(t) dt}{(\cos x - \cos t)^{1/2}}, 1 < x < 2$.
5. Solve the integral equation

$$\frac{1}{x} \int_0^x \frac{t^2 y(t) dt}{(x^2 - t^2)^{1/2}} = \begin{cases} -f_1(x), & 0 < x \leq b, \\ f_2(x), & b < x < \infty. \end{cases} \text{ ; where } b \text{ is a constant}$$

6. Solve the integral equation
$$x \int_x^\infty \frac{y(t) dt}{(t^2 - x^2)^{1/2}} = \begin{cases} f_1(x), & 0 \leq x \leq a, \\ -f_2(x), & a < x < \infty. \end{cases}$$

7. Prove that the solution of the integral equation

$$f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^x t^{2\eta+1} (x^2 - t^2)^{\alpha-1} y(t) dt, \quad 0 < \alpha < 1,$$

is
$$y(t) = \frac{t^{-2\eta-1}}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_0^1 u^{2\alpha+2\eta+1} (t^2 - u^2)^{-\alpha} f(u) du \right].$$

8. Prove that the solution of the integral equation

$$f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^\infty (t^2 - x^2)^{\alpha-1} t^{-2\alpha-2\eta-1} y(t) dt, \quad 0 < \alpha < 1,$$

is
$$y(t) = -\frac{t^{2\alpha+2\eta-1}}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_t^\infty u^{-2\eta+1} (u^2 - t^2)^{-\alpha} f(u) du \right].$$

9. Show that the solution of the integral equation

$$(a^2 + b^2)y(x) - \frac{b^2}{2\pi} \int_0^{2\pi} y(t) dt = af(x) - \frac{b}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \quad \text{is}$$

$$y(x) = \frac{a}{a^2 + b^2} f(x) - \frac{b}{2\pi(a^2 + b^2)} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt + \frac{b^2}{2\pi a(a^2 + b^2)} \int_0^{2\pi} f(t) dt,$$

proved $(a^2 + b^2) \neq 0$.

10. Show that the solution of the integral equation
$$F(v) = \int_0^{*1} w f(w) \frac{e^{-i\alpha(w-v)}}{w-v} dw$$

is given by
$$w f(w) = -\frac{1}{\pi^2} \left(\frac{w}{1-w} \right)^{1/2} \int_0^{*1} \left(\frac{1-v}{v} \right)^{1/2} F(v) \frac{e^{-i\alpha(v-w)}}{v-w} dv.$$

CHAPTER 9

Integral Transform Methods

9.1 INTRODUCTION

The integral transform methods are very convenient in solving integral equations of some special forms. Suppose that a relationship of the form

$$y(x) = \int_a^b \int_a^b \Gamma(x, z) K(z, t) y(t) dt dz \quad \dots (1)$$

be known to be valid and that this double integral can be evaluated as an iterated integral. Then,

from (1), it follows that if
$$F(x) = \int_a^b K(x, t) y(t) dt \quad \dots (2)$$

we also have
$$y(x) = \int_a^b \Gamma(x, t) F(t) dt. \quad \dots (3)$$

Thus, if (2) is regarded as an integral equation in y , a solution is given by (3), whereas if (3) is regarded as an integral equation in F a solution is given by (2). It is conventional to refer to one of the function as the *transform* of the second function, and to the second function as an *inverse transform* of the first. Thus, for example, the Fourier integral

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ipx} e^{-ipt} y(t) dt dp.$$

leads to the reciprocal realtions

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt} y(t) dt \quad \dots (4)$$

and

$$y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixt} F(t) dt. \quad \dots (5)$$

The function $F(x)$ is known as the *Fourier transform* of $y(t)$ and $y(x)$ is called the inverse Fourier transform of $F(t)$.

9.2 SOME USEFUL RESULTS ABOUT LAPLACE TRANSFORM*.

(1) Laplace transform. Definition. Given a function $F(t)$ defined for all real $t \geq 0$, the Laplace transform of $F(t)$ is a function of a new variable p given by

$$L\{F(t)\} = \bar{F}(p) = f(p) = \int_0^{\infty} e^{-pt} F(t) dt. \quad \dots (1)$$

The Laplace transform of $F(t)$ is said to exist if the integral (1) converges for some value of p , otherwise it does not exist.

* For more details please refer “Advanced Differential equations” or “Integral transform” by Dr. M.D. Raisinghania, published by S. Chand & Co., New Delhi.

(2) Table of Laplace transform of some elementary functions :

S. No.	$F(t)$	$L\{F(t)\}$ or $\bar{F}(p)$ or $f(p)$
1.	1	$1/p, p > 0.$
2.	$t^n, n > -1$	$\Gamma(n+1)/p^{n+1}, p > 0.$
3.	t^n (n is positive integer)	$n!/p^{n+1}, p > 0.$
4.	e^{at}	$1/(p-a), p > a.$
5.	$\sin at$	$a/(p^2 + a^2), p > 0.$
6.	$\cos at$	$p/(p^2 + a^2), p > 0.$
7.	$\sinh at$	$a/(p^2 - a^2), p > a .$
8.	$\cosh at$	$p/(p^2 - a^2), p > a .$
9.	$J_0(at)$	$1/\sqrt{p^2 + a^2}$
10.	$J_n(at)$	$[\sqrt{p^2 + a^2} - p]^n / a^n \sqrt{p^2 + a^2}$
11.	$\delta(t-a)$	e^{-ap}
12.	$\operatorname{erf}(\sqrt{t})$	$1/[p\sqrt{p+1}]$

(3) Linearity property of Laplace transforms. If c_1 and c_2 be constants, then

$$L\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}.$$

(4) First translation (or shifting) theorem.

$$\text{If } L\{F(t)\} = \bar{F}(p), \quad \text{then} \quad L\{e^{at} F(t)\} = \bar{F}(p-a).$$

(5) Unit step function or Heaviside's unit function. Definition. It is denoted and defined as

$$H(t-a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t \geq a. \end{cases}$$

Note : $L\{H(t-a)\} = (1/p) \times e^{-ap}$

(6) Second translation (or shifting) theorem.

$$\text{If } L\{F(t)\} = \bar{F}(p), \quad \text{then} \quad L\{F(t-a)H(t-a)\} = e^{-ap} \bar{F}(p).$$

OR

$$\text{If } L\{F(t)\} = \bar{F}(p) \quad \text{and} \quad G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

then

$$L\{G(t)\} = e^{-ap} \bar{F}(p).$$

(7) Change of scale property

$$\text{If } L\{F(t)\} = \bar{F}(p), \quad \text{then} \quad L\{F(at)\} = (1/a) \times \bar{F}(p/a).$$

(8) Laplace transform of derivatives :

$$(i) L\{F'(t)\} = p L\{F(t)\} - F(0) : \text{In particular, if } F(0) = 0, \text{ then } L\{F'(t)\} = p L\{F(t)\}$$

$$(ii) L\{F''(t)\} = p^2 L\{F(t)\} - p F(0) - F'(0),$$

and so on.

(9) Multiplication by positive integral powers of t .

$$(i) \text{ If } L \{F(t)\} = \bar{F}(p), \quad \text{then} \quad L \{t F(t)\} = -\frac{d}{dp} \bar{F}(p)$$

$$(ii) \text{ If } L \{F(t)\} = \bar{F}(p), \quad \text{then} \quad L \{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} \bar{F}(p).$$

(10) Division by t .

$$\text{If } L \{F(t)\} = \bar{F}(p), \text{ then } L \left\{ \frac{F(t)}{t} \right\} = \int_p^\infty \bar{F}(p) dp, \text{ provided the integral exists.}$$

(11) Initial value theorem :

$$\lim_{t \rightarrow 0} F(t) = \lim_{p \rightarrow \infty} p \bar{F}(p).$$

Final value theorem :

$$\lim_{t \rightarrow \infty} F(t) = \lim_{p \rightarrow 0} p \bar{F}(p).$$

(12) Laplace transform of periodic function. Given that $F(t)$ is a *periodic function* with period a , that is, $F(t + na) = F(t)$, for $n = 1, 2, 3, \dots$. Then, we have

$$L \{F(t)\} = \frac{1}{1 - e^{-pa}} \int_0^a e^{-ap} F(t) dt.$$

(13) Inverse Laplace transform. Definition. Let $L \{F(t)\} = \bar{F}(p)$. Then $F(t)$ is called an *inverse Laplace transform* of $\bar{F}(p)$, and we write $F(t) = L^{-1} \{\bar{F}(p)\}$. L^{-1} is known as the *inverse Laplace transformation operator*.

(14) Table of inverse Laplace transform of some functions

S. No.	$\bar{F}(p)$	$L^{-1} \{\bar{F}(p)\}$
1	$1/p$	1
2	$1/p^{n+1}, n > -1$	$t^n / \Gamma(n+1)$
3	$1/p^{n+1}$ (n is positive integer)	$t^n / n!$
4	$1/(p-a)$	e^{at}
5	$1/(p^2 + a^2)$	$(\sin at)/a$
6	$p/(p^2 + a^2)$	$\cos at$
7	$1/(p^2 - a^2)$	$(\sinh at)/a$
8	$p/(p^2 - a^2)$	$\cosh at$
9	$1/\sqrt{p^2 + a^2}$	$J_0(at)$
10	$[\sqrt{p^2 + a^2} - p]^n / a^n \sqrt{p^2 + a^2}$	$J_n(at)$
11	e^{-ap}	$\delta(t-a)$
12	$1/[p \sqrt{p+1}]$	$\text{erf}(\sqrt{t})$

(14) Linearity property of inverse Laplace transforms.

$$L^{-1} \{c_1 \bar{F}_1(p) + c_2 \bar{F}_2(p)\} = c_1 L^{-1} \{\bar{F}_1(p)\} + c_2 L^{-1} \{\bar{F}_2(p)\}.$$

(15) Heaviside expansion theorem (or formula). Given that $\bar{F}(p)$ and $\bar{G}(p)$ are polynomials in p , the degree of $\bar{F}(p)$ being less than that of $\bar{G}(p)$, and if $\bar{G}(p) = (p - \alpha_1)(p - \alpha_2)\dots(p - \alpha_n)$,

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are distinct constants, real or complex, then $L^{-1} \left\{ \frac{\bar{F}(p)}{\bar{G}(p)} \right\} = \sum_{r=1}^n \frac{\bar{F}(\alpha_r)}{\bar{G}'(\alpha_r)} e^{\alpha_r t}$.

(16) First translation (or shifting) theorem for inverse Laplace transform.

If $L^{-1} \{\bar{F}(p)\} = F(t)$, then $L^{-1} \{\bar{F}(p - a)\} = e^{at} F(t)$.

(17) Second translation (or shifting) theorem for inverse Laplace transform.

If $L^{-1} \{\bar{F}(p)\} = F(t)$, then $L^{-1} \{e^{-ap} \bar{F}(p)\} = F(t - a) H(t - a)$

or
$$L^{-1} \{e^{-ap} \bar{F}(p)\} = \begin{cases} F(t - a), & t > a \\ 0, & t < a \end{cases}$$

Here $H(t - a)$ is Heaviside's unit function defined in result (5).

(18) Change of scale property for inverse Laplace transform.

If $L^{-1} \{\bar{F}(p)\} = F(t)$, then $L^{-1} \{\bar{F}(ap)\} = (1/a) \times F(t/a)$.

(19) Inverse Laplace transform of derivatives

If $L^{-1} \{\bar{F}(p)\} = F(t)$, then $L^{-1} \left\{ \frac{d^n}{dp^n} \bar{F}(p) \right\} = (-1)^n t^n F(t)$, where $n = 1, 2, 3, \dots$

(20) Inverse Laplace transform of integrals.

If $L^{-1} \{\bar{F}(p)\} = F(t)$, then $L^{-1} \left\{ \int_p^\infty \bar{F}(p) dp \right\} = \frac{F(t)}{t}$.

(21) Multiplication by p . Let $F(0) = 0$ and if $L^{-1} \{\bar{F}(p)\} = F(t)$, then $L^{-1} \{p \bar{F}(p)\} = F'(t)$.

Again, if $F(0) = F'(0) = \dots = F^{(n-1)}(0) = 0$, then $L^{-1} \{p^n \bar{F}(p)\} = \frac{d^n}{dt^n} F(t)$.

(22) Division by powers of p . If $L^{-1} \{\bar{F}(p)\} = F(t)$, then

$$(i) L^{-1} \left\{ \frac{\bar{F}(p)}{p} \right\} = \int_0^t F(t) dt \quad (ii) L^{-1} \left\{ \frac{\bar{F}(p)}{p^n} \right\} = \int_0^t \int_0^t \dots \int_0^t F(t) dt^n.$$

(23) Convolution (or Faltung). Definition. The convolution of $F(t)$ and $G(t)$ is denoted and

defined as $F * G = \int_0^t F(x) G(t - x) dx$ or $F * G = \int_0^t F(t - x) G(x) dx$.

(24) Convolution theorem or Convolution property.

If, $L^{-1} \{\bar{F}(p)\} = F(t)$ and $L^{-1} \{\bar{G}(p)\} = G(t)$,

then $L^{-1} \{\bar{F}(p) \bar{G}(p)\} = \int_0^t F(x) G(t - x) dx = F * G$

OR

$$L^{-1} \{\bar{F}(p) \bar{G}(p)\} = \int_0^t F(t - x) G(x) dx = F * G.$$

Note : $L \{F * G\} = \bar{F}(p) \bar{G}(p).$

i.e. $L \left\{ \int_0^t F(x) G(t-x) dx \right\} = L \left\{ \int_0^t F(t-x) G(x) dx \right\} = \bar{F}(p) \bar{G}(p).$

(25) Complex inversion formula (or integral) for the Laplace transform.

Let $L \{ \bar{F}(t) \} = \bar{F}(p),$ then $L^{-1} \{ \bar{F}(p) \} = \int_{\gamma-i\infty}^{\gamma+i\infty} e^{jpt} \bar{F}(p) dp, \quad t > 0,$

where the integration is to be performed along a line $p = \gamma$ in the complex plane where $p = x + jy$. The real number γ is chosen so that $p = \gamma$ lies to the right of all the singularities.

9.3. SOME SPECIAL TYPES OF INTEGRAL EQUATIONS

(i) Integro-differential equation. Definition.

[Meerut 2008]

An integral equation in which various derivatives of the unknown function $y(t)$ can also be present is said to be an *integro-differential equation*.

For example, the following integral equation is an integro-differential equation.

$$y'(t) = y(t) + f(t) + \int_0^t \sin(t-x) y(x) dx.$$

(ii) Integral equation of convolution type. Definition.

The integral equation

$$y(t) = f(t) + \int_0^t K(t-x) y(x) dx,$$

in which the kernel $K(t-x)$ is a function of the difference $(t-x)$ only, is known as integral equation of the convolution type. Using the definition of convolution, we may re-write it as

$$y(t) = f(t) + K(t) * y(t).$$

9.4. APPLICATION OF LAPLACE TRANSFORM TO DETERMINE THE SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS WITH CONVOLUTION-TYPE KERNELS. WORKING RULE.

(i) Consider the Volterra integral equation of the first kind

$$F(t) = \int_0^t K(t-x) Y(x) dx, \quad \dots (1)$$

where the kernel $K(t-x)$ depends only on the difference $(t-x)$.

Let $L\{Y\} = \bar{Y}(p), \quad L\{K\} = \bar{K}(p) \quad \text{and} \quad L\{F\} = \bar{F}(p). \quad \dots (2)$

Applying the Laplace transform to both sides of (1), we get

$$L(F) = L\{K(t) * Y(t)\}$$

or

$$\bar{F}(p) = \bar{K}(p) \bar{Y}(p), \quad \text{by the convolution theorem}$$

or

$$\bar{Y}(p) = \bar{F}(p) / \bar{K}(p). \quad \dots (3)$$

Applying the inverse Laplace transform to both sides of (3), we get

$$Y(t) = L^{-1} \{ \bar{F}(p) / \bar{K}(p) \}. \quad \dots (4)$$

(ii) Consider the Volterra integral equation of the second kind

$$Y(t) = F(t) + \int_0^t K(t-x) Y(x) dx. \quad \dots (5)$$

Applying the Laplace transform to both sides of (3), we have

$$L\{y\} = L\{F\} + L\{K(t) * Y(t)\}$$

or $\bar{Y}(p) = \bar{F}(p) + \bar{K}(p) \bar{Y}(p)$, using (2) and the convolution theorem

or $\bar{Y}(p) [1 - \bar{K}(p)] = \bar{F}(p)$ or $\bar{Y}(p) = \frac{\bar{F}(p)}{1 - \bar{K}(p)}$ (6)

Applying the inverse Laplace transform to both sides of (6), we have

$$Y(t) = L^{-1} \left\{ \frac{\bar{F}(p)}{1 - \bar{K}(p)} \right\}. \quad \dots (7)$$

(iii) Suppose we want the resolvent kernel of (5) in which the kernel $K(t-x)$ depends only on the difference $(t-x)$. Before doing so, we first show that, if the original kernel $K(t, x)$ is a difference kernel, then so is the resolvent kernel.

We know that the resolvent kernel $R(t, x)$ is given by (refer Art 5.11, Chapter 5)

$$R(t, x) = \sum_{m=1}^{\infty} K_m(t, x) = K_1(t, x) + K_2(t, x) + \dots \quad \dots (8)$$

[Note that here $\lambda = 1$. So we use symbol $R(t, x)$ in place of usual symbol $R(t, x; \lambda)$.]

We know that the iterated kernels are given by (refer Art 5.11, Chapter 5)

$$K_1(t, x) = K(t, x) \quad \dots (9)$$

and

$$K_n(t, x) = \int_x^t K(t, z) K_{n-1}(z, x) dz. \quad \dots (10)$$

Here, by assumption, we have

$$K(t, x) = K(t-x)$$

$\therefore$ by (9),

$$K_1(t, x) = K(t, x) = K(t-x). \quad \dots (11)$$

Putting $n = 2$ in (10), we have

$$\begin{aligned} K_2(t, x) &= \int_x^t K(t, z) K_1(z, x) dz = \int_x^t K(t-z) K(z-x) dz, \text{ using (11)} \\ &= \int_0^{t-x} K(t-x-u) K(u) du, \text{ putting } u = z-x \end{aligned}$$

showing that $K_2(t, x)$ depends only on the difference $(t-x)$. Proceeding likewise we can show that $K_3(t, x)$, $K_4(t, x)$, ... also depend only on the difference $(t-x)$. From (8), it now follows that the resolvent kernel will also depend only on the difference $(t-x)$ and so we may write

$$R(t, x) = R(t-x). \quad \dots (12)$$

We know that solution of (5) is given by [Refer Art 11.5, Chapter 5]

$$Y(t) = F(t) + \int_0^t R(t, x) F(x) dx$$

or

$$Y(t) = F(t) + \int_0^t R(t-x) F(x) dx, \text{ by (12)} \quad \dots (13)$$

Let

$$L\{Y\} = \bar{Y}(p), \quad L\{F\} = \bar{F}(p) \quad \text{and} \quad L\{R\} = \bar{R}(p). \quad \dots (14)$$

Applying the Laplace transform to both sides of (13), we have

$$L\{Y\} = L\{F\} + L\{R * F(t)\}$$

or

$$\bar{Y}(p) = \bar{F}(p) + \bar{R}(p) \bar{F}(p), \text{ using (14) and the convolution theorem} \quad \dots (15)$$

Using (6), (15) reduces to

$$\frac{\bar{F}(p)}{1 - \bar{K}(p)} = \bar{F}(p) [1 + \bar{R}(p)]$$

or
$$\bar{R}(p) = \frac{1}{1 - \bar{K}(p)} - 1 = \frac{\bar{K}(p)}{1 - \bar{K}(p)}. \quad \dots (16)$$

Applying the inverse Laplace transform to both sides of (14), we have

$$R(t-x) = L^{-1} \left\{ \frac{\bar{K}(p)}{1 - \bar{K}(p)} \right\} \quad \dots (17)$$

Substituting the value of $R(t-x)$ given by (17) in (13) we shall get the desired solution of (5).

9.5. SOLVED EXAMPLES BASED ON ARTICLES 9.2 TO 9.4.

Ex. 1. Solve the Abel's equation
$$\int_0^t \frac{Y(x) dx}{(t-x)^{1/3}} = t(1+t).$$

Sol. Given
$$\int_0^t \frac{Y(x) dx}{(t-x)^{1/3}} = t(1+t). \quad \text{or} \quad Y(t) * t^{-1/3} = t + t^2. \quad \dots (1)$$

[using the definition of convolution]

Applying the Laplace transform to both sides of (1), we have

$$L\{Y(t)\} L\{t^{-1/3}\} = L\{t\} + L\{t^2\}, \text{ using convolution theorem.}$$

or
$$L\{Y(t)\} \times \frac{\Gamma(-1/3+1)}{p^{-(1/3)+1}} = \frac{1}{p^2} + \frac{2!}{p^3} \quad \text{or} \quad L\{Y(t)\} \times \frac{\Gamma(2/3)}{p^{2/3}} = \frac{1}{p^2} + \frac{2}{p^3}$$

or
$$L\{Y(t)\} = \frac{1}{\Gamma(2/3)} \left\{ \frac{1}{p^{4/3}} + \frac{2}{p^{7/3}} \right\}. \quad \dots (2)$$

Taking inverse Laplace transform of both sides of (2), we get

$$\begin{aligned} Y(t) &= \frac{1}{\Gamma(2/3)} \left[L^{-1} \left\{ \frac{1}{p^{4/3}} \right\} + 2L^{-1} \left\{ \frac{1}{p^{7/3}} \right\} \right] = \frac{1}{\Gamma(2/3)} \left[\frac{t^{(4/3)-1}}{\Gamma(4/3)} + 2 \times \frac{t^{(7/3)-1}}{\Gamma(7/3)} \right] \\ &= \frac{1}{\Gamma(2/3)} \left\{ \frac{t^{1/3}}{(1/3)\Gamma(1/3)} + \frac{2t^{4/3}}{(4/3) \times (1/3)\Gamma(1/3)} \right\} = \frac{3t^{1/3}}{\Gamma(2/3)\Gamma(1/3)} \left(1 + \frac{3t}{2} \right) = \frac{3t^{1/3}}{\pi/\sin(\pi/3)} \left(1 + \frac{3t}{2} \right) \\ & \quad [\because \Gamma(n)\Gamma(1-n) = \pi/\sin n\pi \Rightarrow \Gamma(1/3)\Gamma(2/3) = \pi/\sin(\pi/3)] \\ &= (3\sqrt{3}/4\pi) \times t^{1/3} (2+3t) \end{aligned}$$

Ex. 2. Solve the Abel's integral equation :
$$\int_0^t \frac{Y(x) dx}{\sqrt{t-x}} = 1+t+t^2.$$

Sol. Re-writing the given equation, we have
$$Y(t) * t^{-1/2} = 1+t+t^2.$$

Applying the Laplace transform to both sides of (1) and using the convolution theorem, we have

$$L\{Y(t)\} L\{t^{-1/2}\} = L\{1\} + L\{t\} + L\{t^2\}$$

or
$$L\{Y(t)\} \times \frac{\Gamma(1/2)}{p^{1/2}} = \frac{1}{p} + \frac{1}{p^2} + \frac{2!}{p^3}$$

or
$$L\{Y(t)\} = \left(1/\sqrt{\pi} \right) \times \left(1/p^{1/2} + 1/p^{3/2} + 2/p^{5/2} \right) \quad \dots (2)$$

Applying the inverse Laplace transform to both sides of (2), we have

$$Y(t) = \frac{1}{\sqrt{\pi}} \left[L^{-1} \left\{ \frac{1}{p^{1/2}} \right\} + L^{-1} \left\{ \frac{1}{p^{3/2}} \right\} + 2L^{-1} \left\{ \frac{1}{p^{5/2}} \right\} \right] = \frac{1}{\sqrt{\pi}} \left\{ \frac{t^{-1/2}}{\Gamma(1/2)} + \frac{t^{1/2}}{\Gamma(3/2)} + \frac{2t^{3/2}}{\Gamma(5/2)} \right\}$$

$$= \frac{1}{\sqrt{\pi}} \left\{ \frac{t^{-1/2}}{\sqrt{\pi}} + \frac{t^{1/2}}{(1/2) \times \sqrt{\pi}} + \frac{2t^{3/2}}{(3/2) \times (1/2) \times \sqrt{\pi}} \right\} = \frac{1}{\pi} \left(t^{-1/2} + 2t^{1/2} + \frac{8t^{3/2}}{3} \right)$$

Ex. 3. Solve the Volterra integral equation of the first kind : $\int_0^t Y(x) Y(t-x) dx = 16 \sin 4t$.
(Kanpur 2007, 09; Meerut 2004, 05, 12)

Sol. Given integral equation can be re-written as $Y(t) * Y(t) = 16 \sin 4t$ (1)

Applying the Laplace transform to (1) and using convolution theorem, we have
 $L\{Y(t) L\{Y(t)\} = 16 L\{\sin 4t\}$

$$\text{or } [L\{Y(t)\}]^2 = 16 \times \frac{4}{p^2 + 16} = \frac{64}{p^2 + 16} \quad \text{or} \quad L\{Y(t)\} = \pm \frac{8}{\sqrt{(p^2 + 4^2)}} \dots (2)$$

Taking inverse Laplace transform, (2) gives $Y(t) = \pm 8L^{-1}\left\{\frac{1}{\sqrt{(p^2 + 4^2)}}\right\} = \pm 8J_0(4t)$.

Ex. 4. Solve the Volterra integral equation of the second kind $Y(t) = t^2 + \int_0^t Y(u) \sin(t-u) du$.
(Kanpur 2010; Meerut 2008, 2011)

Sol. Re-writing the given integral equation, $Y(t) = t^2 + Y(t) * \sin t$ (1)

Applying the Laplace transform to (1) and using the convolution theorem, we have

$$L\{Y(t)\} = L\{t^2\} + L\{Y(t)\} L\{\sin t\} \quad \text{or} \quad L\{Y(t)\} = \frac{2!}{p^3} + L\{Y(t)\} \times \frac{1}{p^2 + 1^2}$$

$$\text{or } L\{Y(t)\} \left(1 - \frac{1}{p^2 + 1}\right) = \frac{2}{p^3} \quad \text{or} \quad L\{Y(t)\} \times \frac{p^2}{1 + p^2} = \frac{2}{p^3}$$

$$\text{or } L\{Y(t)\} = 2/p^3 + 2/p^5 \dots (2)$$

Inverting, (2) reduces to

$$Y(t) = 2L^{-1}\left\{\frac{1}{p^3}\right\} + 2L^{-1}\left\{\frac{1}{p^5}\right\} \quad \text{or} \quad Y(t) = 2 \times \frac{t^2}{2!} + 2 \times \frac{t^4}{4!} = t^2 + \frac{t^4}{12}.$$

Ex. 5. Solve the integral equation $Y(t) = 1 + \int_0^t Y(x) \sin(t-x) dx$ and verify your solution.
(Meerut 2007)

Sol. The given integral equation can be re-written as $y(t) = 1 + Y(t) * \sin t$ (1)

Applying the Laplace Transform to both sides of (1), we have

$$L\{Y(t)\} = L\{1\} + L\{Y(t) * \sin t\}$$

$$\text{or } L\{Y(t)\} = (1/p) + L\{Y(t)\} \cdot L\{\sin t\}, \text{ by the convolution theorem}$$

$$\text{or } L\{Y(t)\} = \frac{1}{p} + L\{Y(t)\} \times \frac{1}{p^2 + 1} \quad \text{or} \quad \left(1 - \frac{1}{p^2 + 1}\right) L\{Y(t)\} = \frac{1}{p}$$

$$\text{or } \frac{p^2}{p^2 + 1} \times L\{Y(t)\} = \frac{1}{p} \quad \text{or} \quad L\{Y(t)\} = \frac{p^2 + 1}{p^3} = \frac{1}{p} + \frac{1}{p^3} \dots (2)$$

$$\text{Inverting, (2) reduces to } Y(t) = L^{-1}\left\{\frac{1}{p}\right\} + L^{-1}\left\{\frac{1}{p^3}\right\} = 1 + \frac{t^2}{2!} = 1 + \frac{t^2}{2}. \dots (3)$$

Verification of solution (3) : We wish to show that the solution (3) satisfies the given integral equation

$$Y(t) = 1 + \int_0^t Y(x) \sin(t-x) dx. \quad \dots (4)$$

From (3), $Y(x) = 1 + (x^2/2)$

$$\therefore \text{R.H.S. of (4)} = 1 + \int_0^t (1 + x^2/2) \sin(t-x) dx = 1 + \left[(1 + x^2/2) \cos(t-x) \right]_0^t - \int_0^t x \cos(t-x) dx$$

$$= 1 + 1 + (t^2/2) - \cos t - \{[-x \sin(t-x)]_0^t + \int_0^t \sin(t-x) dx\}$$

$$= 2 + (t^2/2) - \cos t - [\cos(t-x)]_0^t = 2 + (t^2/2) - \cos t - (1 - \cos t) = 1 + (t^2/2) = Y(t), \text{ by (3)}$$

$$= \text{L.H.S. of (4),}$$

showing that (3) is solution of given integral equation (4).

Ex. 6. Solve: $Y(t) = a \sin t - 2 \int_0^t Y(x) \cos(t-x) dx.$

Sol. Re-writing the given integral equation, $Y(t) = a \sin t - 2 Y(t) * \cos t. \quad \dots (1)$

Taking the Laplace transform of both sides of (1), we get

$$L\{Y(t)\} = a L\{\sin t\} - 2L\{Y(t) * \cos t\}$$

or $L\{Y(t)\} = \frac{a}{p^2+1} - 2L\{Y(t)\} \times \frac{p}{p^2+1},$ by the convolution theorem

or $\left(1 + \frac{2p}{p^2+1}\right) L\{Y(t)\} = \frac{a}{p^2+1} \quad \text{or} \quad L\{Y(t)\} = \frac{a}{(p+1)^2} \quad \dots (2)$

Inverting, (2) reduces to

$$Y(t) = a L^{-1} \left\{ \frac{1}{(p+1)^2} \right\} = a e^{-t} L^{-1} \left\{ \frac{1}{p^2} \right\}, \text{ by first shifting theorem}$$

or $Y(t) = a t e^{-t}.$

Ex. 7. Solve the equation $Y(t) = e^{-t} - 2 \int_0^t \cos(t-x) Y(x) dx$ by using Laplace transform

(Meerut 2006, 09)

Sol. Re-writing the given integral equation, $Y(t) = e^{-t} - 2 Y(t) * \cos t. \quad \dots (1)$

Applying the Laplace transform to (1) and using the convolution theorem, we have

$$L\{Y(t)\} = L\{e^{-t}\} - 2L\{Y(t)\} L\{\cos t\} \quad \text{or} \quad L\{Y(t)\} = \frac{1}{p+1} - 2L\{Y(t)\} \times \frac{p}{p^2+1}$$

or $\left(1 + \frac{2p}{p^2+1}\right) L\{Y(t)\} = \frac{1}{p+1} \quad \text{or} \quad \frac{(p+1)^2}{p^2+1} L\{Y(t)\} = \frac{1}{p+1}$

or $L\{Y(t)\} = \frac{p^2+1}{(p+1)^3} = \frac{[(p+1)-1]^2+1}{(p+1)^3} \quad \dots (2)$

Inverting, (2) yields
$$Y(t) = L^{-1} \left\{ \frac{[(p+1)-1]^2 + 1}{(p+1)^3} \right\} = e^{-t} L^{-1} \left\{ \frac{(p-1)^2 + 1}{p^3} \right\}$$
[by first shifting theorem]
$$= e^{-t} L^{-1} \left\{ \frac{p^2 - 2p + 2}{p^3} \right\} = e^{-t} L^{-1} \left\{ \frac{1}{p} - \frac{2}{p^2} + \frac{2}{p^3} \right\} = e^{-t} \left[L^{-1} \left\{ \frac{1}{p} \right\} - 2L^{-1} \left\{ \frac{1}{p^2} \right\} + 2L^{-1} \left\{ \frac{1}{p^3} \right\} \right]$$

$$= e^{-t} [1 - 2t + 2 \times (t^2 / 2!)] = e^{-t} (1 - 2t + t^2) = e^{-t} (1 - t)^2$$

Ex. 8. Solve : $Y(t) = t + 2 \int_0^t \cos(t-x) Y(x) dx$ (Kanpur 2005, 08)

Sol. Re-writing given equation, $Y(t) = t + 2 Y(t) * \cos t$ (1)

Taking Laplace transform of (1) and using the convolution theorem, we get

$$L\{Y(t)\} = L\{t\} + 2L\{Y(t)\}L\{\cos t\} = \frac{1}{p^2} + \frac{2p}{p^2 + 1} L\{Y(t)\}$$

or
$$\left(1 - \frac{2p}{p^2 + 1}\right) L\{Y(t)\} = \frac{1}{p^2} \quad \text{or} \quad L\{Y(t)\} = \frac{p^2 + 1}{p^2(p-1)^2}. \quad \dots (2)$$

Inverting, (2) gives

$$Y(t) = L^{-1} \left\{ \frac{1}{(p-1)^2} + \frac{1}{p^2(p-1)^2} \right\} = L^{-1} \left\{ \frac{1}{(p-1)^2} \right\} + L^{-1} \left\{ \frac{1}{p^2(p-1)^2} \right\}. \quad \dots (3)$$

Now,
$$L^{-1} \left\{ \frac{1}{(p-1)^2} \right\} = e^t L^{-1} \left\{ \frac{1}{p^2} \right\}, \text{ by first shifting theorem}$$

$$= e^t \times t \quad \dots (4)$$

Using result (22) of Art. 9.2, we have

$$\therefore L^{-1} \left\{ \frac{1}{p} \frac{1}{(p-1)^2} \right\} = \int_0^t u e^u du = [e^u u]_0^t - \int_0^t e^u du = t e^t - [e^u]_0^t$$

$$= t e^t - (e^t - 1) = e^t(t-1) + 1.$$

$$\therefore L^{-1} \left\{ \frac{1}{p} \frac{1}{p(p-1)^2} \right\} = \int_0^t [e^u(u-1) + 1] du = \int_0^t u e^u du - \int_0^t e^u du + \int_0^t du$$

$$= e^t(t-1) + 1 - (e^t - 1) + t = e^t(t-1) + t + 2. \quad \dots (5)$$

Using (4) and (5) in (3), $Y(t) = t e^t + e^t(t-2) + t + 2 = 2e^t(t-1) + 2 + t.$

Ex. 9. Solve the integro-differential equation : $Y'(t) = \sin t + \int_0^t Y(t-x) \cos x dx$, where $Y(0) = 0$. (Kanpur 2009)

Sol. Re-writing the given equation, we have $Y'(t) = \sin t + Y(t) * \cos t$ (1)

Also given that $Y(0) = 0$ (2)

Applying the Laplace transform to both sides of (1), we get

$$L\{Y'(t)\} = L\{\sin t\} + L\{Y(t) * \cos t\}$$

or
$$pL\{Y(t)\} - Y(0) = \frac{1}{p^2 + 1} + L\{Y(t)\}L\{\cos t\}, \text{ by the convolution theorem}$$

or
$$pL\{Y(t)\} = \frac{1}{p^2 + 1} + L\{Y(t)\} \times \frac{p}{p^2 + 1}, \text{ using (2)}$$

$$\begin{aligned} \text{or} \quad & \left(1 - \frac{1}{p^2 + 1}\right) p L\{Y(t)\} = \frac{1}{p^2 + 1} \quad \text{or} \quad \frac{p^3}{p^2 + 1} L\{Y(t)\} = \frac{1}{p^2 + 1} \\ \text{or} \quad & L\{Y(t)\} = 1/p^3. \quad \dots (3) \\ \text{Inverting, (3) reduces to} \quad & Y(t) = L^{-1}\{1/p^3\} = t^2/2! = t^2/2. \end{aligned}$$

Ex. 10. Solve $Y'(t) = t + \int_0^t Y(t-x) \cos x \, dx$, $Y(0) = 4$. (Kanpur 2005, 08)

Sol. Re-writing the given integral equation, we have $Y'(t) = t + Y(t) * \cos t \quad \dots (1)$

Also, given that $Y(0) = 4$. $\dots (2)$

Applying the Laplace transform to (1) and using the convolution theorem, we have

$$L\{Y'(t)\} = L\{t\} + L\{Y(t)\} L\{\cos t\} \quad \text{or} \quad p L\{Y(t)\} - Y(0) = \frac{1}{p^2} + L\{Y(t)\} \times \frac{p}{p^2 + 1}$$

$$\text{or} \quad \left(1 - \frac{1}{p^2 + 1}\right) p L\{Y(t)\} - 4 = \frac{1}{p^2}, \text{ using (2)}$$

$$\text{or} \quad \frac{p^3}{p^2 + 1} L\{Y(t)\} = 4 + \frac{1}{p^2} \quad \text{or} \quad L\{Y(t)\} = \frac{p^2 + 1}{p^3} \left(4 + \frac{1}{p^2}\right)$$

$$\text{or} \quad L\{Y(t)\} = \frac{4(p^2 + 1)}{p^3} + \frac{p^2 + 1}{p^5} = \frac{4}{p} + \frac{5}{p^3} + \frac{1}{p^5}. \quad \dots (3)$$

$$\text{Inverting, (3) yields} \quad Y(t) = 4L^{-1}\{1/p\} + 5L^{-1}\{1/p^3\} + L^{-1}\{1/p^5\}$$

$$\text{or} \quad Y(t) = 4 + 5 \times (t^2/2!) + (t^4/4!) = 4 + (5t^2/2) + (t^4/24).$$

Ex. 11. Solve the integral equation $t = \int_0^t e^{t-x} Y(x) \, dx$.

Sol. Re-writing the given integral equation, $t = Y(t) * e^t. \quad \dots (1)$

Taking the Laplace transform of both sides of (1) and using the convolution theorem, we get

$$L\{t\} = L\{Y(t)\} \cdot L\{e^t\}$$

$$\text{or} \quad \frac{1}{p^2} = L\{Y(t)\} \times \frac{1}{p-1} \quad \text{or} \quad L\{Y(t)\} = \frac{p-1}{p^2} \quad \text{or} \quad L\{Y(t)\} = \frac{1}{p} - \frac{1}{p^2} \quad \dots (2)$$

$$\text{Inverting,} \quad Y(t) = L^{-1}\{1/p\} - L^{-1}\{1/p^2\} = 1 - t.$$

Ex. 12. Solve : $\sin t = \int_0^t J_0(t-x) Y(x) \, dx$. (Meerut, 2010; Kanpur 2005)

Sol. Re-writing the given integral equation, $\sin t = Y(t) * J_0(t). \quad \dots (1)$

Taking the Laplace transform of both sides of (1) and using the convolution theorem, we get

$$L\{\sin t\} = L\{Y(t)\} \cdot L\{J_0(t)\} \quad \text{or} \quad \frac{1}{p^2 + 1} = L\{Y(t)\} \times \frac{1}{\sqrt{p^2 + 1}}$$

$$\text{or} \quad L\{Y(t)\} = 1/\sqrt{p^2 + 1} \quad \dots (2)$$

$$\text{Inverting (2),} \quad Y(t) = L^{-1}\{1/\sqrt{p^2 + 1}\} = J_0(t).$$

Ex. 13. Solve the Able integral equation $F(t) = \int_0^t \frac{Y(x)}{(t-x)^\alpha} dx$, $0 < \alpha < 1$.

Sol. Re-writing the given Able integral equation, we have $F(t) = Y(t) * t^{-\alpha}$ (1)

Taking the Laplace transform of both sides of (1) and using the convolution theorem, we get

$$L\{F(t)\} = L\{Y(t)\} L\{t^{-\alpha}\}$$

or
$$\bar{F}(p) = L\{Y(t)\} \times \frac{\Gamma(1-\alpha)}{p^{1-\alpha}}, \text{ where } \bar{F}(p) = L\{F(t)\}$$

or
$$\begin{aligned} L\{Y(t)\} &= \frac{p^{1-\alpha} \bar{F}(p)}{\Gamma(1-\alpha)} = \frac{p}{\Gamma(\alpha)\Gamma(1-\alpha)} \{\Gamma(\alpha) p^{-\alpha} \bar{F}(p)\} \\ &= \frac{p}{(\pi/\sin \pi\alpha)} \{F(\alpha) p^{-\alpha} \bar{F}(p)\}, \quad \text{as } \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha} \\ &= \frac{p \sin \pi\alpha}{\pi} L\{t^{\alpha-1}\} \cdot L\{F(t)\} = \frac{p \sin \pi\alpha}{\pi} L\{t^{\alpha-1} * F(t)\}, \text{ by convolution theorem,} \\ &= \frac{p \sin \pi\alpha}{\pi} L\left\{\int_0^t (t-x)^{\alpha-1} F(x) dx\right\}, \text{ by definition of convolution} \\ &= \frac{\sin \pi\alpha}{\pi} p L\left\{\int_0^t (t-x)^{\alpha-1} F(x) dx\right\}. \end{aligned} \quad \dots (2)$$

Let
$$G(t) = \int_0^t (t-x)^{\alpha-1} F(x) dx \quad \dots (3)$$

From (3),
$$G(0) = 0. \quad \dots (4)$$

Now,
$$L\{G'(t)\} = p L\{G(t)\} - G(0) = p L\{G(t)\}, \text{ using (4)}$$

$$\therefore p L\{G(t)\} = L\left\{\frac{d}{dt} G(t)\right\}$$

or
$$p L\left\{\int_0^t (t-x)^{\alpha-1} F(x) dx\right\} = L\left\{\frac{d}{dx} G(t)\right\}, \text{ by (3)} \quad \dots (5)$$

Using (5), (2) reduces to
$$L\{Y(t)\} = \frac{\sin \pi\alpha}{\pi} L\left\{\frac{d}{dt} G(t)\right\}$$

Inverting,
$$Y(t) = \frac{\sin \pi\alpha}{\pi} \frac{d}{dt} G(t) = \frac{\sin \pi\alpha}{\pi} \frac{d}{dt} \left[\int_0^t (t-x)^{\alpha-1} F(x) dx \right]. \quad \dots (6)$$

Note : We have already got the above solution in Art. 8.2 of chapter 8 by a different method.

Ex. 14. Find the resolvent kernel of the Volterra integral equation and hence its solution

$$Y(t) = F(t) + \int_0^t (t-x) Y(x) dx. \quad (\text{Kanpur 2006, Meerut 2005})$$

Sol. Re-writing the given integral equation,
$$Y(t) = F(t) + Y(t) * t. \quad \dots (1)$$

Applying the Laplace transform to both sides of (1) and using the convolution theorem, we get

$$L\{Y(t)\} = L\{F(t)\} + L\{Y(t)\} \cdot L\{t\} \quad \text{or} \quad L\{Y(t)\} = L\{F(t)\} + L\{Y(t)\} \times (1/p^2)$$

or
$$\left(1 - \frac{1}{p^2}\right) L\{Y(t)\} = L\{F(t)\} \quad \text{or} \quad L\{Y(t)\} = \frac{p^2}{p^2 - 1} L\{F(t)\}. \quad \dots (2)$$

Let $R(t-x)$ be the resolvent kernel of the given integral equation. Then, we know that the required solution is given by

$$Y(t) = F(t) + \int_0^t R(t-x)F(t) dt \quad \dots (3)$$

or
$$Y(t) = F(t) + R(t) * F(t). \quad \dots (3)'$$

Applying the Laplace transform to both sides of (3)' and using the convolution theorem, we have

$$L\{Y(t)\} = L\{F(t)\} + L\{R(t)\} \cdot L\{F(t)\}$$

or
$$\frac{p^2}{p^2-1} L\{F(t)\} = [1 + L\{R(t)\}] L\{F(t)\}, \text{ by (2)}$$

or
$$1 + L\{R(t)\} = \frac{p^2}{p^2-1} \quad \text{or} \quad L\{R(t)\} = \frac{p^2}{p^2-1} - 1 = \frac{1}{p^2-1} \quad \dots (4)$$

Inverting, $R(t) = \sinh t$ so that $R(t-x) = \sinh(t-x), \quad \dots (5)$
giving the required resolvent kernel,

Substituting the above value of $R(t-x)$ in (3), the required solution is

$$Y(t) = F(t) + \int_0^t \sinh(t-x)F(t) dt.$$

Ex. 15. Determine the resolvent kernel and hence solve the integral equation

$$Y(t) = F(t) + \int_0^t e^{t-x}Y(x) dx.$$

Sol. Re-writing the giving integral equation,
$$Y(t) = F(t) + Y(t) * e^t. \quad \dots (1)$$

Applying the Laplace transform to both sides of (1) and using the convolution theorem, we get
 $L\{Y(t)\} = L\{F(t)\} + L\{Y(t)\} \cdot L\{e^t\}$ or $L\{Y(t)\} = L\{F(t)\} + L\{Y(t)\} \cdot \frac{1}{p-1}$

or
$$\left(1 - \frac{1}{p-1}\right) L\{Y(t)\} = L\{F(t)\} \quad \text{or} \quad L\{Y(t)\} = \frac{p-1}{p-2} L\{F(t)\}. \quad \dots (2)$$

Let $R(t-x)$ be the resolvent kernel of the given integral equation. Then, we know that the required solution is given by

$$Y(t) = F(t) + \int_0^t R(t-x)F(t) dt \quad \dots (3)$$

or
$$Y(t) = F(t) + R(t) * F(t). \quad \dots (3)'$$

Applying the Laplace transform to both sides of (3)' and using the convolution theorem, we get

$$L\{Y(t)\} = L\{F(t)\} + L\{R(t)\} \cdot L\{F(t)\}$$

or
$$\frac{p-1}{p-2} L\{F(t)\} = [1 + L\{R(t)\}] L\{F(t)\}, \text{ by (2)}$$

or
$$1 + L\{R(t)\} = \frac{p-1}{p-2} \quad \text{or} \quad L\{R(t)\} = \frac{p-1}{p-2} - 1 = \frac{1}{p-2}$$

Inverting, $R(t) = L^{-1}\{1/(p-2)\} = e^{2t}$ so that $R(t-x) = e^{2(t-x)} \quad \dots (4)$
giving the required resolvent kernel.

Substituting the above value of $R(x-t)$ in (3), the required solution is

$$Y(t) = F(t) + \int_0^t e^{2(t-x)} F(x) dx$$

Ex. 16. Solve the integral equation $f(s) = \int_0^s K(s^2 - u^2) y(u) du, s > 0$

Sol. To re-write the given equation in standard form, we introduce two new variables t and x as follows :

$$s = t^{1/2} \quad \text{and} \quad u = x^{1/2} \quad \text{so that} \quad ds = (1/2) \times t^{-1/2} dt \quad \text{and} \quad du = (1/2) \times x^{-1/2} dx \quad \dots (1)$$

$$\text{and obtain} \quad f(t^{1/2}) = \int_0^t \{K(t-x) y(x^{1/2}) \times (1/2) \times x^{-1/2}\} dx \quad \dots (2)$$

$$\text{Let} \quad f(t^{1/2}) = F(t) \quad \text{and} \quad (1/2) \times x^{-1/2} \times y(x^{1/2}) = Y(x) \quad \dots (3)$$

Using (3), (2) reduces to standard form

$$F(t) = \int_0^t K(t-x) Y(x) dx, \quad t > 0$$

where the kernel $K(t-x)$ depends only on the difference $(t-x)$. Re-writing it, we get

$$F(t) = K(t) * Y(t), \quad \text{by definition of convolution} \quad \dots (4)$$

$$\text{Let} \quad L\{Y(t)\} = \bar{Y}(p), \quad L\{K(t)\} = \bar{K}(p) \quad \text{and} \quad L\{F(t)\} = \bar{F}(p)$$

Applying the Laplace transform to both sides of (4), we get

$$L\{F(t)\} = L\{K(t) * Y(t)\} \quad \text{or} \quad L\{F(t)\} = L\{K(t)\} L\{Y(t)\}, \quad \text{by the convolution theorem}$$

$$\text{or} \quad \bar{F}(p) = \bar{K}(p) \bar{Y}(p) \quad \text{or} \quad \bar{Y}(p) = \frac{p \bar{F}(p)}{p \bar{K}(p)} \quad \dots (5)$$

$$\text{Let} \quad \bar{H}(p) = 1 / p \bar{K}(p) \quad \dots (6)$$

$$\text{and let} \quad L^{-1}\{\bar{H}(p)\} = H(t) \quad \dots (7)$$

$$\text{Using (6), (5) reduces to} \quad \bar{Y}(p) = p \bar{H}(p) \bar{F}(p)$$

$$\text{or} \quad \bar{Y}(p) = p L \left\{ \int_0^t H(t-x) F(x) dx \right\}, \quad \text{by result (24) of Art. 9.2}$$

$$\text{Thus,} \quad \bar{Y}(p) = L \left\{ \frac{d}{dt} \int_0^t H(t-x) F(x) dx \right\}, \quad \text{by result 8 of Art 9.2}$$

which by inversion yields the solution in terms of t and x

$$Y(t) = \frac{d}{dt} \int_0^t H(t-x) F(x) dx \quad \dots (8)$$

In order to get the solution in terms of the old variables s and u , we use (1) and (3) and get

$$\frac{1}{2t^{1/2}} y(t^{1/2}) = \frac{1}{2s} \frac{d}{ds} \int_0^s \{H(s^2 - u^2) f(x^{1/2}) \times 2u\} du$$

$$\text{or} \quad \frac{y(s)}{2s} = \frac{1}{s} \frac{d}{ds} \int_0^s u H(s^2 - u^2) f(u) du, \quad \text{by (3)}$$

$$\text{or} \quad y(s) = 2 \times \frac{d}{ds} \int_0^s u f(u) H(s^2 - u^2) du, \quad \dots (9)$$

which is the required solution of the given integral equation.

Solution of the given equation for particular values of the general kernel $K(s^2 - u^2)$.

Particular Case I. Let $K(t) = t^{-\alpha}$, $0 < \alpha < 1$. Then the given integral equation takes the form

$$f(s) = \int_0^s \frac{y(u)}{(s^2 - u^2)^\alpha} du, \quad 0 < \alpha < 1. \quad \dots (10)$$

Here

$$\bar{K}(p) = L\{K(t)\} = L\{t^{-\alpha}\} = \Gamma(1-\alpha) / p^{1-\alpha}$$

$\therefore$

$$\bar{H}(p) = \frac{1}{p \bar{K}(p)} = \frac{1}{p} \times \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} = \frac{p^{-\alpha}}{\Gamma(1-\alpha)}$$

$$\Rightarrow H(t) = L^{-1}\{\bar{H}(p)\} = \frac{1}{\Gamma(1-\alpha)} L^{-1}\left\{\frac{1}{p^\alpha}\right\} = \frac{1}{\Gamma(1-\alpha)} \times \frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

or $H(t) = \frac{t^{\alpha-1}}{(\pi / \sin \pi \alpha)},$ as $\Gamma(\alpha) \Gamma(1-\alpha) = \frac{\pi}{\sin \pi \alpha}$

$$\therefore H(t) = (\sin \pi \alpha) / \pi t^{1-\alpha} \quad \text{so that} \quad H(s^2 - u^2) = (\sin \pi \alpha) / \pi (s^2 - u^2)^{1-\alpha} \quad \dots (11)$$

Substituting the value of $H(s^2 - u^2)$ given by (11) in (9), the required solution of (10) is

$$y(s) = \frac{2 \sin \pi \alpha}{\pi} \frac{d}{ds} \left\{ \int_0^s \frac{u f(u)}{(s^2 - u^2)^{1-\alpha}} du \right\} \quad \dots (12)$$

which has already been obtained by using different method in Ex. 3 on page 8.7.

Particular Case II. Let $K(t) = t^{-1/2} \cos(\beta t^{1/2})$, where β is a constant. Proceed as usual and verify that

$$\bar{K}(p) = \pi^{1/2} p^{-1/2} e^{-\beta^2/4p}$$

and

$$H(t) = L^{-1}\{\bar{H}(p)\} = L^{-1}\{\pi^{-1/2} p^{-1/2} e^{\beta^2/4p}\} = \pi^{-1} t^{-1/2} \cosh(\beta t^{1/2})$$

Hence we find that solution of the integral equation

$$f(s) = \int_0^s \frac{\cos[\beta(s^2 - u^2)^{1/2}]}{(s^2 - u^2)^{1/2}} y(u) du, \quad s > 0 \quad \dots (13)$$

is given by

$$y(s) = \frac{2}{\pi} \frac{d}{ds} \int_0^s \frac{\cosh[\beta(s^2 - u^2)^{1/2}]}{(s^2 - u^2)^{1/2}} u f(u) du \quad \dots (14)$$

Here (13) and (14) are valid for $0 < s < \infty$

Ex. 17. Solve the integral equation $Y(t) = F(t) + \lambda \int_0^t J_0(t-x) Y(x) dx$

Solution. Given $Y(t) = F(t) + \lambda \int_0^t J_0(t-x) Y(x) dx \quad \dots (1)$

Here we shall use the method explained in Art. 10.4 (ii) for solving

$$Y(t) = F(t) + \int_0^t K(t-x) Y(x) dx \quad \dots (2)$$

Comparing (2) with (1), we have

$$K(t) = \lambda J_0(t) \quad \text{so that} \quad \bar{K}(p) = L\{K(t)\} = \lambda / (1 + p^2)^{1/2}. \quad \dots (3)$$

We know that solution of (2) is given by

$$Y(t) = F(t) + \int_0^t R(t-x)F(x) dx \quad \dots (4)$$

where $R(t-x)$ is resolvent kernel of (2).

$$(4) \text{ may be written as } Y(t) = F(t) + R(t) * F(t).$$

Applying the Laplace transform to both sides, we get

$$L\{Y(t)\} = L\{F(t)\} + L\{R(t) * F(t)\}$$

or

$$\bar{Y}(p) = \bar{F}(p) + \bar{R}(p)\bar{F}(p) \quad \dots (5)$$

$$\text{Here } \bar{Y}(p) = L\{Y(t)\}, \bar{F}(p) = L\{F(t)\}, \bar{K}(p) = L\{K(t)\} \text{ and } \bar{R}(p) = L\{R(t)\}$$

Similarly, (2) yields

$$\bar{Y}(p) = \bar{F}(p) + \bar{K}(p)\bar{Y}(p)$$

so that

$$\bar{Y}(p) = \bar{F}(p)/(1 - \bar{K}(p)) \quad \dots (6)$$

From (5) and (6),

$$\bar{F}(p)/(1 - \bar{K}(p)) = \bar{F}(p) + \bar{R}(p)\bar{F}(p)$$

or

$$1 + \bar{R}(p) = \frac{1}{1 - \bar{K}(p)} \quad \text{so that} \quad \bar{R}(p) = \frac{\bar{K}(p)}{1 - \bar{K}(p)} \quad \dots (7)$$

From (3) and (7),

$$\bar{R}(p) = \frac{\lambda/(1+p^2)^{1/2}}{1 - \lambda/(1+p^2)^{1/2}} = \frac{\lambda}{(1+p^2)^{1/2} - \lambda}$$

$$\therefore R(t) = L^{-1}\{\bar{R}(p)\} = L^{-1}[\lambda/\{(1+p^2)^{1/2} - \lambda\}]$$

$$= \frac{\lambda}{(1-\lambda^2)^{1/2}} \int_0^t \{\sin(1-\lambda^2)^{1/2}\} \times (s-u) \frac{J_1(u)}{u} du + \lambda \{[\cos(1-\lambda^2)^{1/2}t] + \frac{\lambda^2}{(1-\lambda^2)^{1/2}} \sin[(1-\lambda^2)^{1/2}t] \} \dots (8)$$

Replacing t by $t-x$ in (8), we get $R(t-x)$ and then solution of (1) can be easily obtained

Ex. 18. Solve the inhomogeneous Abel integral equation

$$Y(t) = F(t) + \lambda \int_0^t \frac{Y(x)}{(t-x)^\alpha} dx, \quad 0 < \alpha < 1$$

Hint. Proceed as in Ex. 17 by a similar method. Here $K(t) = \lambda t^{-\alpha}$ and hence

$$\bar{K}(p) = L\{K(t)\} = \lambda \Gamma(1-\alpha) p^{\alpha-1}$$

and

$$\bar{R}(p) = \lambda \Gamma(1-\alpha) p^{\alpha-1} / [1 - \lambda \Gamma(1-\alpha) p^{\alpha-1}]$$

so that

$$R(t) = L^{-1}\{\bar{R}(p)\} = \sum_{n=1}^{\infty} \frac{[\lambda \Gamma(1-\alpha) t^{1-\alpha}]^n}{t \Gamma[n(1-\alpha)]}$$

Hence the required solution is given by

$$Y(t) = F(t) + \int_0^t R(t-x) F(x) dx$$

i.e.,

$$Y(t) = F(t) + \int_0^t \frac{\sum_{n=1}^{\infty} [\lambda \Gamma(1-\alpha) (t-x)^{1-\alpha}]^n}{(t-x) \Gamma[n(1-\alpha)]} F(x) dx$$

9.6. SOME USEFUL RESULTS ABOUT FOURIER TRANSFORMS.

(1) The (complex) Fourier transform. Definition. Given a function $Y(x)$ defined for all x in the interval $-\infty < x < \infty$, the Fourier transform of $Y(x)$ is a function of a new variable p given by

$$F\{Y(x)\} = \bar{Y}(p) = \int_{-\infty}^{\infty} e^{ipx} Y(x) dx. \quad \dots (1A)$$

The function $Y(x)$ is then called inverse Fourier transform of $F\{Y(x)\}$ or $\bar{Y}(p)$ and is written as $Y(x) = F^{-1}\{F(Y(x))\}$ or $Y(x) = F^{-1}\{\bar{Y}(p)\}$, and is given by

$$Y(x) = F^{-1}\{\bar{Y}(p)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} \bar{Y}(p) dp \quad \dots (2A)$$

Remark 1. Some authors also define (1A) and (2A) in the following manner :

$$F\{Y(x)\} = \bar{Y}(p) = \int_{-\infty}^{\infty} e^{-ipx} Y(x) dx. \quad \dots (1B)$$

and

$$Y(x) = F^{-1}\{\bar{Y}(p)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \bar{Y}(p) dp. \quad \dots (2B)$$

Remark 2. Some authors define (1A) and (2A) in the so called *symmetric form* as follows.

$$F\{Y(x)\} = \bar{Y}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} Y(x) dx \quad \dots (1C)$$

and

$$Y(x) = F^{-1}\{\bar{Y}(p)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \bar{Y}(p) dp. \quad \dots (2C)$$

(2) The (infinite) Fourier sine transform. Definition. The Fourier sine transform of $Y(x)$, $0 < x < \infty$ is denoted and defined as follows :

$$F_s\{Y(x)\} = \bar{Y}_s(p) = \int_0^{\infty} Y(x) \sin px dx. \quad \dots (1A)$$

Then, the corresponding inversion formula is given by

$$Y(x) = F_s^{-1}\{\bar{Y}_s(p)\} = \frac{2}{\pi} \int_0^{\infty} \bar{Y}_s(p) \sin px dp. \quad \dots (2A)$$

Remark. Some authors define (1A) and (2A) in the so called symmetric form :

$$F_s\{Y(x)\} = \bar{Y}_s(p) = (2/\pi)^{1/2} \int_0^{\infty} Y(x) \sin px dx \quad \dots (1B)$$

and

$$Y(x) = F_s^{-1}\{\bar{Y}_s(p)\} = (2/\pi)^{1/2} \int_0^{\infty} \bar{Y}_s(p) \sin px dp. \quad \dots (2B)$$

(3) The (infinite) Fourier cosine transform. Definition. The Fourier cosine transform of $Y(x)$, $0 < x < \infty$ is denoted and defined as follows :

$$F_c\{Y(x)\} = \bar{Y}_c(p) = \int_0^{\infty} Y(x) \cos px dx. \quad \dots (1A)$$

Then, the corresponding inversion formula is given by

$$Y(x) = F_c^{-1}\{\bar{Y}_c(p)\} = \frac{2}{\pi} \int_0^{\infty} \bar{Y}_c(p) \cos px dp. \quad \dots (2A)$$

Remark. Some authors define (1A) and (2A) in the so called symmetric form :

$$F_c \{Y(x)\} = \bar{Y}_c(p) = (2/\pi)^{1/2} \int_0^\infty Y(x) \cos px \, dx \quad \dots (1B)$$

and

$$Y(x) = F_c^{-1} \{\bar{Y}_c(p)\} = (2/\pi)^{1/2} \int_0^\infty \bar{Y}_c(p) \cos px \, dp. \quad \dots (2B)$$

(4) Linearity property of Fourier transforms :

- (i) $F \{c_1 Y_1(x) + c_2 Y_2(x)\} = c_1 F \{Y_1(x)\} + c_2 F \{Y_2(x)\}.$
- (ii) $F_s \{c_1 Y_1(x) + c_2 Y_2(x)\} = c_1 F_s \{Y_1(x)\} + c_2 F_s \{Y_2(x)\}.$
- (iii) $F_c \{c_1 Y_1(x) + c_2 Y_2(x)\} = c_1 F_c \{Y_1(x)\} + c_2 F_c \{Y_2(x)\}.$

(5) Change of scale property.

- (i) If $F \{Y(x)\} = \bar{Y}(p)$, then $F \{Y(ax)\} = (1/a) \times \bar{Y}(p/a).$
- (ii) If $F_s \{Y(x)\} = \bar{Y}_s(p)$, then $F_s \{Y(ax)\} = (1/a) \times \bar{Y}_s(p/a).$
- (iii) If $F_c \{Y(x)\} = \bar{Y}_c(p)$, then $F_c \{Y(ax)\} = (1/a) \times \bar{Y}_c(p/a).$

(6) Convolution or Faltung. The convolution of two functions $G(x)$ and $H(x)$, where $-\infty < x < \infty$, is denoted and defined as

$$G * H = \int_{-\infty}^{\infty} G(x)H(t-x) \, dx \quad \text{or} \quad G * H = \int_{-\infty}^{\infty} G(t-x)H(x) \, dx.$$

(7) Convolution theorem or convolution property. The Fourier transform of the convolution of $G(x)$ and $H(x)$ is the product of the Fourier transforms of $G(x)$ and $H(x)$, i.e.,

$$F \{G * H\} = F \{G(x)\} \cdot F \{H(x)\}.$$

(8) Shifting property. If $\bar{Y}(p)$ is the complex Fourier transform of $Y(x)$, then complex Fourier transform of $Y(x-a)$ is $e^{ipa} \bar{Y}(p).$

9.7. APPLICATION OF FOURIER TRANSFORM TO DETERMINE THE SOLUTIONS OF INTEGRAL EQUATIONS :

The whole procedure will be clear from the following examples.

Ex. 1. Solve the integral equation
$$\int_0^\infty F(x) \cos px \, dx = \begin{cases} 1-p, & 0 \leq p \leq 1 \\ 0, & p > 1. \end{cases}$$

by using Fourier transform. (Meerut 2006, 08, 09, 11; Kanpur 2005, 09)

Sol. Let
$$\bar{F}_c(p) = \begin{cases} 1-p, & 0 \leq p \leq 1 \\ 0, & p > 1. \end{cases} \quad \dots (1)$$

Then the given integral equation can be re-written as
$$\bar{F}_c(p) = \int_0^\infty F(x) \cos px \, dx. \quad \dots (2)$$

By definition of Fourier cosine transform, we see that $\bar{F}_c(p)$ is Fourier cosine transform of $F(x)$. Hence, using the corresponding inversion formula, we have

$$F(x) = \frac{2}{\pi} \int_0^\infty \bar{F}_c(p) \cos px \, dp = \frac{2}{\pi} \left[\int_0^1 \bar{F}_c(p) \cos px \, dp + \int_1^\infty \bar{F}_c(p) \cos px \, dp \right]$$

$$\begin{aligned}
&= \frac{2}{\pi} \left[\int_0^1 (1-p) \cos px \, dp + \int_1^\infty (0) \times \cos px \, dp \right], \text{ by (1)} \\
&= \frac{2}{\pi} \left\{ \left[(1-p) \frac{\sin px}{x} \right]_0^1 - \int_0^1 (-1) \times \frac{\sin px}{x} \, dp \right\} = \frac{2}{\pi x} \int_0^1 \sin px \, dp \\
&= \frac{2}{\pi x} \left[-\frac{\cos px}{x} \right]_0^1 = \frac{2}{\pi x^2} (-\cos x + 1) = \frac{2(1 - \cos x)}{\pi x^2}.
\end{aligned}$$

Ex. 2. Solve the integral equation :
$$\int_0^\infty F(x) \sin xp \, dx = \begin{cases} 1, & 0 \leq p < 1 \\ 2, & 1 \leq p < 2. \\ 0, & p \geq 2. \end{cases}$$
 (Meerut 2005)

Sol. Let
$$\bar{F}_s(p) = \begin{cases} 1, & 0 \leq p < 1 \\ 2, & 1 \leq p < 2. \\ 0, & p \geq 2. \end{cases} \quad \dots (1)$$

Then the given integral equation can be re-written as
$$\bar{F}_s(p) = \int_0^\infty F(x) \sin px \, dx. \quad \dots (2)$$

By definition of Fourier sine transform, we see that $\bar{F}_s(p)$ is Fourier sine transform of $F(x)$. Hence, using the corresponding inversion formula, we have

$$\begin{aligned}
F(x) &= \frac{2}{\pi} \int_0^\infty \bar{F}_s(p) \sin px \, dp \\
&= \frac{2}{\pi} \left[\int_0^1 \bar{F}_s(p) \sin px \, dp + \int_1^2 \bar{F}_s(p) \sin px \, dp + \int_2^\infty \bar{F}_s(p) \sin px \, dp \right] \\
&= \frac{2}{\pi} \left[\int_0^1 \sin px \, dp + \int_1^2 (2) \times \sin px \, dp + \int_2^\infty (0) \times \sin px \, dp \right], \text{ using (1)} \\
&= \frac{2}{\pi} \left\{ \left[-\frac{\cos px}{x} \right]_0^1 + 2 \left[-\frac{\cos px}{x} \right]_1^2 \right\} = \frac{2}{\pi} \left\{ \frac{-\cos x + 1}{x} + 2 \frac{-\cos 2x + \cos x}{x} \right\} \\
&= (2/\pi x) \times (1 + \cos x - 2 \cos 2x).
\end{aligned}$$

Ex. 3. Solve : $\int_0^\infty F(x) \cos px \, dx = e^{-p}$. [Meerut 2010, 12; Kanpur 2007, 08, 10]

Sol. By definition of the Fourier cosine transform, we see e^{-p} is Fourier cosine transform of $F(x)$. Hence, using the corresponding inversion formula, we have

$$\begin{aligned}
F(x) &= \frac{2}{\pi} \int_0^\infty \bar{F}_c(p) \cos px \, dp = \frac{2}{\pi} \int_0^\infty e^{-p} \cos px \, dp \quad [\because \text{Here } e^{-p} = \bar{F}_c(p) = F\{F(x)\}] \\
&= \frac{2}{\pi} \left[\frac{e^{-p}}{1+x^2} (-\cos px + x \sin px) \right]_0^\infty \quad \text{as} \quad \int e^{ax} \cos bx \, dx = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2}
\end{aligned}$$

Thus,
$$F(x) = 2/\pi (1+x^2).$$

9.8. HILBERT TRANSFORM.

The finite Hilbert transform of a function $y(\phi)$ is defined as

$$f(\theta) = \frac{1}{\pi} \int_0^{*\pi} \frac{\sin \theta}{\cos \theta - \cos \phi} y(\phi) \, d\phi \quad \dots (1)$$

with the inverse
$$y(\theta) = \frac{1}{\pi} \int_0^{*\pi} \frac{\sin \phi}{\cos \phi - \cos \theta} f(\phi) d\phi + \frac{1}{\pi} \int_0^\pi y(\phi) d\phi \quad \dots (2)$$

Deduction of various forms of the Hilbert transform pairs from equations (1) and (2)

First alternative form of Hilbert transform pair

Using the well known principle of mathematical induction, we can easily prove that

$$\int_0^{*\pi} \frac{\cos n \phi d\phi}{\cos \theta - \cos \alpha} = \pi \frac{\sin n \alpha}{\sin \alpha} \quad \dots (3)$$

It is to be noted that for $n = 0$ and $n = 1$, (3) can be easily verified.

Now, (1) and (2) $\Rightarrow f(-\theta) = -f(\theta)$ and $y(-\theta) = y(\theta) \quad \dots (4)$

Let us assume that $F(\theta) = -f(\theta)$ so that $F(-\theta) = -F(\theta) \quad \dots (5)$

$$\frac{1}{\pi} \int_0^\pi y(\theta) d\theta = C, \text{ where } C = \text{constant.} \quad \dots (6)$$

and $Y(\theta) = y(\theta) - C$ so that $Y(-\theta) = y(\theta) - C \quad \dots (7)$

From (7), it follows that

$$\frac{1}{\pi} \int_0^\pi Y(\theta) d\theta = \frac{1}{\pi} \int_0^\pi y(\theta) d\theta - \frac{1}{\pi} \int_0^\pi C d\theta = C - C = 0, \text{ using (6)} \quad \dots (8)$$

Now,
$$\begin{aligned} \frac{1}{\pi} \int_0^{*\pi} \frac{\sin \theta}{\cos \phi - \cos \theta} Y(\phi) d\phi &= \frac{1}{\pi} \int_0^{*\pi} \frac{\sin \theta}{\cos \phi - \cos \theta} (y(\phi) - C) d\phi, \text{ by (7)} \\ &= -\frac{1}{\pi} \int_0^{*\pi} \frac{\sin \theta y(\phi) d\phi}{\cos \theta - \cos \phi} - \frac{C}{\pi} \int_0^{*\pi} \frac{\sin \theta d\phi}{\cos \phi - \cos \theta} = -f(\theta) - \frac{C}{\pi} \int_0^{*\pi} \frac{\sin \theta d\phi}{\cos \phi - \cos \theta}, \text{ by (1)} \\ &= F(\theta), \text{ by above relations} \end{aligned}$$

Thus,
$$F(\theta) = \frac{1}{\pi} \int_0^{*\pi} \frac{\sin \theta}{\cos \phi - \cos \theta} Y(\phi) d\phi \quad \dots (9)$$

Replacing θ by $(\theta - \phi)/2 + (\theta + \phi)/2$ in (9), we get

$$F(\theta) = \frac{1}{2\pi} \int_0^{*\pi} \cot \frac{\theta + \phi}{2} Y(\phi) d\phi + \frac{1}{2\pi} \int_0^{*\pi} \cot \frac{\theta - \phi}{2} Y(\phi) d\phi \quad \dots (10)$$

Replace ϕ by $-\phi$ in the first integral on R.H.S. of (10) and use the relation (7). Then, on combining the resulting integral with the second integral on the R.H.S. of (10), we obtain

$$F(\theta) = \frac{1}{2\pi} \int_{-\pi}^{*\pi} \cot \frac{\theta - \phi}{2} Y(\phi) d\phi. \quad \dots (11)$$

Likewise, now start with (2) in place of (1) and proceed exactly as before, Then, we shall get

$$Y(\theta) = \frac{1}{2\pi} \int_{-\pi}^{*\pi} \cot \frac{\phi - \theta}{2} F(\phi) d\phi + \frac{1}{2\pi} \int_0^{*\pi} \cot \frac{\phi + \theta}{2} F(\phi) d\phi \quad \dots (12)$$

Replace ϕ by $-\phi$ in the second integral on the R.H.S. of (12) and use the relation (5). Then on combining with the first integral on the R.H.S. of (12), we have

$$Y(\theta) = \frac{1}{2\pi} \int_{-\pi}^{*\pi} \cot \frac{\phi - \theta}{2} F(\phi) d\phi \quad \dots (13)$$

The relations (11) and (13) give us the first alternative form of finite Hilbert transform pair.

Second alternative form of Hilbert transform pair

Re-writing (10), we have

$$F(\theta) = \frac{1}{2\pi} \int_{-\pi}^{*\pi} \left\{ 1 + \cot \frac{\phi + \theta}{2} \right\} Y(\phi) d\phi + \frac{1}{2\pi} \int_0^{*\pi} \left\{ 1 + \cot \frac{\theta - \phi}{2} \right\} Y(\phi) d\phi - \frac{1}{\pi} \int_0^{*\pi} Y(\phi) d\phi$$

or
$$F(\theta) = \frac{1}{2\pi} \int_0^{*\pi} \left\{ 1 + \cot \frac{\theta + \phi}{2} \right\} Y(\phi) d\phi + \frac{1}{2\pi} \int_0^{*\pi} \left\{ 1 + \cot \frac{\theta - \phi}{2} \right\} Y(\phi) d\phi, \text{ by (8)}$$

Combining the first integral with the second integral by replacing ϕ by $-\phi$ and using (7), the above equation reduces to

$$F(\theta) = \frac{1}{2\pi} \int_{-\pi}^{*\pi} \left\{ 1 + \cot \frac{\theta - \phi}{2} \right\} Y(\phi) d\phi \quad \dots (14)$$

Starting with (12) and proceeding as before, we get

$$Y(\theta) = \frac{1}{2\pi} \int_{-\pi}^{*\pi} \left\{ 1 + \cot \frac{\phi - \theta}{2} \right\} F(\phi) d\phi \quad \dots (15)$$

It may be noticed that the transform pair (14) – (15) is exactly the reciprocal pair of Hilbert type singular integral equations already discussed in Art. 8.10 of chapter 8 except for a trivial adjustment of the symbols and the range of integration. The relations (14) and (15) give us a second alternative form of finite Hilbert transform pair.

Third alternative form of Hilbert transform pair

This transform pair, which is nonsingular, is deduced from the pair (1) – (2) by introducing new variables u and v as follows :

$$u = \cos \theta, \quad v = \cos \phi. \quad p(u) = \frac{f(\theta)}{\sin \theta} = \frac{f(\cos^{-1} u)}{(1 - u^2)^{1/2}} \quad \text{and} \quad q(u) = \frac{y(\theta)}{\sin \theta} = \frac{y(\cos^{-1} u)}{(1 - u^2)^{1/2}},$$

Then (1) takes the form
$$\frac{f(\theta)}{\sin \theta} = \frac{1}{\pi} \int_0^{*\pi} \frac{1}{\cos \theta - \cos \phi} \frac{y(\phi)}{\sin \phi} \sin \phi d\phi$$

or
$$p(u) = \frac{1}{\pi} \int_{-1}^{*1} \frac{q(v) dv}{u - v}, \quad -1 < u < 1 \quad \dots (16)$$

(6) is also known as *airfoil equation*.

Similarly, (2) takes the form

$$q(u) = \frac{1}{\pi} \int_{-1}^{*1} \left(\frac{1 - v^2}{1 - u^2} \right)^{1/2} \frac{p(v)}{v - u} dv + \frac{C}{(1 - u^2)^{1/2}}, \quad -1 < u < 1 \quad \dots (17)$$

where
$$C = \frac{1}{\pi} \int_{-1}^1 q(v) dv, \quad \text{which is an arbitrary constant.}$$

It may be noted that the above pair (16) – (17) is a particular case of the pair of integral equations (24) – (25) of Art. 8.8 of chapter 8. The relations (16) and (17) give us a third alternative form of finite Hilbert transform pair.

9.9. INFINITE HILBERT TRANSFORM. DEFINITION

The infinite Hilbert transform is defined as

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{*\infty} \frac{y(t)}{t - x} dt$$

and its inverse is defined by

$$y(x) = -\frac{1}{\pi} \int_{-\infty}^{*\infty} \frac{f(t)}{t - x} dt$$

Example 1. Solved the homogeneous integral equation $\int_{-1}^{*1} \frac{q(v) dv}{u-v} = 0$

Sol. Using the third alternative form of Hilbert transform pair (16) - (17), we see that here $p(u) = 0$ and hence the required solution from (17) is given by $q(u) = C/(1-u^2)^{1/2}$.

EXERCISE

1. Solve $Y(t) = 1 - \int_0^t (t-x) Y(x) dx$. (Meerut 2012) Ans. $Y(x) = \cos x$

2. Solve the integral equation $F(t) = t \int_t^\infty \frac{Y'(x)}{(x^2 - t^2)^{1/2}} dx$

3. Show that the solution of the integral equation

$$F(t) = 2 \int_t^1 \frac{x Y(x) dx}{(x^2 - t^2)^{1/2}} \quad \text{is} \quad Y(t) = -\frac{1}{\pi t} \frac{d}{dt} \int_t^1 \frac{x F(x) dx}{(x^2 - t^2)^{1/2}}$$

Find the solution for the following two special cases : (i) $F(t) = 2t^2/(1-t^2)^{1/2}$ (ii) $F(t) = t^2$

4. Solve the Abel integral equation of the second kind $Y(t) = t^{-1/2} e^{-a/4t} + \frac{i}{\sqrt{\pi}} \int_0^t \frac{Y(x) dx}{(t-x)^{1/2}}$

5. Solve the integral equation $Y(t) = F(t) + \lambda \int_0^t J_1(t-x) Y(x) dx$

6. Find the resolvent kernel of the integral equation $Y(t) = F(t) + \int_0^t (t^2 - x^2) Y(x) dx$

Hint : Use the method of Art. 9.4.

7. Use the infinite Hilbert transform pair and solve the integral equation

$$\frac{1}{1+x^2} = \int_{-\infty}^{*\infty} \frac{Y(t) dt}{x-t}$$

8. With the help of finite Hilbert transform, solve

$$x^2 = \int_{-l}^l \frac{2tY(t)}{x^2 - t^2} dx, \text{ assuming that } Y(t) = -Y(-t)$$

9. With the help of finite Hilbert transform, solve

$$ax + b + \sigma_l (\log |l-x|) - \sigma_0 \log |x| = \int_0^l \frac{Y'(t) dt}{t-x}$$

subject to the conditions : $Y'(0) = \sigma_0$, $Y'(l) = \sigma_l$, $Y(0) = Y_0$, $Y(l) = Y_l$

10. Solve the integral equation $\sin x = \frac{1}{\pi} \int_{-\infty}^{*\infty} \frac{Y(t) dt}{t-x}$

[Hint : Let us consider the following integral :

$$\begin{aligned} \int_{-\infty}^{*\infty} \frac{e^{it} dt}{x-t} &= \pi i \sum (\text{residues of the poles on the } t\text{-axis}) \\ \Rightarrow \int_{-\infty}^{*\infty} \frac{(\cos t + i \sin t) dt}{x-t} &= \pi i (\cos x + i \sin x) \end{aligned}$$

Separating the real parts, we have
$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos t}{x-t} dt = -\sin x \quad \dots (i)$$

Re-writing the given integral equation,
$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{Y(t) dt}{x-t} = -\sin x \quad \dots (ii)$$

Comparing (i) and (ii), the required solution is given by $Y(t) = \cos t$

11. Solve the integral equation $\sum_{r=0}^{\infty} a_r x^r = \frac{1}{\pi} \int_0^{*l} \frac{Y(t) dt}{t-x}$, where a_r are given constants.

Hint. Use the substitutions : $x = (l/2) \times (1 - \cos \theta)$, $t = (l/2) \times (1 - \cos \phi)$

9.10. *MELLIN TRANSFORM. DEFINITION.

Given a function $Y(x)$, defined for $0 < x < \infty$, the Mellin transform of $Y(x)$ is a function of new variable p and is denoted and defined by

$$M[Y(x); p] = y(p) = \int_0^{\infty} x^{p-1} Y(x) dx$$

Clearly, if C is a constant, then

$$M[C Y(x); p] = C M[Y(x); p]$$

Inverse Mellin transform. If $y(p)$ be the Mellin transform of $Y(x)$, then $Y(x)$ is called the inverse Mellin transform of $y(p)$ and then we write $M^{-1}\{y(p)\} = Y(x)$.

Convolution theorem for Mellin transform. Let $k(p)$ and $y(p)$ be the Mellin transforms of $K(x)$ and $Y(x)$ respectively. Then

$$M\left\{\int_0^{\infty} K(xt) Y(t) dt\right\} = k(p) y(1-p). \quad \dots (1)$$

Proof L.H.S. of (1)

$$\begin{aligned} &= \int_0^{\infty} x^{p-1} \left\{ \int_0^{\infty} K(xt) Y(t) dt \right\} dx, \text{ by definition of Mellin transform} \\ &= \int_0^{\infty} \left(\frac{u}{t}\right)^{p-1} \left\{ \int_0^{\infty} K(u) Y(t) dt \right\} \frac{du}{t} \quad [\text{on taking new variable } u \text{ such that} \\ &\quad x t = u \text{ so that } du = t dx \text{ and so } x = u/t \text{ and } dx = (1/t)du] \\ &= \left(\int_0^{\infty} u^{p-1} K(u) du \right) \times \left(\int_0^{\infty} t^{(1-p)-1} Y(t) dt \right) \\ &= k(p) y(1-p), \text{ by definition of Mellin transform} \\ &= \text{R.H.S. of (1)} \end{aligned}$$

Hence the result

9.11 SOLUTION OF FOX'S INTEGRAL EQUATION, NAMELY

$$Y(x) = F(x) + \int_0^{\infty} K(xt) Y(t) dt, \quad 0 < x < \infty. \quad \dots (1)$$

containing a special type of kernel $K(xt)$.

Let $y(p)$, $f(p)$ and $k(p)$ be Mellin transforms of $Y(x)$, $F(x)$ and $K(x)$ respectively. Again, from result (1) of Art. 9.10, we have

$$M\left\{\int_0^{\infty} K(xt) Y(t) dt\right\} = k(p) y(1-p) \quad \dots (2)$$

* For details, please refer Author's Integral transform, published by S.Chand & Co., New Delhi

Taking the Mellin transform of both sides of (1) and using (2), we have

$$y(p) = f(p) + k(p) y(1-p) \quad \dots (3)$$

Replacing p by $1-p$ in (3), we have

$$y(1-p) = f(1-p) + k(1-p) y(p) \quad \dots (4)$$

From (3),

$$y(1-p) = \{y(p) - f(p)\} / k(p) \quad \dots (5)$$

Substituting the value of $y(1-p)$ given by (5) in (4), we get

$$\{y(p) - f(p)\} / k(p) = f(1-p) + k(1-p) y(p)$$

$$\text{or } y(p) \left\{ \frac{1}{k(p)} - k(1-p) \right\} = f(1-p) + \frac{f(p)}{k(p)}$$

$$\text{so that } y(p) = \frac{f(p) + k(p) f(1-p)}{1 - k(p) k(1-p)} \quad \dots (6)$$

from which we will get the required solution of (1) provided we can compute $Y(x)$ from its Mellin transform $y(p)$.

Solved example. Solve the Fox's integral equation

$$Y(x) = F(x) + \int_0^\infty K(xt) Y(t) dt, \quad 0 < x < \infty \quad \dots (1)$$

with the kernel

$$K(x) = \lambda(2/\pi)^{1/2} \sin x \quad (\text{Kanpur 2005}) \quad \dots (2)$$

Solution. Let $y(p)$, $f(p)$ and $k(p)$ be Mellin transform of $Y(x)$, $F(x)$ and $K(x)$ respectively. Then, solution of (1) can be obtained from the equation (refer eq. (equation) of Art. 9.11)

$$y(p) = \frac{f(p) + k(p) f(1-p)}{1 - k(p) k(1-p)} \quad \dots (3)$$

By definition,

$$k(p) = \int_0^\infty x^{p-1} K(x) dx = \lambda \sqrt{\frac{2}{\pi}} \int_0^\infty x^{p-1} \sin x dx, \text{ by (2)}$$

$$\text{or } k(p) = \lambda \sqrt{\frac{2}{\pi}} \Gamma(p) \sin \frac{\pi p}{2} \quad \dots (4)$$

$$^* [\cdot \cdot M[\sin x] = \int_0^\infty x^{p-1} \sin x dx = \Gamma(p) \sin(\pi p/2)]$$

$$\text{From (4) } k(1-p) = \lambda \sqrt{\frac{2}{\pi}} \Gamma(1-p) \sin \frac{\pi(1-p)}{2} = \lambda \sqrt{\frac{2}{\pi}} \Gamma(1-p) \sin \left(\frac{\pi}{2} - \frac{\pi p}{2} \right)$$

$$\text{so that } k(1-p) = \lambda \sqrt{\frac{2}{\pi}} \Gamma(1-p) \cos \frac{\pi p}{2} \quad \dots (5)$$

$$\begin{aligned} \therefore k(p) k(1-p) &= \frac{2\lambda^2}{\pi} \Gamma(p) \Gamma(1-p) \sin \frac{\pi p}{2} \cos \frac{\pi p}{2}, \text{ by (4) and (5)} \\ &= \frac{\lambda^2}{\pi} \times \frac{\pi}{\sin p \pi} \times \sin \pi p, \text{ as } \Gamma(p) \Gamma(1-p) = \frac{\pi}{\sin p \pi} \end{aligned}$$

$$\text{Thus, } k(p) k(1-p) = \lambda^2 \quad \dots (6)$$

Using (4) and (6), (3) reduces to

$$y(p) = \frac{f(p)}{1-\lambda^2} + \frac{\lambda}{1-\lambda^2} \sqrt{\frac{2}{\pi}} \Gamma(p) \sin \frac{\pi p}{2} f(1-p) \quad \dots (7)$$

Taking inverse Mellin transform of both sides of (7), we have,

$$Y(x) = \frac{F(x)}{1-\lambda^2} + \frac{\lambda}{1-\lambda^2} \sqrt{\frac{2}{\pi}} M^{-1} \left\{ \Gamma(p) \sin \frac{\pi p}{2} f(1-p) \right\} \quad \dots (8)$$

We can easily verify that

$$\begin{aligned} M \left[\int_0^\infty \sin(xt) F(t) dt \right] &= \Gamma(p) \sin \frac{\pi p}{2} f(1-p) \\ \Rightarrow M^{-1} \left\{ \Gamma(p) \sin \frac{\pi p}{2} f(1-p) \right\} &= \int_0^\infty \sin(xt) F(t) dt \quad \dots (9) \end{aligned}$$

Using (9), (8) reduces to $Y(x) = \frac{F(x)}{1-\lambda^2} + \frac{\lambda}{1-\lambda^2} \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(xt) F(t) dt, 0 < x < \infty$

which is the required solution of the given equation (1).

Miscellaneous problems on Chapter 9

1. Solve the integral equation by using Laplace transform $\phi(x) = 1 - \int_0^x (x-t)\phi(t) dt$.
(Kanpur 2009)
2. Solve the Abel's equation $\int_0^x \frac{\phi(t)}{\sqrt{x-t}} dt = 1+x$.
(Kanpur 2009,10)

CHAPTER 10

Self Adjoint Operator, Dirac Delta Function and Spherical Harmonics

10.1 INTRODUCTION

In this chapter we propose to introduce some important concepts which will be used in the subsequent chapters. We shall often refer to the definitions and associated results of self adjoint operator, Dirac delta function and spherical harmonics.

10.2. ADJOINT EQUATION OF SECOND ORDER LINEAR DIFFERENTIAL EQUATION

Consider the second order homogeneous linear differential equation

$$a_0(x) (d^2y/dx^2) + a_1(x) (dy/dx) + a_2(x) y = 0, \quad \dots (1)$$

where $a_0(x)$ has a continuous second order derivative, $a_1(x)$ has a continuous first order derivative, $a_2(x)$ is continuous and $a_0(x) > 0$ on $a \leq x \leq b$.

Let L be a differential operator defined by

$$L \equiv a_0(x) (d^2 / dx^2) + a_1(x) (d / dx) + a_2(x) \quad \dots (2)$$

$$\text{Then (1) can be re-written as} \quad L y(x) = 0, \quad a \leq x \leq b \quad \dots (3)$$

and the adjoint operator M of L is defined as

$$M y(x) \equiv \frac{d^2}{dx^2} \{a_0(x) y(x)\} - \frac{d}{dx} \{a_1(x) y(x)\} + a_2(x) y(x) \quad \dots (4)$$

$$\text{Also, } M y(x) = 0, \text{ i.e., } \frac{d^2}{dx^2} \{a_0(x) y(x)\} - \frac{d}{dx} \{a_1(x) y(x)\} + a_2(x) y(x) = 0 \quad \dots (5)$$

is known as a *adjoint of (1)*.

10.3. SELF ADJOINT EQUATION.

If the adjoint of any linear homogeneous equation is identical with the equation itself, then the given equation is known as *self adjoint equation*.

Theorem I. *The necessary and sufficient condition that the second order homogeneous linear differential equation $a_0(x) (d^2y/dx^2) + a_1(x) (dy/dx) + a_2(x) y = 0$, where $a_0(x)$ is continuously differentiable positive function and $a_1(x)$ has a continuous first order derivative on $a \leq x \leq b$ to be self adjoint is that $a_0'(x) = a_1(x)$ on $a \leq x \leq b$, where prime denotes differentiation w.r.t. 'x'.*

Proof. By definition, the adjoint equation of

$$a_0(x) (d^2y/dx^2) + a_1(x) (dy/dx) + a_2(x) y = 0 \quad \dots (1)$$

is
$$\frac{d^2 y}{dx^2} [a_0(x) y] - \frac{d}{dx} [a_1(x) y] + a_2(x) y = 0 \quad \dots (2a)$$

i.e.,
$$a_0(x) (d^2 y / dx) + \{2a_0'(x) - a_1(x)\} (dy / dx) + \{a_0''(x) - a_1'(x) + a_2(x)\} y = 0, \quad \dots (2b)$$

where prime denotes differentiation with respect to x .

The condition is necessary. Let (1) be a self adjoint equation. Then (2b) must be identical with (1) and hence we must have
$$2a_0'(x) - a_1(x) = a_1(x) \quad \dots (3)$$

and
$$a_0''(x) - a_1'(x) + a_2(x) = a_2(x) \quad \dots (4)$$

From (4), $a_0''(x) = a_1'(x)$ so that $a_0'(x) = a_1(x) + C$, where C is a constant. $\dots (5)$

Substituting the value of $a_0'(x)$ as given by (5) in (3), we have

$$2 \{a_1(x) + C\} - a_1(x) = a_1(x) \quad \text{so that} \quad C = 0.$$

Hence (5) yields $a_0'(x) = a_1(x)$ Hence the condition is necessary.

The condition is sufficient. Suppose that for (1), we have $a_0'(x) = a_1(x) \quad \dots (6)$

Then,
$$2a_0'(x) - a_1(x) = 2a_1(x) - a_1(x) = a_1(x), \text{ by (6)} \quad \dots (7)$$

and
$$a_0''(x) - a_1'(x) + a_2(x) = a_1'(x) - a_1'(x) + a_2(x) = a_2(x), \text{ by (6)} \quad \dots (8)$$

Using (7) and (8), (2b) reduces to

$$a_0(x) (d^2 y / dx^2) + a_1(x) (dy / dx) + a_2(x) y = 0$$

which is identical with (1). Hence the condition is sufficient.

Corollary. If $a_0(x) (d^2 y / dx^2) + a_1(x) (dy / dx) + a_2(x) y = 0$ is self adjoint, then it can be

re-written as
$$\frac{d}{dx} \left\{ a_0(x) \frac{dy}{dx} \right\} + a_2(x) y = 0.$$

Proof. Given that
$$a_0(x) (d^2 y / dx^2) + a_1(x) (dy / dx) + a_2(x) y = 0 \quad \dots (1)$$

is self adjoint equation. For this, the necessary condition is
$$a_0'(x) = a_1(x). \quad \dots (2)$$

Substituting the value of $a_1(x)$ from (2) in (1), we obtain

$$a_0(x) \frac{d^2 y}{dx^2} + a_0'(x) \frac{dy}{dx} + a_2(x) y = 0, \quad \text{i.e.,} \quad \frac{d}{dx} \left\{ a_0(x) \frac{dy}{dx} \right\} + a_2(x) y = 0$$

Theorem II. If the coefficients $a_0(x)$, $a_1(x)$, $a_2(x)$ in the equation $a_0(x) (d^2 y / dx^2) + a_1(x) (dy / dx) + a_2(x) y = 0$ are continuous on $a \leq x \leq b$ and $a_0(x) \neq 0$, then it can be trans-

formed into the equivalent self adjoint equation
$$\frac{d}{dx} \left\{ p(x) \frac{dy}{dx} \right\} + q(x) y = 0,$$

where
$$p(x) = \exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\} \quad \text{and} \quad q(x) = \frac{a_2(x)}{a_1(x)} \exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\}$$

Here $\exp(a)$ stands for e^a

Proof. Given
$$a_0(x) (d^2 y / dx^2) + a_1(x) (dy / dx) + a_2(x) y = 0 \quad \dots (1)$$

Multiplying both sides of (1) by $\frac{1}{a_0(x)} \exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\}$, we get

$$\exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\} \frac{d^2 y}{dx^2} + \frac{a_1(x)}{a_0(x)} \exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\} \frac{dy}{dx} + \frac{a_2(x)}{a_0(x)} \exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\} y = 0 \quad \dots (2)$$

Comparing (2) with $A_0(x)(d^2y/dx^2) + A_1(x)(dy/dx) + A_2(x)y = 0$, we have ... (3)

$$A_0(x) = \exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\} \quad \text{and} \quad A_1(x) = \frac{a_1(x)}{a_0(x)} \exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\}$$

Then, we have $A'_0(x) = A_1(x)$,

which is the condition for (3) to be a self adjoint equation. Hence (2) is a self adjoint equation and it can be re-written as

$$\frac{d}{dx} \left[\exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\} \frac{dy}{dx} \right] + \frac{a_2(x)}{a_0(x)} \exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\} y = 0 \quad \text{or} \quad \frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + q(x) y = 0,$$

where $p(x) = \exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\}$ and $q(x) = \frac{a_2(x)}{a_0(x)} \exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\}$

Theorem III. If M is the adjoint operator of operator L , then L is the adjoint of M , i.e., the adjoint of the adjoint operator is the operator itself.

Proof. Left as an exercise for the reader.

10.4 SOLVED EXAMPLES BASED ON ART. 10.2 AND 10.3

Ex. 1. Find the adjoint equation of $x^2(d^2y/dx^2) + (2x^3 + 1)(dy/dx) + y = 0$.

Sol. Given $x^2(d^2y/dx^2) + (2x^3 + 1)(dy/dx) + y = 0$... (1)

Comparing (1) with $a_0(x)(d^2y/dx^2) + a_1(x)(dy/dx) + a_2(x)y = 0$,

we have $a_0(x) = x^2$, $a_1(x) = 2x^3 + 1$ and $a_2(x) = 1$... (2)

Now, the required adjoint equation is of the form

$$\frac{d^2}{dx^2} [a_0(x)y] - \frac{d}{dx} [a_1(x)y] + a_2(x)y = 0 \quad \text{or} \quad \frac{d^2}{dx^2} (x^2y) - \frac{d}{dx} \{ (2x^3 + 1)y \} + y = 0$$

$$\text{or} \quad \frac{d}{dx} \left\{ \frac{d}{dx} (x^2y) \right\} - \left\{ 6xy + (2x^3 + 1) \frac{dy}{dx} \right\} + y = 0 \quad \text{or} \quad \frac{d}{dx} \left(2xy + x^2 \frac{dy}{dx} \right) - 6x^2y - (2x^3 + 1) \frac{dy}{dx} + y = 0$$

$$\text{or} \quad 2y + 2x(dy/dx) + 2x(dy/dx) + x^2(d^2y/dx^2) - 6x^2y - (2x^3 + 1)(dy/dx) + y = 0$$

$$\text{or} \quad x^2(d^2y/dx^2) + (4x - 2x^3 - 1)(dy/dx) + 3y(1 - 2x^2) = 0$$

Ex. 2. Show that $x^2(d^2y/dx^2) - 2x(dy/dx) + 2y = 0$ is not a self adjoint equation. Transform it into an equivalent self adjoint equation.

Sol. Given $x^2(d^2y/dx^2) - 2x(dy/dx) + 2y = 0$... (1)

Comparing (1) with $a_0(x)(d^2y/dx^2) + a_1(x)(dy/dx) + a_2(x)y = 0$, ... (2)

here $a_0(x) = x^2$, $a_1(x) = -2x$ and $a_2(x) = 2$... (3)

Since $a'_0(x) = 2x \neq a_1(x)$, it follows that (1) is not a self adjoint equation.

We know that (2) can be transformed into an equivalent self adjoint equation by multiplying its both sides by a factor

$$\frac{1}{a_0(x)} \exp \left\{ \int \frac{a_1(x)}{a_0(x)} dx \right\} \quad \text{i.e.,} \quad \frac{1}{x^2} \exp \left\{ \int \left(\frac{-2x}{x^2} \right) dx \right\}, \quad \text{i.e.,} \quad \frac{1}{x^2} \exp (-2 \log x),$$

$$\text{i.e., } (1/x^2) \times e^{\log x^{-2}} \quad \text{i.e.,} \quad (1/x^2) \times x^{-2}, \quad \text{i.e., } 1/x^4$$

Now, multiplying both sides of (1) by $1/x^4$, we get

$$x^{-2}(d^2y/dx^2) - 2x^{-3}(dy/dx) + 2x^{-4}y = 0 \quad \text{... (4)}$$

Comparing (4) with $A_0(x)(d^2y/dx^2) + A_1(x)(dy/dx) + A_2(x)y = 0$,

we have $A_0(x) = x^{-2}$ and $A_1(x) = -2x^{-3}$ so that $A'_0(x) = -2x^{-3} = A_1(x)$.

Hence (4) is the required self adjoint equation.

EXERCISE

1. Find the adjoint equation of $x^2 y'' + 7xy' + 8y = 0$, where $y' = dy/dx$
2. Show that $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ is self adjoint.
3. Transform $y'' - (\tan x)y' + y = 0$ into an equivalent self adjoint equation.
4. Show that each of the following equations are self adjoint.

$$(i) \sin x \frac{d^2 y}{dx^2} + \cos x \frac{dy}{dx} + 2x = 0$$

$$(ii) \frac{x+1}{x} \frac{d^2 y}{dx^2} - \frac{1}{x^2} \frac{dy}{dx} + \frac{1}{x^3} y = 0$$

Answers 1. $x^2 y'' - 3x y' + 3y = 0$ 3. $(\cos x) y'' - (\sin x) y' + (\cos x) y = 0$

10.5 GREEN'S FORMULA.

We know that if L is the differential operator

$$L u(x) = [a_0(x)(d^2/dx^2) + a_1(x)(d/dx) + a_2(x)] u(x), \quad a < x < b \quad \dots (1)$$

where $A(x)$ is continuously differentiable, positive function, then its adjoint operator M is defined as

$$M v(x) = \frac{d^2}{dx^2} \{a_0(x)v(x)\} - \frac{d}{dx} \{a_1(x)v(x)\} + a_2(x)v(x), \quad a < x < b \quad \dots (2)$$

From (1) and (2), we have

$$\begin{aligned} v Lu - u Mv &= v[a_0 \frac{d^2 u}{dx^2} + a_1 \frac{du}{dx} + a_2 u] - u[\frac{d^2}{dx^2} (a_0 v) - \frac{d}{dx} (a_1 v) + a_2 v] \\ &= v a_0 \frac{d^2 u}{dx^2} + v a_1 \frac{du}{dx} - u \frac{d}{dx} \{ \frac{d}{dx} (a_0 v) \} + u \frac{d}{dx} (a_1 v) \\ &= v a_0 u'' + v a_1 u' - u \frac{d}{dx} (a_0' v + a_0 v') + u (a_1' v + a_1 v') \\ &= v a_0 u'' + v a_1 u' - u (a_0'' v + 2a_0' v' + a_0 v'') + u (a_1' v + a_1 v') \\ &= a_0' (v u' - u v') + a_0 (v u'' - u v'') + u v (a_1' - a_0'') + (a_1 - a_0') (u v' + u' v) \\ &= \frac{d}{dx} \{ a_0 (v u' - u v') \} + \frac{d}{dx} \{ u v (a_1 - a_0') \} \end{aligned}$$

$$\text{Thus, } \int_a^b (v Lu - u Mv) dx = \int_a^b \frac{d}{dx} \{ a_0 (v u' - u v') + u v (a_1 - a_0') \} dx$$

$$\text{or } \int_a^b (v Lu - u Mv) dx = [a_0 (v u' - u v') + u v (a_1 - a_0')]_a^b \quad \dots (3)$$

(3) is called *Green's formula* for the operator L .

An important particular case : Let L be a self adjoint operator defined by

$$L = \frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x) = p(x) \frac{d^2}{dx^2} + \frac{dp}{dx} \frac{d}{dx} + q(x) \quad \dots (4)$$

Let u and v be two solutions of $Ly(x) = 0$, Then since $M = L$ for self adjoint operator, we have $Lv(x) = Mv(x)$. Also, here $a_0(x) = p(x)$ and $a_1(x) = p'(x)$. Hence the Green's formula (3) for self adjoint operator given by (4) takes the following form

$$\int_a^b (v Lu - u Lv) dx = [p(x) (vu' - v'u)]_a^b, \quad \text{as } a_1 - a'_0 = p'(x) - p'(x) = 0 \quad \dots (5)$$

$$\text{Thus, } \int_a^b v Lu dx = \int_a^b u Lv dx + [p(x) (vu' - v'u)]_a^b \quad \dots (6)$$

which is known as *Green's formula* for the operator L given by (4).

10.6 THE DIRAC DELTA FUNCTION.

The Dirac delta function is of general interest, and is frequently encountered in highly abstract mathematics, in theoretical physics, and in the description of concentrated forces in solid and fluid mechanics, and in practical subjects such as electrical engineering. The Dirac delta function corresponds to the physical ideal of something concentrated into a point in space or an instant in time; for example Dirac delta function can be used to represent a point mass or a point charge, an impulse given to a dynamical system or a pulse produced by a sudden electrical discharge in some circuit. The Dirac delta function and its derivative play an important role in the solution of initial and boundary value problems.

Dirac Delta function Definition

(Kanpur 2010, 11)

Consider the function having the following property :

$$\delta_\epsilon(x) = \begin{cases} 1/2\epsilon, & |x| < \epsilon \\ 0, & |x| > \epsilon \end{cases} \quad \dots (1)$$

$$\text{Then, } \int_{-\infty}^{\infty} \delta_\epsilon(x) dx = \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon} dx = 1 \quad \dots (2)$$

The limit of $\delta_\epsilon(x)$ as $\epsilon \rightarrow 0$ is denoted by $\delta(x)$, that is,

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \delta_\epsilon(x) \quad \dots (3)$$

Then $\delta(x)$ is known as the *Dirac delta function*.

With help of (1), (2) and (3), the Dirac delta functions is defined as follows :

$$\delta(x) = \begin{cases} \infty, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases} \quad \dots (4)$$

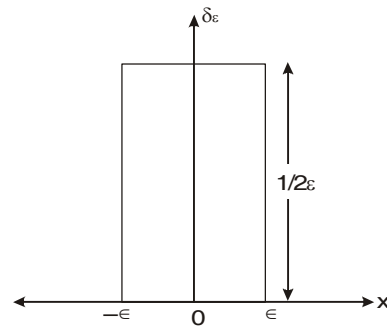
and

$$\int_{-\infty}^{\infty} \delta(x) dx = 1 \quad \dots (5)$$

The adjoining figure shows the profile of $\delta(x)$. We generally call $\delta(x)$ as Dirac delta function for historical reasons while it is not a function in the usual mathematical sense, which requires a function to have a definite value (or values) at each point of a certain domain. For this reason Dirac has called the delta function as “*improper function*” and has emphasised that it may be used in mathematical analysis only when no inconsistency can possibly arise from its application. For application in initial and boundary value problems, we now proceed to derive the formal properties of the Dirac delta function. It should be clearly noted, however, that these properties are purely formal.

From the definition of the Dirac delta function, we have

$$\delta(x-a) = \begin{cases} \infty, & \text{if } x = a \\ 0, & \text{if } x \neq a \end{cases} \quad \dots (6)$$



and

$$\int_{-\infty}^{\infty} \delta(x-a) dx = 1 \quad \dots (7)$$

We also note that
$$\int_a^b \delta(x-c) dx = \begin{cases} 1, & \text{if } c \in (a, b) \\ 0, & \text{if } c \notin (a, b) \end{cases} \quad \dots (8)$$

10.7. SHIFTING PROPERTY OF DIRAC DELTA FUNCTION.

To show that $\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)$, where $f(x)$ is a continuous and bounded function.

[Himachal Pradesh, 2008]

Proof We have
$$\delta_{\varepsilon}(x) = \begin{cases} 1/2\varepsilon, & \text{if } |x| < \varepsilon \\ 0, & \text{if } |x| > \varepsilon \end{cases} \quad \dots (1)$$

Here
$$\int_{-\infty}^{\infty} f(x) \delta_{\varepsilon}(x) dx = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon} f(x) dx, \text{ using (1)}$$

$$= (1/2\varepsilon) \times \{\varepsilon - (-\varepsilon)\} f(\xi), \text{ where } -\varepsilon < \xi < \varepsilon$$

[using mean value theorem]

Clearly, when $\varepsilon \rightarrow 0$, we have $\xi \rightarrow 0$. Hence, taking limits of both sides of the above equation as $\varepsilon \rightarrow 0$ and $\xi \rightarrow 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} f(x) \delta_{\varepsilon}(x) dx = \lim_{\xi \rightarrow 0} f(\xi) \quad \text{or} \quad \int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0). \quad \dots (2)$$

With a simple change of variable, (2) transforms to

$$\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a), \quad \dots (3)$$

showing that the operation of multiplying $f(x)$ by $\delta(x-a)$ and integrating over all x is merely equivalent to substituting a for x in the original function. Symbolically we may write

$$f(x) \delta(x-a) = f(a) \delta(x-a) \quad \dots (4)$$

While using relation (4), we must clearly understand that (4) has a meaning only in the sense that its two sides give equivalent results when used as factors in an integrand. As a particular case of (4) with $a = 0$ and $f(x) = x$, we have
$$x \delta(x) = 0 \quad \dots (5)$$

In a similar way we can prove the relations
$$\delta(-x) = \delta(x) \quad \dots (6)$$

and
$$\delta(ax) = (1/a) \times \delta(x), \quad a > 0 \quad \dots (7)$$

We now prove that
$$\int_{-\infty}^{\infty} \delta(x-y) \delta(y-a) dy = \delta(x-a). \quad \dots (8)$$

Proof. We have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} \delta(x-y) \delta(y-a) dy \right\} dx &= \int_{-\infty}^{\infty} \delta(y-a) \left\{ \int_{-\infty}^{\infty} f(x) \delta(x-y) dx \right\} dy \\ &\quad \text{[Changing the order of integration]} \\ &= \int_{-\infty}^{\infty} \delta(y-a) f(y) dy, \text{ using shifting property (3)} \end{aligned}$$

Thus,
$$\int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} \delta(x-y) \delta(y-a) dy \right\} dx = \int_{-\infty}^{\infty} f(x) \delta(x-a) dx,$$

which formally shows that
$$\int_{-\infty}^{\infty} \delta(x-y) \delta(y-a) dy = \delta(x-a)$$

Remark It is worthwhile to note that the range of integration in relation (3) is not necessary to be $-\infty$ to ∞ . It can be over any domain (a, b) surrounding the point at which $\delta(x)$ is not zero. Clearly limits in these integration need not be mentioned and one may understand that the domain of integration is a subtle one.

10.8.DERIVATIVES OF DIRAC DELTA FUNCTION

We now propose to examine the interpretation that has to be kept in mind while dealing with the “derivatives” of $\delta(x)$. Suppose that $\delta'(x)$ exists and that both it and $\delta(x)$ can be treated as ordinary functions in the rule for integration by parts. Then, we have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \delta'(x) dx &= [f(x) \delta(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \delta(x) dx \\ &= 0 - f'(0), \text{ using shifting theorem and definition of } \delta(x) \end{aligned}$$

$$\text{Thus,} \quad \int_{-\infty}^{\infty} f(x) \delta'(x) dx = -f'(0) \quad \dots (1)$$

$$\text{Repeating this process, we obtain} \quad \int_{-\infty}^{\infty} f(x) \delta^{(n)}(x) dx = (-1)^n f^{(n)}(0) \quad \dots (2)$$

$$\text{Also, we have} \quad \int_{-\infty}^{\infty} f(x) \delta'(x-a) dx = -f'(a) \quad \dots (3)$$

$$\text{and} \quad x \delta'(x) = -\delta(x) \quad \dots (4)$$

10.9.RELATION BETWEEN DIRAC DELTA FUNCTION AND THE HEAVISIDE UNIT FUNCTION.

By definition, the Heaviside unit function $H(x-a)$ is given by

$$H(x-a) = \begin{cases} 0, & \text{if } x < a \\ 1, & \text{if } x > a \end{cases} \quad \dots (1)$$

$$\begin{aligned} \text{Now,} \quad \int_{-\infty}^{\infty} U'(x-a) f(x) dx &= [U(x-a) f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} U(x-a) f'(x) dx, \text{ integrating by parts} \\ &= f(\infty) - \left[\int_{-\infty}^a U(x-a) f'(x) dx + \int_a^{\infty} U(x-a) f'(x) dx \right], \text{ using definition (1)} \\ &= f(\infty) - \left[0 + \int_a^{\infty} f'(x) dx \right], \text{ using definition (1) again} \\ &= f(\infty) - [f(\infty) - f(a)] = f(a), \quad \dots (2) \end{aligned}$$

where we have assumed that $f(x)$ is both continuous and bounded.

$$\text{Again, we have} \quad \int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a) \quad \dots (3)$$

$$\text{From (2) and (3),} \quad \int_{-\infty}^{\infty} U'(x-a) f(x) dx = \int_{-\infty}^{\infty} \delta(x-a) f(x) dx$$

$$\text{from which we formally obtain the required relation} \quad U'(x-a) = \delta(x-a) \quad \dots (4)$$

$$\text{which with a simple change of variable transforms to} \quad U'(x) = \delta(x) \quad \dots (5)$$

10.10 ALTERNATIVE FORMS OF REPRESENTING DIRAC DELTA FUNCTION $\delta(x)$.

Sometimes it is useful to employ an explicit expression for $\delta(x)$ as the limit of a sequence of analytic function, for example,

$$\delta(x) = \lim_{L \rightarrow \infty} \frac{\sin x L}{\pi x} \quad \dots (1)$$

Then, at $x = 0$, the limiting value of $(\sin x L) / \pi x$ is equal to L / π and its value oscillates with period $2\pi / L$ when x increases. Its integrand, taken from $-\infty$ to ∞ is unity and is independent of the value of L . Clearly, $\lim_{L \rightarrow \infty} \{(\sin x L) / \pi x\}$ has all the properties of Dirac delta function defined by the equation (3) of Art. 10.6

An integral representation of the Dirac delta function is $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} du$... (2)

We now proceed to prove (2). Indeed, we have

$$\lim_{L \rightarrow \infty} \int_{-L}^L e^{iux} du = \lim_{L \rightarrow \infty} \frac{e^{ixL} - e^{-ixL}}{i x} = 2\pi \lim_{L \rightarrow \infty} \frac{\sin Lx}{\pi x} = 2\pi \delta(x), \text{ using (1)}$$

$$\therefore \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} du \quad \text{or} \quad \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\cos ux + i \sin ux) du \quad \dots (3)$$

$$\text{Separating the real parts, (3) gives} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos ux du = \delta(x) \quad \dots (4)$$

(4) is most commonly used as an explicit expression for $\delta(x)$.

Again, $\delta(x)$ is given by a limit of the functions

$$f_n(x) = \begin{cases} n, & 0 < |x| < 1/n \\ 0, & \text{for all other } x \end{cases}, \text{ where } n = 1, 2, 3, 4 \dots \quad \dots (5)$$

10.11 SPHERICAL HARMONICS

In terms of spherical coordinates (refer figure),

Laplace's equation $\nabla^2 u = 0$ takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0,$$

which can be re-written as

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0. \quad \dots (1)$$

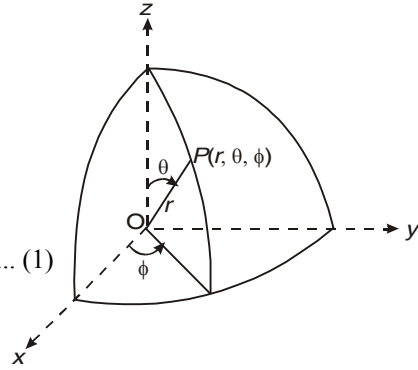
Let a solution of (1) be of the form

$$u(r, \theta, \phi) = R(r) \Psi(\theta, \phi), \quad \dots (2)$$

where R is function r only and Ψ is function of θ and ϕ only. Substituting the value of u as given by (2) in (1), we obtain

$$\frac{\partial}{\partial r} \left(r^2 \Psi \frac{\partial R}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta R \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \left(R \frac{\partial^2 \Psi}{\partial \phi^2} \right) = 0$$

$$\text{or} \quad \frac{1}{R} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right) = - \frac{1}{\Psi \sin \theta} \left[\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 \Psi}{\partial \phi^2} \right] \quad \dots (3)$$



Now the left-hand side of (3) is a function only of the variable r , while the right-hand side is a function only of the variables θ and ϕ . Since these variables are independent, it follows that left hand side and right hand side must be a constant which we shall denote by λ . We thus, obtain

$$-\frac{1}{\Psi} \frac{1}{\sin \theta} \left\{ \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 \Psi}{\partial \phi^2} \right\} = \lambda$$

$$\text{or} \quad -\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi}{\partial \theta} \right) - \frac{1}{\sin \theta} \frac{\partial^2 \Psi}{\partial \phi^2} = \lambda \sin \theta \Psi \quad \dots (4)$$

$$\text{or} \quad S \Psi = \lambda \sin \theta \Psi, \quad \dots (5)$$

where S is the differential operator defined by

$$S = -\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \quad \dots (6)$$

The same equation (4) also arises when separating Helmholtz's equation in spherical coordinates. It can be proved that S is a formally self-adjoint operator.

We now discuss the boundary conditions to be associated with (4). Suppose it is required that (4) must hold on the entire surface of a sphere ($0 < \theta < \pi$, $0 < \phi \leq 2\pi$). Then $\Psi(\theta, \phi)$ should be continuous and have a continuous gradient on the surface. To this end, we impose the following boundary conditions:

$$\left. \begin{aligned} \Psi(\theta, 0+) = \Psi(\theta, 2\pi-), \quad [\partial \Psi / \partial \phi]_{\phi=0+} &= [\partial \Psi / \partial \phi]_{\phi=2\pi-0} \\ \text{and} \quad \Psi(0, \phi) = \Psi(\pi, \phi) &= \text{finite} \end{aligned} \right\} \quad \dots (7)$$

The first two conditions imply that the planes $\phi = 0$ and $\phi = 2\pi$ are on the same physical plane. The third condition of finiteness at $\theta = 0$ and $\theta = \pi$ is imposed because these values of θ are singular for the differential equation. It follows that some solutions are infinite at $\theta = 0$ and $\theta = \pi$ and hence these solutions must be rejected. As usual, it can be proved that the eigenvalues of (5) are real and non-negative, and that eigenfunctions corresponding to different values of eigenvalues are orthogonal on the surface of unit sphere.

In many branches of physics and engineering there is interest in the equation (4) solutions of which are known as *spherical harmonics*. We now proceed to find the solutions (eigenfunctions) of (4) by using the well known method of separation of variables.

$$\text{Let a solution of (4) be of the form} \quad \Psi(\theta, \phi) = \Theta(\theta) \Phi(\phi), \quad \dots (8)$$

Inserting this expression into (4), we have

$$-\frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \Phi \right) - \frac{1}{\sin \theta} \Theta \frac{d^2 \Phi}{d\phi^2} = \lambda \sin \theta \Theta \Phi$$

$$\text{or} \quad -\frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \lambda \sin^2 \theta = \frac{\Phi''}{\Phi} \quad \dots (9)$$

Now the left hand side of (9) is a function only of the variable θ , while the right-hand side is a function only of the variable ϕ . Since these two variables are independent, it follows that left-hand side and right hand side must separately be a constant which we shall denote by $-\mu$. Thus, we obtain

$$\begin{aligned} \frac{\Phi''}{\Phi} = -\mu \quad \quad \quad \text{and} \quad \quad \quad -\frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \lambda \sin^2 \theta = -\mu \\ \text{Thus,} \quad \quad \quad \Phi'' + \mu \Phi = 0 \quad \quad \quad \dots (10) \end{aligned}$$

and
$$\frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \lambda \sin^2 \theta = \mu \quad \dots (11)$$

In view of (7) and (8), we see that the boundary conditions for (10) are

$$\Phi(0) = \Phi(2\pi) \quad \text{and} \quad \Phi'(0) = \Phi'(2\pi) \quad \dots (12)$$

Hence, in order to get non-trivial solutions of (10), we set

$$\mu = m^2, \quad m \equiv 0, \pm 2, +3, \dots \quad \dots (13)$$

and then the corresponding solution is given by

$$\Phi(\phi) = e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \dots \quad \dots (14)$$

Putting $\mu = m^2$ in (11) and re-writting it, we obtain

$$-\frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \lambda \sin \theta \Theta + \frac{m^2}{\sin \theta} \Theta = 0, \quad 0 < \theta < \pi \quad \dots (15)$$

In (15), we make the change of variable $x = \cos \theta$ so that $dx = -\sin \theta d\theta$. Then, the interval $0 < \theta < \pi$ will transform into $-1 < x < 1$. We, also observe that

$$\frac{1}{\sin \theta} \frac{d}{d\theta} = -\frac{d}{dx} \quad \text{so that} \quad \sin \theta \frac{d}{d\theta} = -\sin^2 \theta \frac{d}{dx} = -(1-x^2) \frac{d}{dx}.$$

Accordingly, (15) takes the form

$$-\frac{d}{dx} \left\{ (1-x^2) \frac{d\Theta}{dx} \right\} + \frac{m^2}{1-x^2} \Theta = \lambda \Theta, \quad -1 < x < 1 \quad \dots (16)$$

With m as a given integer, this is an eigenvalue problem in the parameter λ . For number of values of λ , it is known that there is no solution which is finite at $x = 1$, and $x = -1$ (these values of x correspond, respectively, to $\theta = 0$ and $\theta = \pi$). Only by setting $\lambda = n(n+1)$, $n = |m|, |m| + 1, |m| + 2, \dots$, we arrive at a solution which is finite at $x = 1$ and at $x = -1$. This solution is the

*associated Legendre function $P_n^{|m|}(\cos \theta)$

In particular, if $m = 0$, the permissible values of n are $0, 1, 2, \dots$, and the corresponding eigenfunctions are the ordinary *Legendre polynomials $P_n(\cos \theta)$ i.e., $P_n^0(\cos \theta)$.

Thus, the only eigenvalues of (4) are

$$\lambda = n(n+1), \quad n = 0, 1, 2, \dots \quad \dots (17)$$

and to the eigenvalue $n(n+1)$ correspond $2n+1$ eigenfunctions

$$Y_n^m(\theta, \phi) = e^{im\phi} P_n^{|m|}(\cos \theta), \quad |m| \leq n \quad \dots (18)$$

For example, for $\lambda = 2$, we have the three eigenfunctions

$$e^{i\phi} P_1^1(\cos \theta), \quad e^{-i\phi} P_1^1(\cos \theta) \quad \text{and} \quad P_1^0(\cos \theta)$$

As already discussed, it follows from (4) and the boundary conditions that the set

$$Y_n^m(\theta, \phi), \quad |m| \leq n, \quad n = 0, 1, 2, \dots \quad \dots (19)$$

is an orthogonal set on the surface of the unit sphere. Moreover, it can be proved that (19) is complete and hence there can be no other eigenfunctions of (4).

*For Legendre function and associated Legendre function, refer authors "Advanced Differential Equations" or 'Ordinary and Partial Differential equations', published by S.Chand & Co. New Delhi.

Since the element of area on the surface is $\sin \theta d\theta d\phi$, we obtain

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta Y_n^m(\theta, \phi) \bar{Y}_\beta^\alpha(\theta, \phi) d\theta = 0, \text{ unless } \beta = n \text{ and } \alpha = m \quad \dots (20)$$

where bar denotes the complex conjugate. The normalization integral is defined as

$$N_{m,n} = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \left| Y_n^m(\theta, \phi) \right|^2 d\theta \quad \dots (21)$$

and we can easily verify that
$$N_{m,n} = \frac{4\pi}{2n+1} \times \frac{(n+|m|)!}{(n-|m|)!} \quad \dots (22)$$

If $f(\theta, \phi)$ is any function regular on the surface of the unit sphere, we obtain

$$f(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n f_{mn} Y_n^m(\theta, \phi), \quad \dots (23)$$

where
$$f_{mn} = \frac{1}{N_{m,n}} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta f(\theta, \phi) \bar{Y}_n^m(\theta, \phi) d\theta \quad \dots (24)$$

As a particular case, when f is independent of ϕ , we obtain expansion involving ordinary Legendre polynomials

$$f(\theta) = \sum_{n=0}^{\infty} f_n P_n(\cos \theta), \quad \text{where} \quad f_n = \frac{2n+1}{2} \int_0^\pi f(\theta) P_n(\cos \theta) \sin \theta d\theta \quad \dots (25)$$

Spherical expansion of the free-space. * Green's function for Laplace's equation.

Suppose a unit point source be placed at the point $P_0(\mathbf{x}_0)$ (whose spherical polar coordinates are (r_0, θ_0, ϕ_0) where $r_0 \neq 0, \theta_0 \neq 0$ and $\phi_0 \neq \pi$) and $P(\mathbf{x})$ be an arbitrary point in free space. Let (r, θ, ϕ) be spherical polar coordinates of P . Then the free-space Green's function $E(\mathbf{x}; \mathbf{x}_0)$ for Laplace's equation is given by

$$E(\mathbf{x}; \mathbf{x}_0) = \frac{1}{4\pi |\mathbf{x} - \mathbf{x}_0|} = E(r, \theta, \phi; r_0, \theta_0, \phi_0) \quad \dots (26)$$

such that $E(\mathbf{x}; \mathbf{x}_0)$ satisfies

$$-\frac{\partial}{\partial r} \left(r^2 \frac{\partial E}{\partial r} \right) + \frac{1}{\sin \theta} S E = \frac{\delta(r-r_0) \delta(\theta-\theta_0) \delta(\phi-\phi_0)}{\sin \theta}. \quad \dots (27)$$

where S is the operator defined by (6).

For $r \neq r_0$, E can be expanded in a series of spherical harmonics

$$E = \sum_{n=0}^{\infty} \sum_{m=-n}^n E_{m,n} Y_n^m(\theta, \phi), \quad \dots (28)$$

where $E_{m,n}$ depends on r, r_0, θ_0, ϕ_0 . Using (25), we find

$$E_{m,n} = \frac{1}{N_{m,n}} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta E \bar{Y}_n^m(\theta, \phi) d\theta \quad \dots (29)$$

To find $E_{m,n}$, we multiply both sides of (27) by $(1/N_{m,n}) \times \sin \theta \bar{Y}_n^m(\theta, \phi)$ and then integrate from $\theta = 0$ to π and for $\phi = 0$ to 2π . Thus, we obtain

*The remaining part of the present article can be understood after reading chapters 12 & 13.

$$-\frac{d}{dr}\left(r^2 \frac{d}{dr} E_{m,n}\right) + \frac{1}{N_{m,n}} \int_0^{2\pi} d\phi \int_0^\pi \bar{Y}_n^m(\theta, \phi) S E d\theta = \frac{1}{N_{m,n}} \bar{Y}_n^m(\theta_0, \phi_0) \quad \dots (30)$$

Using (17) and the fact that $Y_n^m(\theta, \phi)$ is a solution of (5), we have

$$S Y_n^m(\theta, \phi) = n(n+1) \sin \theta Y_n^m(\theta, \phi). \quad \dots (31)$$

Integrating by parts, using the fact that S is symmetric and keeping (31) in view, (30) yields the following ordinary differential equation

$$-\frac{d}{dr}\left(r^2 \frac{d}{dr} E_{m,n}\right) + n(n+1) E_{m,n} = \frac{1}{N_{m,n}} \bar{Y}_n^m(\theta_0, \phi_0) \delta(r-r_0) \quad \dots (32)$$

The functions r^n and r^{-n-1} are linearly independent solutions of the homogeneous equation, first of which is bounded at $r = 0$ and the other at $r = \infty$. Using the standard arguments for one-dimensional problems, we obtain

$$E_{m,n} = \frac{\bar{Y}_n^m(\theta_0, \phi_0)}{(2n+1)N_{m,n}} r_{<}^n r_{>}^{-n-1},$$

where $r_{<} = \min(r, r_0)$ and $r_{>} = \max(r, r_0)$. Therefore, we have

$$E(\mathbf{x}; \mathbf{x}_0) = \sum_{n=0}^{\infty} \sum_{m=-n}^n r_{<}^n r_{>}^{-n-1} \frac{Y_n^m(\theta, \phi) \bar{Y}_n^m(\theta_0, \phi_0)}{(2n+1) N_{m,n}} \quad \dots (33)$$

In particular, if the source is placed at a point where $\theta_0 = 0$, that is, on the positive z -axis, then (33) can be further simplified as shown below.

$$\text{For the present case,} \quad Y_n^m(0, \phi_0) = e^{-im\phi_0} P_n^{|m|}(1),$$

$$\text{where} \quad P_n^{|m|}(1) = \begin{cases} 0, & m \neq 0 \\ 1, & m = 0 \end{cases}$$

$$\text{and hence} \quad E(\mathbf{x}; \mathbf{x}_0) = \frac{1}{4\pi} \sum_{n=0}^{\infty} r_{<}^n r_{>}^{-n-1} P_n(\cos \theta) \quad (\text{source at } \theta_0 = 0, r_0) \quad \dots (34)$$

If we now return the source to its arbitrary position in (33), the angle θ in (34) becomes the angle γ between the points $(1, \theta_0, \phi_0)$ and $(1, \theta, \phi)$. It can be easily shown that

$$\cos \gamma = \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos(\phi - \phi_0) \quad \dots (35)$$

and hence (33) can be re-written as

$$E(\mathbf{x}; \mathbf{x}_0) = \frac{1}{4\pi} \sum_{n=0}^{\infty} r_{<}^n r_{>}^{-n-1} P_n(\cos \gamma) \quad \dots (36)$$

Comparing (33) and (36), we obtain the so called addition theorem for Legendre functions :

$$P_n(\cos \gamma) = \sum_{m=-n}^n \frac{e^{im(\phi-\phi_0)} P_n^{|m|}(\cos \theta) P_n^{|m|}(\cos \theta_0)}{(n+|m|)!(n-|m|)!} \quad \dots (37)$$

Finally, we discuss a special case of result (34). Suppose the source is placed at $r_0 = 1$,

$$\theta_0 = 0. \text{ Then, we have} \quad \frac{1}{(1+r^2-2r\cos\theta)^{1/2}} = \sum_{n=0}^{\infty} r^n P_n(\cos \theta), \quad r < 1 \quad \dots (38)$$

and
$$\frac{1}{(1+r^2-2r\cos\theta)^{1/2}} = \sum_{n=0}^{\infty} r^{-n-1} P_n(\cos\theta), \quad r > 1 \quad \dots (39)$$

Differentiating both sides of (38) with respect to r , we get

$$\frac{1-r^2}{(1+r^2-2r\cos\theta)^{1/2}} = \sum_{n=0}^{\infty} (2n+1) r^n P_n(\cos\theta), \quad r < 1 \quad \dots (40)$$

10.12. BESSEL FUNCTIONS*

Let α^2 be an arbitrary given complex number and let z be a complex variable. Then Bessel's

equation of order α is given by
$$\frac{d}{dz} \left(z \frac{du}{dz} \right) + \left(z - \frac{\alpha^2}{z} \right) u = 0 \quad \dots (1)$$

By virtue of the above definition, the Bessel's equations of order α and $-\alpha$ are exactly the same. Any solution of (1) is known as *cylinder function*.

The Bessel function of order α , $J_\alpha(z)$ is defined by the series

$$J_\alpha(z) = (z/2)^\alpha \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k! (\alpha+k)!}, \quad \dots (2)$$

which is a solution of (1). Infinite series in (2) is convergent for all values of z .

When α is not an integer, the second solution of (1) is given by

$$J_{-\alpha}(z) = (z/2)^{-\alpha} \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k! (-\alpha+k)!} \quad \dots (3)$$

When α is an integer (positive or negative), we have

$$J_{-\alpha}(z) = (-1)^\alpha J_\alpha(z) \quad \dots (4)$$

However when α is not an integer, $J_\alpha(z)$ and $J_{-\alpha}(z)$ are two independent solutions of (1).

It can be shown that the Neumann function
$$N_\alpha(z) = \frac{\cos \alpha \pi}{\sin \alpha \pi} \frac{J_\alpha(z) - J_{-\alpha}(z)}{2} \quad \dots (5)$$

and the Hankel functions
$$H_\alpha^{(1)} = J_\alpha(z) + i N_\alpha(z), \quad \dots (6)$$

$$H_\alpha^{(2)} = J_\alpha(z) - i N_\alpha(z), \quad \dots (7)$$

are all solutions of (1).

The general solution of (1) is a linear combination of two independent solutions. To this end we consider the following two cases :

Case I. If α is not an integer, then any two of the functions J_α , $J_{-\alpha}$, $N_\alpha H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are independent. Accordingly, the general solution (1) can be written as a linear combinations of any two of these five functions.

Case II. If α is an integer (say $\alpha = n$, where n is any positive integer), then J_n and J_{-n} are not independent due to the relation (4). The function $N_\alpha(z)$ defined by (5) by taking the limit

*For more details, refer, chapter dealing with Bessel functions in author's "Advanced Differential Equations" or "Ordinary and Partial Differential Equation" published by S.Chand & Co., New Delhi.

$\alpha \rightarrow n$ is still a solution of (1), and is independent of $J_n(\alpha)$. Thus any two of the functions $J_n, N_n, H_n^{(1)}, H_n^{(2)}$ are independent. Accordingly, the general solution of (1) can be written as a linear combinations of any two of these four functions.

The modified Bessel functions

Bessel's modified equation is given by
$$\frac{d}{dz} \left(z \frac{du}{dz} \right) - \left(z + \frac{\alpha^2}{z} \right) u = 0, \dots (8)$$

which differs from Bessel equation (1) only by the sign of the term zu . Since the formal substitution $t = iz$ reduces (8) to (1), we expect that the solutions of (8) will be cylinder functions of argument iz .

By convention, the modified Bessel function, $I_\alpha(z) = (z/2)^\alpha \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k! (\alpha + 2k)!} \dots (9)$

and the Macdonald function
$$K_\alpha(z) = \frac{\pi}{2} \left[\frac{I_{-\alpha}(z) - I_\alpha(z)}{\sin \alpha \pi} \right] \dots (10)$$

are taken as the basic independent solutions of (8). These solutions are real when α and z are real and positive. The functions $I_\alpha(z)$ and $K_\alpha(z)$ are constant multiplier of $J_\alpha(z)$ and $H_\alpha^{(1)}(z)$ respectively.

The behaviour of cylinder functions at zero and at infinity.

If α has positive real part or if $\alpha = 0$, $J_\alpha(z)$ is the only solution of (1) that is bounded in a neighbourhood of $z = 0$. Similarly, $I_\alpha(z)$ is the only solution of (8) that is bounded at the origin.

Again, as $z \rightarrow \infty$, we have

$$\begin{aligned} (i) \quad J_\alpha(z) &\sim \left(\frac{2}{\pi z} \right)^{1/2} \cos \left\{ z - \left(\alpha + \frac{1}{2} \right) \frac{\pi}{2} \right\} & (ii) \quad N_\alpha(z) &\sim \left(\frac{2}{\pi z} \right)^{1/2} \sin \left\{ z - \left(\alpha + \frac{1}{2} \right) \frac{\pi}{2} \right\} \\ (iii) \quad H_\alpha^{(1)} &\sim \left(\frac{2}{\pi z} \right)^{1/2} \exp \left\{ i \left(z - \frac{\alpha \pi}{2} - \frac{\pi}{4} \right) \right\} & (iv) \quad H_\alpha^{(2)} &\sim \left(\frac{2}{\pi z} \right)^{1/2} \exp \left\{ -i \left(z - \frac{\alpha \pi}{2} - \frac{\pi}{4} \right) \right\} \\ (v) \quad I_\alpha(z) &\sim \frac{1}{(2\pi z)^{1/2}} e^z & (vi) \quad K_\alpha(z) &\sim \left(\frac{\pi}{2z} \right)^{1/2} e^{-z} \end{aligned}$$

EXERCISE

1. Define Dirac delta function and show that

$$\begin{aligned} (i) \quad \int_{-\infty}^{\infty} \delta(t-a) f(t) dt &= f(a) & (ii) \quad \int_{-\infty}^{\infty} \delta(t) f(t) dt &= f(0) \\ (iii) \quad \int_{-\infty}^{\infty} \delta'(t-a) f(t) dt &= -f'(a) & (iv) \quad \int_{-\infty}^{\infty} \delta'(t) f(t) dt &= -f'(0) \quad [\text{Himanchal 2008, 09}] \end{aligned}$$

2. Prove that $\delta(-t) = \delta(t)$

[Himanchal 2009]

Applications of Integral equations and Green's function to ordinary differential equations

11.1 INTRODUCTION.

We have already dealt with conversion of initial and boundary value problems into integral equations in chapter 2. In the present chapter we shall consider the initial and boundary problems again in a different context. We shall introduce the concept of Green's function and utilize it in converting initial and boundary value problems into integral equations. Sometimes we shall be able to solve the given initial and boundary value problems completely with help of Green's function. The reader is advised to study chapter 10 before starting with the present chapter.

11.2 GREEN'S FUNCTION. DEFINITION.

[Meerut 2002]

Consider a linear homogeneous differential equation of order n :

$$L[y] = 0, \quad \dots (1)$$

where L is the differential operator

$$L \equiv p_0(x) \frac{d^n}{dx^n} + p_1(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + p_n(x), \quad \dots (2)$$

where the functions $p_0(x), p_1(x), \dots, p_n(x)$ are continuous on $[a, b]$, $p_0(x) \neq 0$ on $[a, b]$ and the boundary conditions are

$$V_k(y) = 0, \quad (k = 1, 2, 3, \dots, n) \quad \dots (3)$$

where

$$\begin{aligned} V_k(y) = & \alpha_k y(a) + \alpha_k^{(1)} y'(a) + \dots + \alpha_k^{(n-1)} y^{(n-1)}(a) \\ & + \beta_k y(b) + \beta_k^{(1)} y'(b) + \dots + \beta_k^{(n-1)} y^{(n-1)}(b), \end{aligned} \quad \dots (4)$$

where the linear forms $V_1, \dots, V_n$ in $y(a), y'(a), \dots, y^{(n-1)}(a), y(b), y'(b), \dots, y^{(n-1)}(b)$ are linearly independent.

Suppose that the homogeneous boundary-value problem given by (1) to (4) has only a trivial solution $y(x) \equiv 0$. Then Green's function of the boundary value problem (1) to (4) is the function $G(x, t)$ constructed for any point $t, a < t < b$, and which has the following four properties :

(i) In each of the intervals $[a, t]$ and $(t, b]$ the function $G(x, t)$, considered as a function of x , is a solution of (1), that is,

$$L[G] = 0. \quad \dots (5)$$

(ii) $G(x, t)$ is continuous and has continuous derivatives with respect to x upto order $(n-2)$ inclusive for $a \leq x \leq b$.

(iii) $(n-1)$ th derivative of $G(x, t)$ with respect to x at the point $x = t$ has discontinuity of the first kind, *the jump being equal to $-1/p_0(t)$, that is,

$$\left(\frac{\partial^{n-1} G}{\partial x^{n-1}} \right)_{x=t+0} - \left(\frac{\partial^{n-1} G}{\partial x^{n-1}} \right)_{x=t-0} = -\frac{1}{p_0(t)}. \quad \dots (6)$$

(iv) $G(x, t)$ satisfies the boundary conditions (3), that is,

$$V_k(G) = 0. \quad (k = 1, 2, \dots, n) \quad \dots (7)$$

Theorem. If the boundary value problems given by (1) to (4) has only the trivial solution $y(x) \equiv 0$, then the operator L has a unique Green's function $G(x, t)$.

Proof. Suppose that $y_1(x), y_2(x), \dots, y_n(x)$ be linearly independent solutions of the equation (1). Then, by virtue of property (i) of Green's function, the unknown Green's function $G(x, t)$ must have the following representation on the intervals $[a, t]$ and $[t, b]$:

$$G(x, t) = \begin{cases} a_1 y_1(x) + a_2 y_2(x) + \dots + a_n y_n(x), & a \leq x < t, \\ b_1 y_1(x) + b_2 y_2(x) + \dots + b_n y_n(x), & t < x \leq b, \end{cases} \quad \dots (8)$$

where $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are some functions of t .

Again, by virtue of property (ii), the continuity of the function $G(x, t)$ and of its first $n-2$ derivatives with respect to x at the point $x = t$ gives rise to the following relations :

$$[b_1 y_1(t) + \dots + b_n y_n(t)] - [a_1 y_1(t) + \dots + a_n y_n(t)] = 0, \quad \dots (A_1)$$

$$[b_1 y_1'(t) + \dots + b_n y_n'(t)] - [a_1 y_1'(t) + \dots + a_n y_n'(t)] = 0, \quad \dots (A_2)$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$[b_1 y_1^{(n-2)}(t) + \dots + b_n y_n^{(n-2)}(t)] - [a_1 y_1^{(n-2)}(t) + \dots + a_n y_n^{(n-2)}(t)] = 0. \quad \dots (A_{n-1})$$

Finally, by virtue of property (iii), we obtain

$$[b_1 y_1^{(n-1)}(t) + \dots + b_n y_n^{(n-1)}(t)] - [a_1 y_1^{(n-1)}(t) + \dots + a_n y_n^{(n-1)}(t)] = -\frac{1}{p_0(t)}. \quad \dots (A_n)$$

Let $C_k(t) = b_k(t) - a_k(t)$, where $k = 1, 2, \dots, n$ (9)

Re-writing $(A_1), (A_2), \dots, (A_n)$ with help of (9), we obtain a system of linear equations in $C_k(t)$:

$$C_1 y_1(t) + \dots + C_n y_n(t) = 0, \quad \dots (B_1)$$

$$C_1 y_1'(t) + \dots + C_n y_n'(t) = 0, \quad \dots (B_2)$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$C_1 y_1^{(n-2)}(t) + \dots + C_n y_n^{(n-2)}(t) = 0, \quad \dots (B_{n-1})$$

and

$$C_1 y_1^{(n-1)}(t) + \dots + C_n y_n^{(n-1)}(t) = -\frac{1}{p_0(t)}. \quad \dots (B_n)$$

* The reader must note carefully that some authors consider this jump with opposite sign (i.e., $1/p_0(t)$) and this implies a change of $G(x, t)$ to $-G(x, t)$.

The determinant D of the system $(B_1), \dots, (B_n)$ is given by

$$D = \begin{vmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \dots & y_n^{(n-2)}(t) \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{vmatrix} = W(y_1, y_2, \dots, y_n), \quad \dots (10)$$

where $W(y_1, y_2, \dots, y_n)$ is the Wronskian of the functions $y_1, \dots, y_n$. Since $y_1, y_2, \dots, y_n$ are taken as linearly independent, it follows that

$$D = W(y_1, y_2, \dots, y_n) \neq 0. \quad \dots (11)$$

Since $D \neq 0$ it follows that the system of equations $(B_1), \dots, (B_n)$ possess a unique solution for $C_k, k = 1, 2, \dots, n$.

We now proceed to find the functions $a_k(t)$ and $b_k(t)$. For this purpose, we use the property (iv) of the Green's function. We first write $V_k(y)$ in the following form

$$V_k(y) = P_k(y) + Q_k(y), \quad \dots (12)$$

$$\text{where} \quad P_k(y) = \alpha_k y(a) + \alpha_k^{(1)} y'(a) + \dots + \alpha_k^{(n-1)} y^{(n-1)}(a), \quad \dots (13)$$

$$\text{and} \quad Q_k(y) = \beta_k y(b) + \beta_k^{(1)} y'(b) + \dots + \beta_k^{(n-1)} y^{(n-1)}(b). \quad \dots (14)$$

Then, by virtue of the equation (7) of the property (iv), we have

$$V_k(G) = a_1 P_k(y_1) + \dots + a_n P_k(y_n) + b_1 Q_k(y_1) + \dots + b_n Q_k(y_n) = 0, \quad k = 1, 2, \dots, n \quad \dots (15)$$

$$\text{From (9),} \quad a_k = b_k - C_k, \quad k = 1, 2, \dots, n. \quad \dots (16)$$

Using (16), (15) reduces to

$$(b_1 - C_1) P_k(y_1) + \dots + (b_n - C_n) P_k(y_n) + b_1 Q_k(y_1) + \dots + b_n Q_k(y_n) = 0, \quad k = 1, 2, \dots, n \quad \dots (17)$$

$$\text{or} \quad b_1 [P_k(y_1) + Q_k(y_1)] + \dots + b_n [P_k(y_n) + Q_k(y_n)] = C_1 P_k(y_1) + \dots + C_n P_k(y_n)$$

$$\text{or} \quad b_1 V_k(y_1) + \dots + b_n V_k(y_n) = C_1 P_k(y_1) + \dots + C_n P_k(y_n), \quad k = 1, 2, \dots, n, \text{ by (12)} \quad \dots (18)$$

Here (18) is a linear system of n equations for determination of n quantities $b_1, \dots, b_n$. Since we have assumed the linear independence of the forms $V_1, V_2, \dots, V_n$, it follows that the determinant of the system (18) is non-zero, that is,

$$\begin{vmatrix} V_1(y_1) & V_1(y_2) & \dots & V_1(y_n) \\ V_2(y_1) & V_2(y_2) & \dots & V_2(y_n) \\ \dots & \dots & \dots & \dots \\ V_n(y_1) & V_n(y_2) & \dots & V_n(y_n) \end{vmatrix} \neq 0. \quad \dots (19)$$

By virtue of (19), we see that the system of equations (18) possess a unique solution in $b_1(t), b_2(t), \dots, b_n(t)$. Already, we have seen that unique values of $C_1(t), \dots, C_n(t)$ exist. Hence (16) shows that the quantities $a_1(t), \dots, a_n(t)$ are defined uniquely.

Thus we have established the existence and uniqueness of Green's function $G(x, t)$. We have also indicated a procedure for constructing the Green's function in the above proof of the theorem.

Remark 1. If the boundary value problem (1) to (4) is *self-adjoint*, then Green's function is symmetric, that is, $G(x, t) = G(t, x)$. The converse is also true. For definition and examples of self-adjoint equations, refer chapter 10.

Remark 2. If at one of the extremities of an interval $[a, b]$ the coefficient of the highest derivative vanishes, for example $p_0(a) = 0$, then the natural boundary condition for boundedness of the solution at $x = a$ is imposed, and at the other extremity the ordinary boundary condition is specified. Refer solved example 3 in Art. 11.5 for more details.

11.3 CONVERSION OF A BOUNDARY VALUE PROBLEM INTO FREDHOLM INTEGRAL EQUATION. SOLUTION OF A BOUNDARY VALUE PROBLEM

In what follows, we shall use the following notations :

$$L \equiv p_0(x) \frac{d^n}{dx^n} + p_1(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + p_n(x).$$

$$V_k(y) \equiv \alpha_k y(a) + \alpha_k^{(1)} y'(a) + \dots + \alpha_k^{(n-1)} y^{(n-1)}(a) + \beta_k y(b) + \beta_k^{(1)} y'(b) + \dots + \beta_k^{(n-1)} y^{(n-1)}(b).$$

Suppose $G(x, t)$ is Green's function of the boundary value problem

$$L[y] = 0, \quad \dots (1)$$

$$V_k(y) = 0, \quad k = 1, 2, 3, \dots, n, \quad \dots (2)$$

involving homogeneous boundary conditions (2) at the end points $x = a$ and $x = b$ of an interval $a \leq x \leq b$.

Result 1. Consider the boundary value problem

$$L[y] + \Phi(x) = 0, \quad \dots (3)$$

$$V_k(y) = 0, \quad k = 1, 2, \dots, n, \quad \dots (4)$$

involving the same homogeneous boundary conditions as in (2). Here $\Phi(x)$ is a direct function of x .

*Then solution of the boundary value problem (3)—(4) is given by the formula

$$y(x) = \int_a^b G(x, t) \Phi(t) dt. \quad \dots (5)$$

Result 2. Consider the boundary-value problem

$$L[y] + \Phi(x) = 0, \quad \dots (6)$$

$$V_k(y) = 0, \quad k = 1, 2, \dots, n, \quad \dots (7)$$

involving the same homogeneous boundary conditions as in (2).

In this result we assume that $\Phi(x)$ is not a given direct function of x . However, $\Phi(x)$ may also depend upon x indirectly by also involving the unknown function $y(x)$, and so being expressible in the form

$$\Phi(x) = \phi(x, y(x)). \quad \dots (8)$$

Then the boundary-value problem (6)—(7) can be reduced to the following integral equation

$$y(x) = \int_a^b G(x, t) \phi(t, y(t)) dt. \quad \dots (9)$$

Particular case of Result 2 :

Let $\Phi(x) = \lambda r(x) y(x) - f(x)$, where λ is a parameter. Then, we see that the boundary value problem

$$L[y] + \lambda r(x) y(x) = f(x) \quad \dots (10)$$

$$V_k(y) = 0, \quad k = 1, 2, \dots, n \quad \dots (11)$$

reduces to the following integral equation

$$y(x) = \lambda \int_a^b G(x, t) r(t) y(t) dt - \int_a^b G(x, t) f(t) dt, \quad \dots (12)$$

* A proof of this result for a particular case will be presented in Art. 11.4.

where $G(x, t)$ is the relevant Green's function. In (12), $G(x, t) r(t)$ is not symmetric unless the function $r(t)$ is a constant. However, if we write

$$\{r(x)\}^{1/2} y(x) = Y(x),$$

under the assumption that $r(x)$ is non-negative over (a, b) , as is usually the case in practice, the equation (12) can be re-written in the form

$$Y(x) = \lambda \int_a^b K^*(x, t) Y(t) dt - \int_a^b K^*(x, t) \frac{f(t)}{\{r(t)\}^{1/2}} dt,$$

where $K^*(x, t) = G(x, t) \{r(x) r(t)\}^{1/2}$ is a symmetric kernel. We have already seen the importance of a symmetric kernel in Chapter 7.

Result 3. When the prescribed end conditions are not homogeneous, we shall use a modified method as explained below :

In this case, let $G(x, t)$ denote the Green's function corresponding to the associated *homogeneous* end conditions. We now search for a function $P(x)$ such that the relation

$$y(x) = P(x) + \int_a^b G(x, t) \Phi(t) dt \quad \dots (13)$$

is equivalent to the differential equation $L(y) + \Phi(x) = 0$, ... (14)
together with the prescribed *nonhomogeneous* end conditions.

$$\text{Since} \quad L \left[\int_a^b G(x, t) \Phi(t) dt \right] = -\Phi(x), \quad \dots (15)$$

the requirement that (13) imply (14) leads us to

$$L[P(x)] = 0. \quad \dots (16)$$

Furthermore, since the second term in (13) satisfies the associated homogeneous end conditions, we conclude that function $P(x)$ in (13) must be the solution of (16) which satisfies the prescribed nonhomogeneous end conditions. When $G(x, t)$ exists, then $P(x)$ always exists.

11.4 AN IMPORTANT SPECIAL CASE OF RESULTS OF ART. 11.2

Consider a linear homogeneous differential equation of order two :

$$Ly(x) + \phi(x) = 0, \quad a \leq x \leq b \quad \dots (1)$$

where L is the self adjoint differential operator

$$L \equiv \frac{d}{dx} \left(p \frac{d}{dx} \right) + q = p \frac{d^2}{dx^2} + \frac{dp}{dx} \frac{d}{dx} + q, \quad \dots (2)$$

together with homogeneous boundary conditions

$$\alpha_1 y(x) + \beta_1 y'(a) = 0 \quad \dots (3a)$$

$$\text{and} \quad \alpha_2 y(b) + \beta_2 y'(b) = 0 \quad \dots (3b)$$

with the usual assumption that at least one of α_1 and β_1 and one of α_2 and β_2 are non-zero.

Here p, p' and q are given real valued continuous functions defined on $[a, b]$ such that $p(x)$ is non-zero* on $[a, b]$.

* It $p(x) = 0$ at an end point, the corresponding appropriate end condition may require merely that $y(x)$ remains finite at that point. Refer Ex. 3., on page 11.12 of Art 11.5.

11.6 Applications of integral equations and Green's function to ordinary differential equation

Then, by definition, a function $G(x, t)$ is called a Green's function for $Ly(x) = 0$ if, for a given t .

$$G(x, t) = \begin{cases} G_1(x, t), & \text{if } a \leq x < t \\ G_2(x, t), & \text{if } t < x \leq b \end{cases} \quad \dots (4)$$

where G_1 and G_2 are such that

(i) The functions G_1 and G_2 satisfy the equation $LG = 0$ in their respective intervals of definition, that is,

$$LG_1 = 0, \quad \text{when } a \leq x < t \quad \dots (5a)$$

and

$$LG_2 = 0, \quad \text{when } t < x \leq b \quad \dots (5b)$$

(ii) G_1 satisfies the boundary condition (3a) whereas G_2 satisfies the condition (3b).

(iii) The function $G(x, t)$ is continuous at $x = t$, i.e., $G_1(t, t) = G_2(t, t) \quad \dots (6)$

(iv) The derivative of $G(x, t)$ has a discontinuity of magnitude $-1/p(t)$ at $x = t$, that is,

$$\left[\partial G_2 / \partial x - \partial G_1 / \partial x \right]_{x=t} = -1/p(t) \quad \dots (7)$$

Condition (7) can also be re-written as

$$\left(\frac{\partial G}{\partial x} \right)_{x=t+0} - \left(\frac{\partial G}{\partial x} \right)_{x=t-0} = -\frac{1}{p(t)} \quad \dots (7')$$

We now use the above definition to construct the Green's function for boundary value problem given by (2), (3a) and (3b) as follows :

Suppose that $y = u(x)$ is a nontrivial solution of the associated equation

$$Ly = 0. \quad \dots (8)$$

Furthermore, suppose that $y = u(x)$ satisfies the given homogeneous condition (3a). Similarly, let $y = v(x)$ be a nontrivial solution of (8), which satisfies the given condition (3b). It follows that the properties (i) and (ii) are satisfied if we take $G_1(x) = C_1 u(x)$ and $G_2(x) = C_2 v(x)$, where C_1 and C_2 are constants. Thus, we have

$$G(x, t) = \begin{cases} C_1 u(x), & a \leq x < t, \\ C_2 v(x), & t < x \leq b. \end{cases} \quad \dots (9)$$

We shall now determine C_1 and C_2 by fulfilling the properties (iii) and (iv).

By virtue of property (iii), we have

$$C_1 u(t) = C_2 v(t) \quad \text{or} \quad C_2 v(t) - C_1 u(t) = 0. \quad \dots (10)$$

Again by the virtue of property (iv), we have

$$C_2 v'(t) - C_1 u'(t) = -1/p(t). \quad \dots (11)$$

The determinant D of the system of equations (10) and (11) is given by

$$D = \begin{vmatrix} v(t) & -u(t) \\ v'(t) & -u'(t) \end{vmatrix} = W[u(t), v(t)]. \quad \dots (12)$$

Suppose that $u(x)$ and $v(x)$ be two linearly independent solutions of (8). Then, we known that the Wronskian $W[u(t), v(t)]$ of linearly independent functions u and v does not vanish. Accordingly, (12) reduces to

$$D = W[u(t), v(t)] \neq 0, \quad \dots (13)$$

showing that (10) and (11) possess a unique solution.

Now, solving (10) and (11) by cross- multiplication we have

$$\frac{C_2}{-u/p(t)} = \frac{C_1}{-v/p(t)} = \frac{1}{-v u' + u v'}. \quad \dots (14)$$

We shall now prove the Abel's formula, namely,

$$u v' - u' v = A/p(t). \quad \dots (15)$$

By our assumption u and v are solutions of

$$Ly = 0 \quad \text{or} \quad p \frac{d^2 y}{dx^2} + \frac{dp}{dx} \frac{dy}{dx} + q y = 0, \text{ by (2)}$$

$$\text{or} \quad p y'' + p' y' + q y = 0 \quad \text{or} \quad (p y')' + q y = 0. \quad \dots (16)$$

$$\therefore \quad (p u')' + q u = 0 \quad \dots (17)$$

$$\text{and} \quad (p v')' + q v = 0 \quad \dots (18)$$

Multiplying (17) by v and (18) by u , we get

$$v (p u')' + q u v = 0 \quad \dots (19)$$

$$\text{and} \quad u (p v')' + q u v = 0. \quad \dots (20)$$

Subtracting (19) from (20), we have

$$u (p v')' - v (p u')' = 0 \quad \text{or} \quad [p (u v' - v u')] = 0. \quad \dots (21)$$

Integrating (21), $p(u v' - u' v) = A$, when A is a constant of integration.

$$\text{Thus,} \quad u v' - u' v = A/p(t),$$

which proves the *Abel's formula* (15).

$$\text{Using (15), (14) reduces to} \quad \frac{C_2}{-u/p(t)} = \frac{C_1}{-v/p(t)} = \frac{1}{A/p(t)}$$

$$\text{so that} \quad C_1 = -(1/A) v(t) \quad \text{and} \quad C_2 = -(1/A) u(t).$$

Substituting these values of C_1 and C_2 in (9), we obtain

$$G(x, t) = \begin{cases} -(1/A) u(x) v(t), & a \leq x < t \\ -(1/A) u(t) v(x), & t < x \leq b, \end{cases} \quad \dots (22)$$

where A is a constant, independent of x and t , which is determined by Abel's formula (15).

$$\text{From (22), it follows that } G(x, t) \text{ is symmetric, i.e.,} \quad G(x, t) = G(t, x) \quad \dots (23)$$

We now state and prove the main result of this article.

Theorem. Let the Green's function for $Ly(x) = 0$ be given by (22). Then $y(x)$ is a solution of boundary value problem given by (1), (3a) and (3b) if and only if

$$y(x) = \int_a^b G(x, t) \phi(t) dt \quad \dots (24)$$

Proof. Let the relation (24) be true. Then we shall show that $y(x)$ is a solution of the boundary value problem given by (1), (3a) and (3b).

$$\text{Re-writing (24), we have} \quad y(x) = \int_a^x G(x, t) \phi(t) dt + \int_x^b G(x, t) \phi(t) dt$$

$$\text{or} \quad y(x) = -\frac{1}{A} \left[\int_a^x v(x) u(t) \phi(t) dt + \int_x^b u(x) v(t) \phi(t) dt \right], \text{ by (22)} \quad \dots (25)$$

Differentiating both sides of (25) w.r.t. ' x ' and using Leibnitz's rule of differentiation under the sign of integration (refer Art. 1.13), we have

$$y'(x) = -\frac{1}{A} \left[\int_a^x v'(x) u(t) \phi(t) dt + u(x) v(x) \phi(x) + \int_x^b u'(x) v(t) \phi(t) dt - u(x) v(x) \phi(x) \right]$$

or
$$y'(x) = -\frac{1}{A} \left[\int_a^x v'(x) u(t) \phi(t) dt + \int_x^b u'(x) v(t) \phi(t) dt \right] \quad \dots (26)$$

Differentiating both sides of (26) with respect of 'x' and using Leibnitz's rule, we get

$$y''(x) = -\frac{1}{A} \left[\int_a^x v''(x) u(t) \phi(t) dt + v'(x) u(x) \phi(x) + \int_x^b u''(x) v(t) \phi(t) dt - u'(x) v(x) \phi(x) \right]$$

or
$$y''(x) = -\frac{\phi(x)}{A} \{v'(x) u(x) - u'(x) v(x)\} - \frac{1}{A} \left[\int_a^x v''(x) u(t) \phi(t) dt + \int_x^b u''(x) v(t) \phi(t) dt \right]$$

or
$$y''(x) = -\frac{\phi(x)}{p(x)} - \frac{1}{A} \left[\int_a^x v''(x) u(t) \phi(t) dt + \int_x^b u''(x) v(t) \phi(t) dt \right], \text{ by (15)} \quad \dots (27)$$

Now, $L y(x) = p(x) y''(x) + p'(x) y'(x) + q(x) y(x)$, by definition (2)

Substituting the values of $y(x)$, $y'(x)$ and $y''(x)$ given by (25), (26) and (27) in the above equation, we get

$$\begin{aligned} L y(x) = & -\phi(x) - \frac{p(x)}{A} \left[\int_a^x v''(x) u(t) \phi(t) dt + \int_x^b u''(x) v(t) \phi(t) dt \right] \\ & - \frac{p'(x)}{A} \left[\int_a^x v'(x) u(t) \phi(t) dt + \int_x^b u'(x) v(t) \phi(t) dt \right] \\ & - \frac{q(x)}{A} \left[\int_a^x v(x) u(t) \phi(t) dt + \int_x^b u(x) v(t) \phi(t) dt \right] \end{aligned}$$

or
$$\begin{aligned} L y(x) = & -\phi(x) - \frac{1}{A} \left[\int_a^x \{p(x) v''(x) + p'(x) v'(x) + q(x) v(x)\} u(t) \phi(t) dt \right. \\ & \left. + \int_x^b \{p(x) u''(x) + p'(x) u'(x) + q(x) u(x)\} v(t) \phi(t) dt \right] \end{aligned}$$

or
$$L y(x) = -\phi(x) - \frac{1}{A} \left[\int_a^x \{(p v)'\} u(t) \phi(t) dt + \int_x^b \{(p u)'\} v(t) \phi(t) dt \right]$$

or
$$L y(x) = -\phi(x), \text{ using (17) and (18)}$$

or
$$L y(x) + \phi(x) = 0, \quad \dots (28)$$

Further, from the relations (25) and (26), we easily obtain

and
$$\left. \begin{aligned} A y(a) &= -u(a) \int_a^b v(t) \phi(t) dt \\ A y'(a) &= -u'(a) \int_a^b v(t) \phi(t) dt \end{aligned} \right\} \quad \dots (29)$$

Since $u(x)$ satisfies the boundary condition (3a), it follows from (29) that $y(x)$ also satisfies the boundary condition (3a). Similarly we can show that $y(x)$ satisfies the boundary condition (3b).

Thus, we have shown that if $G(x, t)$ is given by (22), then $y(x)$ is a solution of the boundary value problem given by (1), (3a) and (3b).

Conversely, let $y(x)$ satisfy (1) together with boundary conditions (3a) and (3b). Then, we shall prove that $y(x)$ satisfies (24), where $G(x, t)$ is given by (22).

From (1), $L y(x) = -\phi(x)$ so that $-G(x, t) \phi(x) = G(x, t) L y(x)$

$$\therefore -\int_a^b G(x, t) \phi(x) dx = \int_a^b G(x, t) L y(x) dx = \int_a^t G(x, t) L y(x) dx + \int_t^b G(x, t) L y(x) dx$$

or
$$-\int_a^b G(x, t) \phi(x) dx = \int_a^t G_1(x, t) L y(x) dx + \int_t^b G_2(x, t) L y(x) dx, \text{ using (4)} \quad \dots (30)$$

Now, Green's formula for self adjoint operator L (refer result (5) of Art. 10.5 of chapter 10 by using x_1 and x_2 in place of a and b respectively) is given by

$$\int_{x_1}^{x_2} v Lu dx = \int_a^b u Lv dx + [p(x)(vu' - v'u)]_{x_1}^{x_2} \quad \dots (31)$$

Using the Green's formula (31), we have

$$\begin{aligned} \int_a^t G_1(x, t) Ly(x) dx &= \int_a^t Ly(x) LG_1 dx + \left[p(x) \left\{ G_1(x, t) y'(x) - y(x) \frac{\partial G_1(x, t)}{\partial x} \right\} \right]_a^t \\ &= p(t) \left\{ G_1(t, t) y'(t) - y(t) \left[\frac{\partial G_1(x, t)}{\partial x} \right]_{x=t} \right\} \quad \dots (32) \\ &\quad \text{[on using the properties of } G_1(x, t)] \end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_t^b G_2(x, t) Ly(x) dx &= \int_t^b Ly(x) LG_2 dx + \left[p(x) \left\{ G_2(x, t) y'(x) - y(x) \frac{\partial G_2(x, t)}{\partial x} \right\} \right]_t^b \\ &= -p(t) \left\{ G_2(t, t) y'(t) - y(t) \left[\frac{\partial G_2(x, t)}{\partial x} \right]_{x=t} \right\} \quad \dots (33) \\ &\quad \text{[on using the properties of } G_2(x, t)] \end{aligned}$$

Using (32) and (33), we have

$$\begin{aligned} \text{R.H.S. of (30)} &= p(t) y'(t) \{G_1(t, t) - G_2(t, t)\} + p(t) y(t) (\partial G_2 / \partial x - \partial G_1 / \partial x)_{x=t} \\ &= p(t) y'(t) \times 0 + p(t) y(t) \times \{-1/p(t)\}, \text{ using (6) and (7)} \\ &= -y(t) \end{aligned}$$

Hence (30) reduces to

$$-\int_a^b G(x, t) \phi(x) dx = -y(t), \quad \text{i.e.,} \quad y(x) = \int_a^b G(t, x) \phi(t) dt \quad \text{[on interchanging } x \text{ and } t]$$

or
$$y(x) = \int_a^b G(x, t) \phi(t) dt, \text{ using (23)}$$

Thus, we have shown that if $y(x)$ satisfy (1) together with (3a) and (3b), then $y(x)$ must satisfy (24).

Remark 1. Relation (24) defines the solution of the boundary value problem given by (1), (3a) and (3b) when ϕ is a given direct function of x , whereas (24) constitutes an equivalent *integral equation formulation* of the problem when ϕ involves y . In other words, if ϕ involves y , then (24) is not a solution of given boundary value problem. In that case (24) merely is a non-linear integral equation which is equivalent to the given boundary value problem.

Remark 2. Consider inhomogeneous system

$$L y(x) + \phi(x) = 0; \quad \alpha_1 y(a) + \beta_1 y'(a) = 0, \quad \alpha_2 y(b) + \beta_2 y'(b) = 0. \quad \dots (i)$$

and the associated completely homogeneous system.

$$L y(x) = 0, \quad \alpha_1 y(a) + \beta_1 y'(a) = 0, \quad \alpha_2 y(b) + \beta_2 y'(b) = 0, \quad \dots (ii)$$

where
$$L \equiv a_0(x) \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_2(x) \quad \text{and} \quad a < x < b.$$

11.10 Applications of integral equations and Green's function to ordinary differential equation

If (ii) has only the trivial solution, then (i) has always a unique solution and Green's function exists for (ii).

If (ii) has a non-trivial solution, then (i) either has no solution or else has many solutions, but never just one.

For example, consider the completely homogeneous system

$$y''(x) = 0, \quad 0 < x < 1; \quad y'(0) = 0, \quad y'(1) = 0. \quad \dots (iii)$$

Solving $y''(x) = 0$, we have $y'(x) = A$ and $y(x) = Ax + B$.

Using $y'(0) = 0$ and $y'(1) = 0$, we get $A = 0$ and so (iii) has non-trivial solution $y(x) = B$, where B is an arbitrary constant. Now, consider the related inhomogeneous system

$$y''(x) = 1, \quad 0 < x < 1; \quad y'(0) = 0, \quad y'(1) = 0 \quad \dots (iv)$$

Solving $y''(x) = 1$, we have $y'(x) = x + A$ and $y(x) = B + Ax + x^2/2$

Using $y'(0) = 0$, we get $A = 0$ so that $y'(x) = x$

Using $y'(1) = 0$, $y'(x) = x \Rightarrow 0 = 1$, which is absurd. Hence the system (iv) has no solution.

Consider the following another inhomogeneous system related to (iii)

$$y''(x) = \cos \pi x, \quad 0 < x < 1; \quad y'(0) = 0, \quad y'(1) = 0 \quad \dots (v)$$

Solving $y''(x) = \cos \pi x$, $y'(x) = (1/\pi) \sin \pi x + A$, $y(x) = -(1/\pi^2) \cos \pi x + Ax + B$

Using $y'(0) = 0$, we get $A = 0$. So, $y'(x) = (1/\pi) \sin \pi x$, which also satisfies the condition $y'(1) = 0$. Hence the solution of (v) is given by $y(x) = B - (1/\pi^2) \cos \pi x$, where B is an arbitrary constant. Hence the system (v) has the infinite set of solutions.

A *modified Green function* which is appropriate to the above mentioned exceptional situations will be discussed in Art. 11.12.

11.5 SOLVED EXAMPLES BASED ON CONSTRUCTION OF GREEN'S FUNCTION (REFER ART. 11.2 AND 11.4)

Ex. 1. Find the Green's function of the boundary value problem. $y'' = 0$, $y(0) = y(l) = 0$.

Sol. Given boundary value problem is $y'' = 0$, ... (1)

with the boundary conditions : $y(0) = 0$... (2a)

and $y(l) = 0$ (2b)

The general solution of (1) is $y(x) = Ax + B$ (3)

Putting $x = 0$ in (3) and using (2a), we get $B = 0$... (4)

Next, putting $x = l$ in (3) and using (2b), we get $0 = Al + B$ (5)

Solving (4) and (5), we get $A = B = 0$. Hence (3) yields only the trivial solution $y(x) \equiv 0$ for the given boundary value problem. Therefore, the Green's function exists and is given by

$$G(x, t) = \begin{cases} a_1 x + a_2, & 0 \leq x < t, \\ b_1 x + b_2, & t < x \leq l. \end{cases} \quad \dots (6)$$

In addition to the above property (6), the proposed Green's function must satisfy the following three properties :

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 t + b_2 = a_1 t + a_2 \quad \text{or} \quad (b_1 - a_1)t + (b_2 - a_2) = 0 \quad \dots (7)$$

(ii) The derivative of G has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x) = \text{coeff of the highest order derivative in (1)} = 1$.

$$\therefore (\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1 \quad \text{or} \quad b_1 - a_1 = -1. \quad \dots (8)$$

(iii) $G(x, t)$ must satisfy the boundary conditions (2a) and (2b), that is,

$$G(0, t) = 0 \quad \text{so that} \quad a_2 = 0 \quad \dots (9)$$

$$\text{and} \quad G(l, t) = 0 \quad \text{so that} \quad b_1 l + b_2 = 0. \quad \dots (10)$$

$$\text{Using (8), (7) becomes} \quad -t + b_2 - a_2 = 0. \quad \dots (11)$$

Solving (8), (9), (10) and (11), we have

$$a_2 = 0, \quad b_2 = t, \quad b_1 = -t/l, \quad a_1 = 1 - (t/l).$$

$$\therefore a_1 x + a_2 = \left(1 - \frac{t}{l}\right)x = \frac{x}{l}(l - t). \quad \text{and} \quad b_1 x + b_2 = -\frac{t}{l}x + t = \frac{t}{l}(l - x).$$

Substituting the above values in (6), the required Green's function of the given boundary problem is given by

$$G(x, t) = \begin{cases} (x/l)(l - t), & 0 \leq x < t \\ (t/l)(l - x), & t < x \leq l. \end{cases} \quad \dots (12)$$

Ex.2. Find the Green's function for the boundary value problem

$$d^2 y / dx^2 + \mu^2 y = 0, \quad y(0) = y(1) = 0. \quad (\text{Kanpur 2008})$$

Sol. Given boundary value problem is

$$y'' + \mu^2 y = 0 \quad \text{or} \quad (D^2 + \mu^2) y = 0, \quad \text{where } D \equiv d / dx \quad \dots (1)$$

$$\text{with the boundary conditions} \quad y(0) = 0, \quad \dots (2a)$$

$$\text{and} \quad y(1) = 0. \quad \dots (2b)$$

$$\text{The auxiliary equation of (1) is} \quad D^2 + \mu^2 = 0 \quad \text{so that} \quad D = \pm i \mu.$$

$$\text{Hence the general solution of (1) is} \quad y(x) = A \cos \mu x + B \sin \mu x. \quad \dots (4)$$

$$\text{Putting } x = 0 \text{ in (3) and using B.C. (2a), we get} \quad A = 0.$$

Again, putting $x = 1$ in (3) and B.C. (2b), we get

$$0 = A \cos \mu + B \sin \mu. \quad \dots (5)$$

Solving (4) and (5), we get $A = B = 0$. Hence (3) yields only the trivial solution $y(x) \equiv 0$ for the given boundary value problem. Therefore, the Green's function exists and is given by

$$G(x, t) = \begin{cases} a_1 \cos \mu x + a_2 \sin \mu x, & 0 \leq x < t, \\ b_1 \cos \mu x + b_2 \sin \mu x, & t < x \leq 1. \end{cases} \quad \dots (6)$$

In addition to the above property (6), the proposed Green's function must also satisfy the following three properties.

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 \cos \mu t + b_2 \sin \mu t = a_1 \cos \mu t + a_2 \sin \mu t$$

$$\text{or} \quad (b_1 - a_1) \cos \mu t + (b_2 - a_2) \sin \mu t = 0. \quad \dots (7)$$

11.12 Applications of integral equations and Green's function to ordinary differential equation

(ii) The derivative of G has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x) =$ coefficient of highest order derivative in (1) $= 1$.

$$\therefore (\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1$$

or $\mu(-b_1 \sin \mu t + b_2 \cos \mu t) - \mu(-a_1 \sin \mu t + a_2 \cos \mu t) = -1$

or $-(b_1 - a_1) \sin \mu t + (b_2 - a_2) \cos \mu t = -(1/\mu)$ (8)

(iii) $G(x, t)$ must satisfy the boundary conditions (2a) and (2b), that is,

$$G(0, t) = 0 \quad \text{so that} \quad a_1 = 0. \quad \dots (9)$$

and $G(1, t) = 0 \quad \text{so that} \quad b_1 \cos \mu + b_2 \sin \mu = 0. \quad \dots (10)$

Let $b_1 - a_1 = C_1$ and $b_2 - a_2 = C_2$ (11)

Then (7) and (8) may be written as

$$C_1 \cos \mu t + C_2 \sin \mu t + 0 = 0 \quad \dots (12)$$

and $-C_1 \sin \mu t + C_2 \cos \mu t + (1/\mu) = 0. \quad \dots (13)$

Solving (12) and (13) by cross-multiplication rule, we have

$$\frac{C_1}{(1/\mu) \sin \mu t} = \frac{C_2}{-(1/\mu) \cos \mu t} = \frac{1}{\cos^2 \mu t + \sin^2 \mu t}$$

so that $C_1 = (1/\mu) \sin \mu t$ and $C_2 = -(1/\mu) \cos \mu t$.

$\therefore b_1 - a_1 = (1/\mu) \sin \mu t$, by (11) ... (14)

and $b_2 - a_2 = -(1/\mu) \cos \mu t$, by (11) ... (15)

Solving (9), (10), (14), (15), we have

$$a_1 = 0, b_1 = \frac{1}{\mu} \sin \mu t, b_2 = -\frac{\sin \mu t \cos \mu}{\mu \sin \mu}, a_2 = -\frac{\sin \mu t \cos \mu}{\mu \sin \mu} + \frac{\cos \mu t}{\mu} = -\frac{\sin \mu (t-1)}{\mu \sin \mu} \quad \dots (16)$$

Now, $a_1 \cos \mu x + a_2 \sin \mu x = -\frac{\sin \mu (t-1) \sin \mu x}{\mu \sin \mu}$

and $b_1 \cos \mu x + b_2 \sin \mu x = -\frac{\sin \mu t \cos \mu \sin \mu x}{\mu \sin \mu} + \frac{\sin \mu t \cos \mu x}{\mu} = -\frac{\sin \mu t \sin \mu (x-1)}{\mu \sin \mu}$.

Substituting the above values in (6), the required Green's function is given by

$$G(x, t) = \begin{cases} -\frac{\sin \mu (t-1) \sin \mu x}{\mu \sin \mu}, & 0 \leq x < t, \\ -\frac{\sin \mu t \sin \mu (x-1)}{\mu \sin \mu}, & t < x \leq 1. \end{cases}$$

Ex. 3. Construct Green's function for the differential equation $xy'' + y' = 0$ for the following conditions. $y(x)$ is bounded as $x \rightarrow 0$, and $y(1) = \alpha$, $y'(1) = 0$, $\alpha \neq 0$. [Kanpur 2006, Meerut 2005, 10]

Sol. Given boundary value problem is : $xy'' + y' = 0$ or $x^2 y'' + xy' = 0$

$$\text{or } x^2(d^2y/dx^2) + x(dy/dx) = 0 \quad \text{or } (x^2D^2 + xD)y = 0, \quad D \equiv d/dx. \quad \dots (1)$$

$$\text{with the boundary conditions : } y(x) \text{ is bounded as } x \rightarrow 0 \quad \dots (2a)$$

$$\text{and } y(1) = \alpha y'(1), \quad \alpha \neq 0. \quad \dots (2b)$$

To solve linear homogeneous differential equation (1), we proceed by the usual method.

$$\text{Put } x = e^z \quad \text{so that } \log x = z. \quad \dots (3)$$

$$\text{Then } xD = D_1 \quad \text{and} \quad x^2D^2 = D_1(D_1 - 1), \quad \text{where } D_1 \equiv d/dz. \quad \dots (4)$$

$$\text{Using (4), (1) reduces to } [D_1(D_1 - 1) + D_1]y = 0 \quad \text{or} \quad D_1^2y = 0 \quad \dots (5)$$

The auxiliary equation of (5) is $D_1^2 = 0$ so that $D_1 = 0, 0$. Hence the solution is

$$y = Az + B \quad \text{or} \quad y(x) = A \log x + B, \text{ by (3)} \quad \dots (6)$$

$$\text{Now, from (6), } y'(x) = A/x. \quad \dots (7)$$

$$\text{From (6) and (7), } y(1) = B \quad \text{and} \quad y'(1) = A.$$

$$\text{Putting these values in (2b), we get } B = \alpha A$$

In view of B.C. (2a), we must take $A = 0$ in (6). Then $A = 0$ and $B = \alpha A \Rightarrow B = 0$.

Thus $A = B = 0$. Hence (6) yields only the trivial solution $y(x) \equiv 0$. Therefore, the Green's function exists and is given by

$$G(x, t) = \begin{cases} a_1 \log x + a_2, & 0 \leq x < t, \\ b_1 \log x + b_2, & t \leq x < 1. \end{cases} \quad \dots (8)$$

In addition to the above property (8), the proposed Green's function must also satisfy the following three properties :

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 \log t + b_2 = a_1 \log t + a_2 \quad \text{or} \quad (b_1 - a_1) \log t + (b_2 - a_2) = 0. \quad \dots (9)$$

(ii) The derivative of G has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x) =$ coefficient of the highest power of x in the given differential equation $= x$. Thus, we have

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -(1/t) \quad \text{or} \quad (b_1/t) - (a_1/t) = -(1/t)$$

$$\text{or} \quad (b_1 - a_1) = -1. \quad \dots (10)$$

(iii) $G(x, t)$ given by (8) must satisfy the boundary condition (2a) and (2b).

For (2a), $G(x, t)$ must be bounded as $x \rightarrow 0$ i.e., $(a_1 \log x + a_2)$ must be bounded at $x \rightarrow 0$, which is possible only if we take

$$a_1 = 0, \quad \dots (11)$$

For (2b), we must have $G(1, t) = \alpha G'(1, t)$

$$\text{i.e., } b_1 \log 1 + b_2 = \alpha \times (b_1/x)_{x=1} \quad \text{or} \quad b_2 = \alpha b_1. \quad \dots (12)$$

Solving (9), (10), (11) and (12), we get

$$a_1 = 0, \quad b_1 = -1, \quad b_2 = -\alpha \quad \text{and} \quad a_2 = -\alpha - \log t.$$

Substituting the above values in (8), the required Green's function is

$$G(x, t) = \begin{cases} -\alpha - \log t, & 0 \leq x < t, \\ -\alpha - \log x, & t < x < 1. \end{cases}$$

Ex. 4. Construct Green's function for the homogeneous boundary value problem

$$d^4 y / dx^4 = 0, \quad y(0) = y'(0) = y(1) = y'(1) = 0. \quad [\text{Kanpur 2009; Meerut 2000, 03, 04}]$$

Sol. Given $d^4 y / dx^4 = 0$ or $D^4 y = 0, D = d / dx, \dots (1)$
 with the boundary conditions $y(0) = 0, \dots (2a)$
 $y'(0) = 0, \dots (2b)$
 $y(1) = 0, \dots (2c)$
 and $y'(1) = 0. \dots (2d)$

The auxiliary equation of (1) is $D^4 = 0$ so that $D = 0, 0, 0, 0.$
 Hence the general solution of (1) is $y(x) = A + Bx + Cx^2 + Dx^3, \dots (3)$
 where A, B, C, D are arbitrary constants.

From (3), $y'(x) = B + 2Cx + 3Dx^2. \dots (4)$

Putting $x = 0$ and $x = 1$ in (3) and using (2a) and (2c), we have

$y(0) = A$ and $y(1) = A + B + C + D$
 or $A = 0 \dots (5)$

and $A + B + C + D = 0. \dots (6)$

Putting $x = 0$ and $x = 1$ in (4) and using (2b) and (2d), we have

$y'(0) = B$ and $y'(1) = B + 2C + 3D$
 or $B = 0 \dots (7)$

and $B + 2C + 3D = 0 \dots (8)$

Solving (5), (6), (7) and (8), we get $A = B = C = D = 0$. Hence (3) yields only the trivial solution $y(x) \equiv 0$ for the given boundary value problem. Therefore, the Green's functions exists and is given by

$$G(x, t) = \begin{cases} a_1 + a_2 x + a_3 x^2 + a_4 x^3, & 0 \leq x < t, \\ b_1 + b_2 x + b_3 x^2 + b_4 x^3, & t < x \leq 1. \end{cases} \dots (9)$$

In addition to the above property (9), the proposed Green's function must also satisfy the following three properties :

(i) $G(x, t), \partial G / \partial x, \partial^2 G / \partial x^2$ are continuous at $x = t$. Thus, we have

$$b_1 + b_2 t + b_3 t^2 + b_4 t^3 = a_1 + a_2 t + a_3 t^2 + a_4 t^3, \dots (10)$$

$$b_2 + 2b_3 t + 3b_4 t^2 = a_2 + 2a_3 t + 3a_4 t^2, \dots (11)$$

and $2b_2 + 6b_4 t = 2a_2 + 6a_4 t. \dots (12)$

(ii) The derivative $\partial^3 G / \partial x^3$ of G has a discontinuity of magnitude $-1 / p_0(t)$ at the point $x = t$, where $p_0(x) =$ coefficient of the highest order derivative in (1) = 1. Thus,

$$(\partial^3 G / \partial x^3)_{x=t+0} - (\partial^3 G / \partial x^3)_{x=t-0} = -1$$

or $6b_4 - 6a_4 = -1. \dots (13)$

(iii) Green's function must satisfy the boundary conditions (2a), (2b), (2c) and (2d). Thus we have,

$$G(0, t) = 0 \quad \text{so that} \quad a_1 = 0, \quad \dots (14)$$

$$G'(0, t) = 0 \quad \text{so that} \quad a_2 = 0, \quad \dots (15)$$

$$G(1, t) = 0 \quad \text{so that} \quad b_1 + b_2 + b_3 + b_4 = 0, \quad \dots (16)$$

$$G'(1, t) = 0 \quad \text{so that} \quad b_2 + 2b_3 + 3b_4 = 0 \quad \dots (17)$$

Let $C_k = b_k - a_k, k = 1, 2, 3, 4$.

Then (10), (11), (12) and (13) may be re-written as

$$C_1 + C_2 t + C_3 t^2 + C_4 t^3 = 0, \quad \dots (19)$$

$$C_2 + 2C_3 t + 3C_4 t^2 = 0, \quad \dots (20)$$

$$2C_3 + 6C_4 t = 0 \quad \dots (21)$$

$$6C_4 = -1, \quad \dots (22)$$

Solving (19), (20), (21) and (22), we have

$$C_1 = t^3/6, \quad C_2 = -t^2/2, \quad C_3 = t/2, \quad C_4 = -1/6. \quad \dots (23)$$

In view of (18), relations (23) take the forms :

$$b_1 - a_1 = t^3/6, \quad \dots (24)$$

$$b_2 - a_2 = -t^2/2, \quad \dots (25)$$

$$b_3 - a_3 = t/2, \quad \dots (26)$$

$$b_4 - a_4 = -1/6. \quad \dots (27)$$

and

Solving (14), (15), (16), (17), (24), (25), (26) and (27), we have

$$\left. \begin{aligned} a_1 = 0, \quad a_2 = 0, \quad a_3 = -t/2 + t^2 - t^3/2; \quad a_4 = 1/6 - t^2/2 + t^3/3 \\ b_1 = -t^3/6, \quad b_2 = -t^2/2, \quad b_3 = -t^3/2 + t^2, \quad b_4 = -t^2/2 + t^3/3 \end{aligned} \right\} \quad \dots (28)$$

Substituting the above values given by (28) in (9), the required Green's function is

$$G(x, t) = \begin{cases} x^2 \left(t^2 - \frac{t}{2} - \frac{t^3}{2} \right) + x^3 \left(\frac{1}{6} - \frac{t^2}{2} + \frac{t^3}{3} \right), & 0 \leq x < t \\ t^2 \left(x^2 - \frac{x}{2} - \frac{x^3}{2} \right) + t^3 \left(\frac{1}{6} - \frac{x^2}{2} + \frac{x^3}{3} \right), & t < x \leq 1. \end{cases} \quad \dots (29)$$

Ex.5. Construct the Green's function for the boundary value problem $d^2y/dx^2 - y = 0$; $y(0) = y'(0)$ and $y(l) + \lambda y'(l) = 0$.

Sol. Given boundary value problem is

$$d^2y/dx^2 - y = 0 \quad \text{or} \quad (D^2 - 1)y = 0, \quad \text{where } D \equiv d/dx. \quad \dots (1)$$

with boundary conditions :

$$y(0) = y'(0) \quad \dots (2a)$$

and

$$y(l) + \lambda y'(l) = 0 \quad \dots (2b)$$

The general solution of (1) is

$$y(x) = Ae^x + Be^{-x} \quad \dots (3)$$

From (3),

$$y'(x) = Ae^x - Be^{-x} \quad \dots (4)$$

From (3) and (4), $y(0) = A + B$, $y'(0) = A - B$, $y(l) = Ae^l + Be^{-l}$ and $y'(l) = Ae^l - Be^{-l}$

$$\therefore (2a) \Rightarrow A + B = A - B \quad \text{so that} \quad B = 0 \quad \dots (5)$$

$$\text{Using } B = 0, (2b) \Rightarrow Ae^l + \lambda Ae^l = 0 \quad \text{giving} \quad A = 0.$$

Thus, $A = B = 0$ and so (3) yields only the trivial solution $y(x) = 0$ for the given boundary value problem. Therefore, the Green's function $G(x, t)$ exists and is given by

$$G(x, t) = \begin{cases} a_1 e^x + a_2 e^{-x}, & 0 \leq x < t \\ b_1 e^x + b_2 e^{-x}, & t < x \leq l \end{cases} \quad \dots (6)$$

where a_1, a_2, b_1 and b_2 are unknown functions of t . In addition to the above property (6), the proposed Green's function must satisfy the following three properties :

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 e^t + b_2 e^{-t} = a_1 e^t + a_2 e^{-t} \quad \text{so that} \quad (b_1 - a_1) e^t + (b_2 - a_2) e^{-t} = 0 \quad \dots (7)$$

(ii) The derivative of $G(x, t)$ has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x)$ = coefficient of the highest order derivative in (1) = 1, that is,

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1/p_0(t)$$

$$\text{or} \quad (b_1 e^t - b_2 e^{-t}) - (a_1 e^t - a_2 e^{-t}) = -1 \quad \text{or} \quad (b_1 - a_1) e^t - (b_2 - a_2) e^{-t} = -1 \quad \dots (8)$$

(iii) The Green's function must satisfy the prescribed boundary conditions (2a) and (2b).

$$\text{Now, (2a)} \Rightarrow (G(x, t))_{x=0} = (\partial G / \partial x)_{x=0} \Rightarrow [a_1 e^x + a_2 e^{-x}]_{x=0} = [a_1 e^x - a_2 e^{-x}]_{x=0}$$

$$\text{Thus,} \quad a_1 + a_2 = a_1 - a_2 \quad \text{so that} \quad a_2 = 0 \quad \dots (9)$$

$$\text{and (2b)} \Rightarrow (G(x, t))_{x=l} + \lambda (\partial G / \partial x)_{x=l} = 0 \Rightarrow [b_1 e^x + b_2 e^{-x}]_{x=l} + \lambda [b_1 e^x - b_2 e^{-x}]_{x=l} = 0$$

$$\text{Thus,} \quad b_1 e^l + b_2 e^{-l} + \lambda (b_1 e^l - b_2 e^{-l}) = 0 \quad \text{or} \quad (1 + \lambda) b_1 e^l + (1 - \lambda) b_2 e^{-l} = 0 \quad \dots (10)$$

Setting $b_1 - a_1 = C_1$ and $b_2 - a_2 = C_2$, (7) and (8) yield

$$C_1 e^t + C_2 e^{-t} = 0 \quad \text{and} \quad C_1 e^t - C_2 e^{-t} = -1$$

$$\text{Solving for } C_1 \text{ and } C_2, \quad C_1 = -(1/2) \times e^{-t} \quad \text{and} \quad C_2 = (1/2) \times e^t$$

$$\text{i.e.,} \quad b_1 - a_1 = -(1/2) \times e^{-t} \quad \text{and} \quad b_2 - a_2 = (1/2) \times e^t \quad \dots (11)$$

Solving (9), (10) and (11), we have

$$a_2 = 0, \quad b_2 = \frac{1}{2} e^t, \quad b_1 = -\frac{1}{2} \left(\frac{1-\lambda}{1+\lambda} \right) e^{t-2l}, \quad a_1 = -\frac{1}{2} \left(\frac{1-\lambda}{1+\lambda} \right) e^{t-2l} + \frac{1}{2} e^{-t}$$

Substituting these values in (6), the required Green's function $G(x, t)$ is given by

$$G(x, t) = \begin{cases} -\frac{1}{2} \left(\frac{1-\lambda}{1+\lambda} \right) e^{x+t-2l} + \frac{1}{2} e^{x-l}, & 0 \leq x < t \\ -\frac{1}{2} \left(\frac{1-\lambda}{1+\lambda} \right) e^{x+t-2l} + \frac{1}{2} e^{-x+l}, & t < x \leq l \end{cases}, \quad \text{where } |\lambda| \neq 0$$

EXERCISE

In each of the following boundary value problem examine whether a Green's function exists and if it does, construct it.

$$1. \quad y'' = 0; \quad y(0) = y(1), \quad y'(0) = y'(1).$$

[Ans. Since $y''(0) = 0$ has an infinity of solutions $y(x) = C$ satisfying the given boundary condition $y(0) = y(1)$ and $y'(0) = y'(1)$, it follows that Green's function does not exist for the given boundary value problem.]

$$2. \quad y'' = 0; \quad y(0) = 0, \quad y(1) = y'(1)$$

[Ans. No.]

$$3. \quad y'' + y' = 0; \quad y(0) = y(1), \quad y'(0) = y'(1).$$

[Ans. No.]

4. $y'' + y = 0$; $y(0) = y(\pi) = 0$. [Ans. No.]

5. $y''' = 0$; $y(0) = y(1) = 0$, $y'(0) + y'(1) = 0$. [Ans. No.]

6. $y'' = 0$; $y(0) = y'(1)$, $y'(0) = y(1)$. [Kanpur 2006, 10; Meerut 2008, 12]

Ans. $G(x, t) = \begin{cases} 1 - t + x(2 - t), & 0 \leq x < t \\ x(1 - t) + 1, & t < x \leq 1. \end{cases}$

7. $d^4y/dx^4 = 0$, $y(0) = y'(0) = y''(1) = y'''(1) = 0$. Ans. $G(x, t) = \begin{cases} (1/6) \times x^2 (x - 3t), & 0 \leq x < t \\ (1/6) \times t^2 (t - 3x), & t < x \leq 1. \end{cases}$

8. $y''' = 0$; $y(0) = y'(1) = 0$; $y'(0) = y(1)$.

Ans. $G(x, t) = \begin{cases} (1/2) \times (1 - t)(x - xt + 2t), & 0 \leq x < t \\ (1/2) \times t[x(2 - x)(2 - t) - t], & t < x \leq 1. \end{cases}$

9. $y''' = 0$; $y(0) = y(1) = 0$; $y'(0) = y'(1)$. Ans. $G(x, t) = \begin{cases} (x/2) \times (x - t)(1 - t), & 0 \leq x < t \\ (t/2) \times (t - x)(x - 1), & t < x \leq 1. \end{cases}$

10. $y'' - k^2y = 0$ ($k \neq 0$); $y(0) = y(1) = 0$. Ans. $G(x, t) = \begin{cases} -\frac{\sinh k(t-1) \sinh kx}{k \sinh k}, & 0 \leq x < t \\ -\frac{\sinh kt \sinh k(x-1)}{k \sinh k}, & t < x \leq 1. \end{cases}$

11. $y'' + y = 0$; $y(0) = y(1)$, $y'(0) = y'(1)$. Ans. $G(x, t) = \begin{cases} -\frac{1}{2} \cos\left(x - t + \frac{1}{2}\right) \operatorname{cosec} \frac{1}{2}, & 0 \leq x < t \\ -\frac{1}{2} \cos\left(t - x + \frac{1}{2}\right) \operatorname{cosec} \frac{1}{2}, & t < x \leq 1. \end{cases}$

12. $y''(0) = 0$; $y'(0) = Ay(0)$, $y'(1) = -By(1)$. Ans. $G(x, t) = \begin{cases} -\frac{(Ax+1)[B(t-1)-1]}{A+B+AB}, & 0 \leq x < t, \\ -\frac{(At+1)[B(x-1)-1]}{A+B+AB}, & t < x \leq 1. \end{cases}$

13. $x^2y'' + 2xy' = 0$; $y(x)$ is bounded for $x \rightarrow 0$, $y(1) = \alpha y'(1)$.

Ans. $G(x, t) = \begin{cases} (1/t) - \alpha - 1, & 0 \leq x < t, \\ (1/x) - \alpha - 1, & t < x \leq 1. \end{cases}$

14. $x^3(d^4y/dx^4) + 6x^2(d^3y/dx^3) + 6x(d^2y/dx^2) = 0$; $y(x)$ is bounded as $x \rightarrow 0$

and $y(1) + y'(1) = 0$. Ans. $G(x, t) = \begin{cases} \log_e t + 1 + \frac{x(t-1)^2}{2t} - t, & 0 \leq x < t, \\ \log_e x + 1 + \frac{t(x-1)^2}{2x} - x, & t < x \leq 1. \end{cases}$

15. $x^2y'' + xy' - y = 0$, $y(x)$ is bounded as $x \rightarrow 0$, $y(1) = 0$

Ans. $G(x, t) = \begin{cases} (x/2) \times \{(1/t^2) - 1\}, & 0 \leq x < t, \\ (1/2) \times \{(1/x) - x\}, & t < x \leq 1. \end{cases}$

16. $xy'' + y' - (1/x)y = 0$; $y(0)$ is finite, $y(1) = 0$. Ans. $G(x, t) = \begin{cases} (x/2) \times \{(1/t) - 1\}, & 0 \leq x < t, \\ (t/2) \times \{(1/x) - x\}, & t < x \leq 1. \end{cases}$

11.18 Applications of integral equations and Green's function to ordinary differential equation

17. $x^2 y'' + xy' - n^2 y = 0$, $y(0)$ is finite, $y(1) = 0$

$$\text{Ans. } G(x, t) = \begin{cases} (1/2nt) \times \{(x/t)^n - (xt)^n\}, & 0 \leq x < t \\ (t/2nt) \times \{(t/x)^2 - (xt)^n\}, & t < x \leq 1. \end{cases}$$

18. $x^2 (\log_e x - 1) y'' - xy' + y = 0$, $y(0)$ is finite, $y(1) = 0$.

$$\text{Ans. } G(x, t) = \begin{cases} \frac{x \log_e t}{t^2 (\log_e t - 1)^2}, & 0 \leq x < t \\ \frac{\log_e x}{t (\log_e t - 1)^2}, & t < x \leq 1. \end{cases}$$

19. $\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] = 0$; $y(0) = 0$, $y(1)$ is finite. $\text{Ans. } G(x, t) = \begin{cases} \frac{1}{2} \log_e \frac{1+x}{1-x}, & 0 < x \leq t \\ \frac{1}{2} \log_e \frac{1+t}{1-t}, & t < x \leq 1. \end{cases}$

20. $xy'' + y' = 0$; $y(0)$ is bounded, $y(l) = 0$. $\text{Ans. } G(x, t) = \begin{cases} \log_e (l/t), & 0 \leq x < t, \\ \log_e (l/x), & t < x \leq 1. \end{cases}$

11.6 SOLVED EXAMPLES BASED ON RESULT 1 OF ART. 11.3.

Ex.1. Using Green's function, solve the boundary value problem

$$y'' + y = x, \quad y(0) = y(\pi/2) = 0. \quad [\text{Kanpur 2007, Meerut 2006}]$$

Sol. Given boundary value problem $(y'' + y) - x = 0 \quad \dots (1)$

with the boundary conditions : $y(0) = y(\pi/2) = 0 \quad \dots (2)$

Consider the associated boundary value problem :

$$y'' + y = 0 \quad \text{or} \quad (D^2 + 1)y = 0, \quad D \equiv d/dx \quad \dots (3)$$

subject to the boundary conditions $y(0) = 0 \quad \dots (4a)$

and $y(\pi/2) = 0. \quad \dots (4b)$

We shall first find the Green's function of the above mentioned boundary value problem given by (3), (4a) and (4b).

The auxiliary equation of (3) is $D^2 + 1 = 0$ so that $D = \pm i$.

Hence the general solution of (3) is $y(x) = A \cos x + B \sin x. \quad \dots (5)$

Putting $x = 0$ in (5) and using B.C. (4a), we get $A = 0. \quad \dots (6)$

Putting $x = \pi/2$ in (5) and using B.C. (4b), we get $B = 0. \quad \dots (7)$

From (6) and (7), $A = B = 0$. Hence (5) yields only the trivial solution $y(x) \equiv 0$. Therefore, Green's function exists for the boundary value problem given by (3), (4a) and (4b) and it is given by

$$G(x, t) = \begin{cases} a_1 \cos x + a_2 \sin x, & 0 \leq x < t \\ b_1 \cos x + b_2 \sin x, & t < x \leq \pi/2. \end{cases} \quad \dots (8)$$

In addition to the above property (8), the proposed Green's functions must also satisfy the following three properties :

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 \cos t + b_2 \sin t = a_1 \cos t + a_2 \sin t$$

$$\text{or} \quad (b_1 - a_1) \cos t + (b_2 - a_2) \sin t = 0. \quad \dots (9)$$

(ii) The derivative of G has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x) =$ coefficient of the highest order derivative in (3) $= 1$, that is,

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1$$

$$\text{or} \quad -b_1 \sin t + b_2 \cos t - (-a_1 \sin t + a_2 \cos t) = -1$$

$$\text{or} \quad -(b_1 - a_1) \sin t + (b_2 - a_2) \cos t = -1. \quad \dots (10)$$

(iii) $G(x, t)$ must satisfy the boundary conditions (4a) and (4b), that is,

$$G(0, t) = 0 \quad \text{so that} \quad a_1 = 0 \quad \dots (11)$$

$$\text{and} \quad G(\pi/2, t) = 0 \quad \text{so that} \quad b_2 = 0. \quad \dots (12)$$

$$\text{Let} \quad b_1 - a_1 = C_1 \quad \text{and} \quad b_2 - a_2 = C_2. \quad \dots (13)$$

Then (9) and (10) may be written as

$$C_1 \cos t + C_2 \sin t + 0 = 0 \quad \dots (14)$$

$$\text{and} \quad -C_1 \sin t + C_2 \cos t + 1 = 0. \quad \dots (15)$$

Solving (14) and (15) by cross-multiplication method, we have

$$\frac{C_1}{\sin t} = \frac{C_2}{-\cos t} = \frac{1}{\cos^2 t + \sin^2 t}$$

$$\text{Hence} \quad C_1 = \sin t \quad \text{and} \quad C_2 = -\cos t.$$

$$\therefore \quad b_1 - a_1 = \sin t, \text{ by (13)} \quad \dots (16)$$

$$\text{and} \quad b_2 - a_2 = -\cos t, \text{ by (13)} \quad \dots (17)$$

Solving (11), (12), (16) and (17), we have

$$a_1 = 0, \quad b_2 = 0, \quad b_1 = \sin t, \quad a_2 = \cos t.$$

Substituting the above values in (6), we have

$$G(x, t) = \begin{cases} \cos t \sin x, & 0 \leq x < t \\ \sin t \cos x, & t < x \leq \pi/2. \end{cases} \quad \dots (18)$$

Then we known that the solution of the given boundary value problem (1)–(2) is given by

$$y(x) = \int_0^{\pi/2} G(x, t) \Phi(t) dt, \quad \dots (19)$$

where $\Phi(x) = -x$ so that $\Phi(t) = -t$. [Refer result 1 of Art. 11.3]

Hence the required solution is given by

$$\begin{aligned} y(x) &= - \int_0^{\pi/2} G(x, t) t dt = - \left[\int_0^x t G(x, t) dt + \int_x^{\pi/2} t G(x, t) dt \right] \\ &= - \int_0^x t \sin t \cos x dt - \int_x^{\pi/2} t \cos t \sin x dt, \text{ using (18)} \\ &= -\cos x \int_0^x t \sin t dt - \sin x \int_x^{\pi/2} t \cos t dt, \\ &= -\cos x \left[[-t \cos t]_0^x - \int_0^x (-\cos t) dt \right] - \sin x \left[[t \sin t]_x^{\pi/2} - \int_x^{\pi/2} \sin t dt \right] \end{aligned}$$

$$\begin{aligned}
&= -\cos x \left[-x \cos x + \left[\sin t \right]_0^x \right] - \sin x \left[\pi/2 - x \sin x + \left[\cos t \right]_x^{\pi/2} \right] \\
&= -\cos x (-x \cos x + \sin x) - \sin x \{ (\pi/2) - x \sin x - \cos x \} \\
&= x \cos^2 x - \sin x \cos x - (\pi/2) \times \sin x + x \sin^2 x + \sin x \cos x
\end{aligned}$$

Thus, $y(x) = x - (\pi/2) \times \sin x$.

Ex. 2. Using Green's function, solve the boundary value problem $y'' - y = x$, $y(0) = y(1) = 0$.
[Meerut 2004, 09]

Sol. Given $y'' - y = x$, $y(0) = y(1) = 0$.

Consider the associated boundary value problem :

$$y'' - y = 0 \quad \text{or} \quad (D^2 - 1)y = 0, \quad D \equiv d/dx \quad \dots (1)$$

with the boundary conditions : $y(0) = 0$ (2a)

and $y(1) = 0$ (2b)

The auxiliary equation of (1) is $D^2 - 1 = 0$ so that $D = \pm 1$.

Hence the general solution of (1) is $y(x) = A \cosh x + B \sinh x$ (3)

Putting $x = 0$ in (3) and using B.C. (2a), we get $A = 0$ (4)

Again, putting $x = 1$ in (3) and using B.C. (2b), we get

$$0 = A \cosh 1 + B \sinh 1. \quad \dots (5)$$

Solving (4) and (5), we get $A = B = 0$. Hence (3) yields only the trivial solution $y(x) \equiv 0$ for the boundary value problem given by (1), (2a) and (2b). Therefore, Green's function exists and is given by

$$G(x, t) = \begin{cases} a_1 \cosh x + a_2 \sinh x, & 0 \leq x < t, \\ b_1 \cosh x + b_2 \sinh x, & t < x \leq 1. \end{cases} \quad \dots (6)$$

In addition to the above property (6), the proposed Green's function must also satisfy the following three properties :

(i) $G(x, t)$ is continuous at $x = t$, this is,

$$b_1 \cosh t + b_2 \sinh t = a_1 \cosh t + a_2 \sinh t$$

or $(b_1 - a_1) \cosh t + (b_2 - a_2) \sinh t = 0$... (7)

(ii) The derivative of G has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x) =$ coefficient of the highest order derivative in (1) $= 1$, that is,

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1$$

or $b_1 \sinh t + b_2 \cosh t - (a_1 \sinh t + a_2 \cosh t) = -1$.

or $(b_1 - a_1) \sinh t + (b_2 - a_2) \cosh t + 1 = 0$... (8)

(iii) $G(x, t)$ must satisfy the boundary conditions (2a) and (2b), that is,

$$G(0, t) = 0 \quad \text{so that} \quad a_1 = 0 \quad \dots (9)$$

and $G(1, t) = 0$ so that $b_1 \cosh 1 + b_2 \sinh 1 = 0$ (10)

Let $b_1 - a_1 = C_1$ and $b_2 - a_2 = C_2$ (11)

Then (7) and (8) may be written as

$$C_1 \cosh t + C_2 \sinh t + 0 = 0 \quad \dots (12)$$

and $C_1 \sinh t + C_2 \cosh t + 1 = 0$ (13)

Solving (12) and (13) by cross-multiplication method, we get

$$\frac{C_1}{\sinh t} = \frac{C_2}{-\cosh t} = \frac{1}{\cosh^2 t - \sinh^2 t}.$$

Hence $C_1 = \sinh t$ and $C_2 = -\cosh t$. [$\because \cosh^2 t - \sinh^2 t = 1$]

$\therefore b_1 - a_1 = \sinh t$, by (11) ... (14)

and $b_2 - a_2 = -\cosh t$, by (11) ... (15)

Solving (9), (10), (14) and (15), we have

$$a_1 = 0, \quad b_1 = \sinh t, \quad a_2 = -\frac{\sinh t \cosh 1}{\sinh 1} + \cosh t, \quad b_2 = -\frac{\sinh t \cosh 1}{\sinh 1} \quad \dots (16)$$

Now, $a_1 \cosh x + a_2 \sinh x = 0 + \left[-\frac{\sinh t \cosh 1}{\sinh 1} + \cosh t \right] \sinh x$

$$= -\frac{\sinh x (\sinh t \cosh 1 - \cosh t \sinh 1)}{\sinh 1} = -\frac{\sinh x \sinh (t-1)}{\sinh 1}$$

and $b_1 \cosh x + b_2 \sinh x = \sinh t \cosh x - \frac{\sinh t \cosh 1}{\sinh 1} \sinh x$

$$= -\frac{\sinh t (\sinh x \cosh 1 - \cosh x \sinh 1)}{\sinh 1} = -\frac{\sinh t \sinh (x-1)}{\sinh 1}.$$

Substituting the above values in (6), the Green's function of the boundary value problem given by (1), (2a) and (2b) is given by

$$G(x, t) = \begin{cases} -\frac{\sinh x \sinh (t-1)}{\sinh 1}, & 0 \leq x < t, \\ -\frac{\sinh t \sinh (x-1)}{\sinh 1}, & t < x \leq 1. \end{cases} \quad \dots (17)$$

Hence the solution of the given boundary value problem $y'' - y - x = 0$, $y(0) = y(1) = 0$ is

$$y(x) = -\int_0^1 G(x, t) \Phi(t) dt \quad \text{or} \quad y(x) = -\int_0^1 G(x, t) t dt$$

[Refer result 1 of Art. 11.3. Here $\phi(x) = x$ so that $\phi(t) = t$]

or $y(x) = -\left[\int_0^x G(x, t) t dt + \int_x^1 G(x, t) t dt \right]$

$$= \int_0^x \frac{t \sinh t \sinh (x-1)}{\sinh 1} dt + \int_x^1 \frac{t \sinh x \sinh (t-1)}{\sinh 1} dt, \text{ using (17)}$$

$$= \frac{\sinh (x-1)}{\sinh 1} \int_0^x t \sinh t dt + \frac{\sinh x}{\sinh 1} \int_x^1 t \sinh (t-1) dt$$

$$= \frac{\sinh (x-1)}{\sinh 1} \left[[t \cosh t]_0^x - \int_0^x \cosh t dt \right]$$

$$+ \frac{\sinh x}{\sinh 1} \left[[t \cosh (t-1)]_x^1 - \int_x^1 \cosh (t-1) dt \right], \text{ integrating by parts}$$

$$= \frac{\sinh (x-1)}{\sinh 1} \left[x \cosh x - [\sinh t]_0^x \right] + \frac{\sinh x}{\sinh 1} \left[1 - x \cosh (x-1) - [\sinh (t-1)]_x^1 \right]$$

$$\begin{aligned}
&= \frac{\sinh(x-1)}{\sinh 1} (x \cosh x - \sinh x) + \frac{\sinh x}{\sinh 1} [1 - x \cosh(x-1) + \sinh(x-1)] \\
&= \frac{\sinh x}{\sinh 1} + \frac{x}{\sinh 1} [\sinh(x-1) \cosh x - \cosh(x-1) \sinh x] = \frac{\sinh x}{\sinh 1} + \frac{x}{\sinh 1} \sinh(x-1-x) \\
\therefore y(x) &= \frac{\sinh x}{\sinh 1} + \frac{x}{\sinh 1} \sinh(-1) = \frac{\sinh x}{\sinh 1} - \frac{x \sinh 1}{\sinh 1} \quad \text{or} \quad y(x) = \frac{\sinh x}{\sinh 1} - x,
\end{aligned}$$

which is the required solution of the given boundary value problem.

EXERCISE

1. Solve the following boundary-value problems using Green's functions :

(i) $y'' + \pi^2 y = \cos \pi x$; $y(0) = y(1)$, $y'(0) = y'(1)$. **Ans.** $y = (1/4\pi) \times (2x-1) \sin \pi x$

(ii) $y'' + y = x^2$; $y(0) = y(\pi/2) = 0$. **Ans.** $y = 2 \cos x + (2 - \pi^2/4) \sin x + x^2 - 2$

(iii) $y'' - y = 2 \sinh 1$; $y(0) = y(1) = 0$. **Ans.** $y = 2 [\sinh x - \sinh(x-1) - \sinh 1]$

(iv) $y'' - y = -2e^x$; $y(0) = y'(0)$, $y(1) + y'(1) = 0$. **Ans.** $y = \sinh x + e^x(1-x)$

(v) $xy'' + y' = x$; $y(1) = y(e) = 0$. **Ans.** $y = (1/4) \times [(1-e^2) \log_e x + x^2 - 1]$

(vi) $d^4 y / dx^4 = 1$; $y(0) = y'(0) = y''(1) = y'''(1) = 0$. **Ans.** $y = (x^2/24) \times (x^2 - 4x + 6)$

2. Transform the problem $d^4 y / dx^4 + \Phi(x) = 0$, $y(0) = y'(0) = y(1) = y'(1) = 0$ to the relation

$$y(x) = \int_0^1 G(x,t) \Phi(t) dt, \text{ where } G(x,t) = \begin{cases} (1/6) \times x^2 (1-t)^2 (2xt + x - 3t), & \text{when } x < t \\ (1/6) \times t^2 (1-x)^2 (2xt + t - 3x), & \text{when } x > t. \end{cases}$$

3. Find Green's function for the boundary value problem $(d^2 u / dx^2) - (du / dx) = x$, $u(0) = u(1) = 0$. Hence solve it. [Meerut 2002, 07]

11.7 SOLVED EXAMPLES BASED ON RESULT 2 OF ART. 11.3.

Ex.1. Reduce the boundary-value problem $y'' + \lambda y = x$, $y(0) = y(\pi/2) = 0$ to an integral equation.

Sol. Given boundary-value problem is $y'' + \lambda y - x = 0$, $y(0) = y(\pi/2) = 0$ (1)

We shall first find Green's function of the following associated boundary value problem :

$$y'' = 0 \quad \text{or} \quad D^2 y = 0, \quad D \equiv d/dx \quad \dots (2)$$

with boundary conditions. $y(0) = 0$... (3)

and $y(\pi/2) = 0$ (4)

The auxiliary equation of (2) is $D^2 = 0$ so that $D = 0, 0$. So the general solution of (2) is

$$y(x) = Ax + B. \quad \dots (5)$$

Putting $x = 0$ in (5) and using B.C. (3), we get $B = 0$ (6)

Next, putting $x = \pi/2$ in (5) and using B.C. (4), we get $0 = A \times (\pi/2) + B$ (7)

From (6) and (7), $A = B = 0$. Hence (5) yields only the trivial solution $y(x) \equiv 0$. Therefore, Green's function $G(x, t)$ exists for the associated boundary value problem given by (2), (3) and (4) and is given by

$$G(x, t) = \begin{cases} a_1 x + a_2, & 0 \leq x < t \\ b_1 x + b_2, & t < x \leq \pi/2. \end{cases} \quad \dots (8)$$

In addition to the above property (8), the proposed Green's function must also satisfy the following properties :

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 t + b_2 = a_1 t + a_2 \quad \text{or} \quad (b_1 - a_1) t + b_2 - a_2 = 0. \quad \dots (9)$$

(ii) The derivative of G has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x)$ = coefficient of the highest order derivative in (2) = 1, that is,

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1 \quad \text{or} \quad b_1 - a_1 = -1. \quad \dots (10)$$

(iii) $G(x, t)$ must satisfy the boundary condition (2) and (4), that is,

$$G(0, t) = 0 \quad \text{so that} \quad a_2 = 0 \quad \dots (11)$$

$$\text{and} \quad G(\pi/2, t) = 0 \quad \text{so that} \quad b_1(\pi/2) + b_2 = 0. \quad \dots (12)$$

$$\text{Using (10), (9) gives} \quad -t + b_2 - a_2 = 0 \quad \dots (13)$$

Solving (10), (11), (12) and (13), we have

$$a_2 = 0, \quad b_2 = t, \quad b_1 = -(2t/\pi), \quad a_1 = 1 - (2t/\pi).$$

$$\therefore \quad a_1 x + a_2 = \{1 - (2t/\pi)\} x \quad \text{and} \quad b_1 x + b_2 = \{1 - (2x/\pi)\} t.$$

Substituting the above values in (8), we have

$$G(x, t) = \begin{cases} \{1 - (2t/\pi)\} x, & 0 \leq x < t, \\ \{1 - (2x/\pi)\} t, & t < x \leq \pi/2. \end{cases} \quad \dots (14)$$

Comparing $y'' + \lambda y - x = 0$ with $y'' + \Phi(x) = 0$, we have

$$\Phi(x) = \lambda y(x) - x \quad \text{so that} \quad \Phi(t) = \lambda y(t) - t. \quad \dots (15)$$

Also, we known that, if $G(x, t)$ is Green's function of the boundary-value problem given by (2), (3), (4) then the boundary value problem (1) can be reduced to the following integral equation [refer equation (8) in result 2 of Art. 11.3]

$$y(x) = \int_0^{\pi/2} G(x, t) \Phi(t) dt = \int_0^{\pi/2} G(x, t) [\lambda y(t) - t] dt, \text{ using (15)}$$

$$\text{or} \quad y(x) = \lambda \int_0^{\pi/2} G(x, t) y(t) dt - \int_0^{\pi/2} t G(x, t) dt. \quad \dots (16)$$

Now, we have

$$\begin{aligned} \int_0^{\pi/2} t G(x, t) dt &= \int_0^x t G(x, t) dt + \int_x^{\pi/2} t G(x, t) dt \\ &= \int_0^x t^2 \left(1 - \frac{2x}{\pi}\right) dt + \int_x^{\pi/2} t x \left(1 - \frac{2t}{\pi}\right) dt, \text{ using (14)} \\ &= \left(1 - \frac{2x}{\pi}\right) \int_0^x t^2 dt + x \int_x^{\pi/2} \left(t - \frac{2t^2}{\pi}\right) dt = \left(1 - \frac{2x}{\pi}\right) \left[\frac{t^3}{3}\right]_0^x + x \left[\frac{t^2}{2} - \frac{2t^3}{3\pi}\right]_x^{\pi/2} \\ &= \left(1 - \frac{2x}{\pi}\right) \frac{x^3}{3} + x \left(\frac{\pi^2}{8} - \frac{\pi^2}{12} - \frac{x^2}{2} + \frac{2x^3}{3\pi}\right) = -\frac{x^3}{6} + \frac{\pi^2 x}{24} \end{aligned}$$

Substituting the above value in (16), we obtain the required integral equation

$$y(x) = \lambda \int_0^{\pi/2} G(x, t) y(t) dt + \frac{x^3}{6} - \frac{\pi^2 x}{24}, \text{ where } G(x, t) \text{ is given by (14).}$$

Ex.2. Reduce the following boundary value problem into an integral equation with help of Green's function : $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) + v_2 y(1) = 0$

Sol. Given boundary value problem is

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) + v_2 y(1) = 0 \quad \dots (1)$$

We shall first find the Green's function of the following associated boundary value problem :

$$\begin{array}{lll} y'' = 0 & \text{or} & D^2 y = 0, \text{ where } D \equiv d/dx \quad \dots (2) \\ \text{subject boundary condition} & & y(0) = 0 \quad \dots (3) \\ \text{and} & & y'(1) + v_2 y(1) = 0 \quad \dots (4) \end{array}$$

$$\text{The general solution of (2) is} \quad y(x) = Ax + B. \quad \dots (5)$$

$$\text{From (5),} \quad y'(x) = A \quad \dots (6)$$

$$\text{Putting } x = 0 \text{ in (5) and using (3), we have} \quad B = 0$$

$$\text{Hence (5) reduces to} \quad y(x) = Ax \quad \dots (7)$$

$$\text{From (6) and (7),} \quad y'(1) = A \quad \text{and} \quad y(1) = A$$

$$\text{Substituting these values in (1), we get} \quad A(1 + v_2) = 0 \quad \text{so that} \quad A = 0 \text{ if } v_2 \neq -1.$$

Thus, $A = B = 0$. Hence (5) yields only the trivial solution $y(x) = 0$. Therefore, Green's function $G(x, t)$ exist for the associated boundary value given by (2), (3) and (4), and is given by

$$G(x, t) = \begin{cases} a_1 x + a_2, & 0 \leq x < t \\ b_1 x + b_2, & t < x \leq 1. \end{cases} \quad \dots (8)$$

where a_1, a_2, b_1, b_2 are unknown functions of t . In addition to the above property (8), the proposed Green function must satisfy the following three properties :

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 t + b_2 = a_1 t + a_2 \quad \text{or} \quad (b_1 - a_1)t + (b_2 - a_2) = 0 \quad \dots (9)$$

(ii) The derivative of $G(x, t)$ has a discontinuity of the magnitude $-1/p_0(t)$ at $x = t$, where $p_0(x) = \text{coefficient of the highest order derivative in (2)} = 1$. Thus, we must have

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1 / p_0(x)$$

$$\text{i.e.,} \quad b_1 - a_1 = -1 \quad \dots (10)$$

(iii) The Green's function must satisfy the prescribed boundary conditions (2a) and (2b).

$$\text{Now,} \quad (2a) \Rightarrow (G(x, t))_{x=0} = 0 \Rightarrow a_2 = 0 \quad \dots (11)$$

$$\text{and} \quad (2b) \Rightarrow (\partial G / \partial x)_{x=1} + v_2 [G(x, t)]_{x=1} = 0 \Rightarrow [b_1]_{x=1} + v_2 [b_1 x + b_2]_{x=1} = 0$$

$$\text{Thus,} \quad b_1 + v_2 (b_1 + b_2) = 0. \quad \dots (12)$$

Solving (9), (10), (11) and (12), we have

$$a_2 = 0, \quad b_2 = t, \quad b_2 = -(v_2 t) / (1 + v_2), \quad a_1 = \{1 + v_2 (1 - t)\} / (1 + v_2), \quad v_2 \neq -1$$

$$\text{Now,} \quad a_1 x + a_2 = \frac{1 + v_2 (1 - t)}{1 + v_2} x \quad \text{and} \quad b_1 x + b_2 = \frac{1 + v_2 (1 - x)}{1 + v_2} t \quad \dots (13)$$

Using (13), (8) reduces to

$$G(x, t) = \begin{cases} [1 + v_2 (1 - t)]x / (1 + v_2), & 0 \leq x < t \\ [1 + v_2 (1 - x)]t / (1 + v_2), & t \leq x \leq 1 \end{cases} \quad \dots (14)$$

Comparing $y'' + \lambda y = 0$ with $y'' + \phi(x) = 0$, we have

$$\phi(x) = \lambda y(x) \quad \text{so that} \quad \phi(t) = \lambda y(t) \quad \dots (15)$$

Refer equation (8) in result 2 of Art. 11.3. According to this result, if $G(x, t)$ is Green's function of boundary value problem given by (2), (3) and (4), then the given boundary value problem (1) can be transformed into the following integral equation

$$y(x) = \int_0^1 G(x, t) \phi(t) dt \quad \text{i.e.,} \quad y(x) = \int_0^1 t G(x, t) y(t) dt,$$

where $G(x, t)$ is given by (14).

Ex.3. Reduce the boundary-value problem $y'' + y = x$, $y(0) = 0$, $y'(1) = 0$ to a Fredholm integral equation.

Sol. Given boundary value problem is

$$y'' + y - x = 0, \quad y(0) = 0, \quad y'(1) = 0. \quad \dots (1)$$

We shall first find Green's function of the following associated boundary-value problem

$$y'' = 0 \quad \text{or} \quad D^2 y = 0, \quad D \equiv d/dx. \quad \dots (2)$$

with boundary conditions

$$y(0) = 0 \quad \dots (3)$$

and

$$y'(1) = 0. \quad \dots (4)$$

The auxilliary equation of (2) is

$$D^2 = 0 \quad \text{so that} \quad D = 0, 0.$$

Hence the general solution of (2) is

$$y(x) = Ax + B. \quad \dots (5)$$

Putting $x = 0$ in (5) and using B.C. (3), we get

$$B = 0. \quad \dots (6)$$

From (5), we have

$$y'(x) = A.$$

Putting $x = 1$ in above relation and using B.C. (4), we get

$$A = 0. \quad \dots (7)$$

From (6) and (7), $A = B = 0$. Hence (5) yields only the trivial solution $y(x) \equiv 0$. Therefore, Green's function $G(x, t)$ exists for the associated boundary-value problem given by (2), (3) and (4) and is given by

$$G(x, t) = \begin{cases} a_1 x + a_2, & 0 \leq x < t \\ b_1 x + b_2, & t < x \leq 1 \end{cases} \quad \dots (8)$$

In addition to the above property (8), the proposed Green's function must also satisfy the following properties :

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 t + b_2 = a_1 t + a_2 \quad \text{or} \quad t(b_1 - a_1) + b_2 - a_2 = 0. \quad \dots (9)$$

(ii) The derivative of G has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x) = \text{coeff. of the highest order derivative in (2)} = 1$, that is

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1 \quad \text{or} \quad b_1 - a_1 = -1. \quad \dots (10)$$

(iii) $G(x, t)$ must satisfy the boundary conditions (3) and (4), that is,

$$G(0, t) = 0 \quad \text{so that} \quad a_2 = 0 \quad \dots (11)$$

and

$$G(1, t) = 0 \quad \text{so that} \quad b_1 = 0. \quad \dots (12)$$

From (9) and (10),

$$b_2 - a_2 = t. \quad \dots (13)$$

Solving (10), (11), (12) and (13), we have $a_2 = 0$, $b_1 = 0$, $b_2 = t$, $a_1 = 1$.

Substituting these values in (8), we get
$$G(x, t) = \begin{cases} x, & 0 \leq x < t \\ t & t < x \leq 1. \end{cases} \quad \dots (14)$$

Comparing $y'' + y - x = 0$ with $y'' + \Phi(x) = 0$, we get

$$\Phi(x) = y(x) - x \quad \text{so that} \quad \Phi(t) = y(t) - t. \quad \dots (15)$$

Also, we know that, if $G(x, t)$ is Green's function of the associated boundary-value problem (given by (2), (3) and (4)), then the given boundary value problem (1) can be reduced to the following Fredholm integral equation. [Refer equation (8) in result 2 of Art. 11.3]

$$y(x) = \int_0^1 G(x, t) \Phi(t) dt = \int_0^1 G(x, t) [y(t) - t] dt$$

or
$$y(x) = \int_0^1 G(x, t) y(t) dt - \int_0^1 t G(x, t) dt. \quad \dots (16)$$

Now, we have

$$\begin{aligned} \int_0^1 t G(x, t) dt &= \int_0^x t G(x, t) dt + \int_x^1 t G(x, t) dt = \int_0^x t^2 dt + \int_x^1 xt dt, \text{ using (14)} \\ &= \left[\frac{t^3}{3} \right]_0^x + x \left[\frac{t^2}{2} \right]_x^1 = \frac{x^3}{3} + \frac{x}{2} (1 - x^2) = \frac{1}{6} (3x - x^3). \end{aligned}$$

Substituting the above value in (16), we obtain the required Fredholm integral equation

$$y(x) = \int_0^1 G(x, t) y(t) dt - \frac{1}{6} (3x - x^3).$$

Ex.4. Transform the problem $y'' + xy = 1$, $y(0) = y(1) = 0$ to the integral equation

$$y(x) = \int_0^1 G(x, t) t y(t) dt - \frac{1}{2} x (1 - x),$$

where $G(x, t) = x(1 - t)$ when $x < t$ and $G(x, t) = t(1 - x)$ when $x > t$.

Sol. Given boundary-value problem is $y'' + xy - 1 = 0$, $y(0) = y(1) = 0$. $\dots (1)$

We shall first find Green's function of the following associated boundary-value problem

$$y'' = 0 \quad \text{or} \quad D^2 y = 0, \quad D \equiv d/dx \quad \dots (2)$$

with boundary conditions $y(0) = 0$. $\dots (3)$

and $y(1) = 0$. $\dots (4)$

The auxiliary equation of (2) in $D^2 = 0$ so that $D = 0$, 0. So general solution of (2) is

$$y(x) = Ax + B. \quad \dots (5)$$

Putting $x = 0$ in (5) and using B.C. (3), we get $0 = B$. $\dots (6)$

Putting $x = 1$ in (5) and using B.C., (4), we get $0 = A + B$. $\dots (7)$

From (6) and (7), $A = B = 0$. Hence (5) yields only the trivial solution $y(x) \equiv 0$. Therefore, Green's function $G(x, t)$ exists for the associated boundary-values problem (given by (2), (3) and (4)) and is given by

$$G(x, t) = \begin{cases} a_1 x + a_2, & 0 \leq x < t \\ b_1 x + b_2, & t < x \leq 1. \end{cases} \quad \dots (8)$$

In addition to the above property (8), the proposed Green's function must also satisfy the following three properties :

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 t + b_2 = a_1 t + a_2 \quad \text{or} \quad t(b_1 - a_1) + b_2 - a_2 = 0. \quad \dots (9)$$

(ii) The derivative of G has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x)$ = coefficient of the highest order derivative in (2) = 1, that is,

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1 \quad \text{or} \quad b_1 - a_1 = -1. \quad \dots (10)$$

(iii) $G(x, t)$ must satisfy the boundary conditions (3) and (4), that is,

$$G(0, t) = 0 \quad \text{so that} \quad a_2 = 0 \quad \dots (11)$$

$$\text{and} \quad G(1, t) = 0 \quad \text{so that} \quad b_1 + b_2 = 0. \quad \dots (12)$$

$$\text{From (9) and (10),} \quad b_2 - a_2 = t. \quad \dots (13)$$

$$\text{Solving (10), (11), (12), and (13), we have} \quad a_2 = 0, \quad b_2 = t, \quad b_1 = -t, \quad a_1 = 1 - t.$$

Substituting these values in (8), we have

$$G(x, t) = \begin{cases} x(1-t), & 0 \leq x < t \\ t(1-x), & t < x \leq 1. \end{cases} \quad \dots (14)$$

Comparing $y'' + xy - 1 = 0$ with $y'' + \Phi(x) = 0$, we get

$$\Phi(x) = xy(x) - 1 \quad \text{so that} \quad \phi(t) = t y(t) - 1. \quad \dots (15)$$

Also, we know that, if $G(x, t)$ is Green's function of the associated boundary-value problem (given by (2), (3) and (4)), then the given boundary value problem (1) can be reduced to the following integral equation. (refer equation (8) in result 2 of Art. 11.3)

$$y(x) = \int_0^1 G(x, t) \Phi(t) dt = \int_0^1 G(x, t) [ty(t) - 1] dt, \text{ using (15)}$$

$$\text{or} \quad y(x) = \int_0^1 G(x, t) t y(t) dt - \int_0^1 G(x, t) dt. \quad \dots (16)$$

$$\text{Now,} \quad \int_0^1 G(x, t) dt = \int_0^x G(x, t) dt + \int_x^1 G(x, t) dt = \int_0^x t(1-x) dt + \int_x^1 x(1-t) dt, \text{ by (14)}$$

$$= (1-x) \left[\frac{t^2}{2} \right]_0^x + x \left[t - \frac{t^2}{2} \right]_x^1 = (1-x) \frac{x^2}{2} + x \left(1 - \frac{1}{2} - x + \frac{x^2}{2} \right)$$

$$= \frac{x^2}{2} - \frac{x^3}{2} + \frac{x}{2} - x^2 + \frac{x^3}{2} = \frac{x}{2} - \frac{x^2}{2} = \frac{1}{2} x(1-x).$$

Substituting the above value in (16), we obtain the required integral equation

$$y(x) = \int_0^1 G(x, t) t y(t) dt - \frac{1}{2} x(1-x), \text{ where } G(x, t) \text{ is given by (14)}$$

Ex.5. Reduce the Bessel equation $x^2(d^2y/dx^2) + x(dy/dx) + (\lambda x^2 - 1)y = 0$ with end conditions $y(0) = 0, y(1) = 0$, to a Fredholm integral equation. [Kanpur 2005, 08; Meerut 2000, 04, 05]

$$\text{Sol. Given} \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \lambda x^2 y - y = 0 \quad \text{or} \quad x \frac{d^2 y}{dx^2} + \frac{dy}{dx} - \frac{y}{x} + \lambda xy = 0$$

$$\text{or} \quad \left[\frac{d}{dx} \left(x \frac{d}{dx} \right) + \left(-\frac{1}{x} \right) \right] y + \lambda xy = 0. \quad \dots (1)$$

Comparing (1) with $Ly + \lambda r(x)y(x) = f(x)$

$$\text{i.e.,} \quad \left[\frac{d}{dx} \left(p \frac{dy}{dx} \right) + q \right] y + \lambda r(x)y(x) = f(x). \quad \dots (2)$$

$$\text{we have} \quad p(x) = x, \quad q(x) = -1/x, \quad r(x) = x, \quad f(x) = 0. \quad \dots (3)$$

We shall first find Green's function for the boundary value problem

$$Ly = 0 \quad \text{or} \quad \left[\frac{d}{dx} \left(x \frac{d}{dx} \right) + \left(-\frac{1}{x} \right) \right] y = 0, \quad \dots (4)$$

$$\text{with boundary conditions} \quad y(0) = 0 \quad \dots (5)$$

$$\text{and} \quad y(1) = 0 \quad \dots (6)$$

We know that the required Green's function $G(x, t)$ is given by (refer Art. 11.4)

$$G(x, t) = \begin{cases} -(1/A) u(x) v(t), & 0 \leq x < t \\ -(1/A) u(t) v(x), & t \leq x \leq 1. \end{cases} \quad \dots (7)$$

$$\text{where } A \text{ is given by the Abel's formula} \quad A/p(t) = u(t)v'(t) - v(t)u'(t), \quad \dots (8)$$

$u(t)$ and $v(t)$ being two independent solution of (4) satisfying boundary conditions (5) and (6) respectively.

$$\text{Re-writing (4), we have} \quad \frac{d}{dx} \left(x \frac{dy}{dx} \right) - \frac{y}{x} = 0 \quad \text{or} \quad x \frac{d^2 y}{dx^2} + \frac{dy}{dx} - \frac{y}{x} = 0$$

$$\text{or} \quad x^2 (d^2 y/dx^2) + x (dy/dx) - y = 0 \quad \text{or} \quad (x^2 D^2 + xD - 1)y = 0, \quad D \equiv d/dx \quad \dots (9)$$

Equation (9) is homogeneous differential equation. To solve (9), we proceed as follows.

$$\text{Let} \quad x = e^z \quad \text{so that} \quad z = \log x. \quad \dots (10)$$

Also, let $D_1 \equiv d/dz$. Then, we have

$$xD \equiv D_1 \quad \text{and} \quad x^2 D^2 \equiv D_1 (D_1 - 1). \quad \dots (11)$$

Using (11), (9) reduces to

$$[D_1 (D_1 - 1) + D_1 - 1] y = 0 \quad \text{or} \quad (D_1^2 - 1) y = 0. \quad \dots (12)$$

$$\text{The auxiliary equation of (12) is} \quad D_1^2 - 1 = 0 \quad \text{so that} \quad D_1 = 1, -1$$

$$\text{Hence the general solution of (12) is} \quad y = C_1 e^z + C_2 e^{-z} = C_1 e^x + C_2 (e^x)^{-1}$$

$$\text{or} \quad y(x) = C_1 x + C_2 x^{-1} \text{ using (10)} \quad \dots (13)$$

As a solution for which B.C. (5) is satisfied, we may take $y = u(x)$, where

$$u(x) = x. \quad \dots (14)$$

and as a solution for which (6) is satisfied, we may take $y = v(x)$, where

$$v(x) = (1/x) - x. \quad \dots (15)$$

The Wronskian $W[u(x), v(x)]$ of $u(x)$ and $v(x)$ is given by

$$\begin{aligned} W[u(x), v(x)] &= \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} = \begin{vmatrix} x & (1/x) - x \\ 1 & -(1/x^2) - 1 \end{vmatrix} = x \left(-\frac{1}{x^2} - 1 \right) - \left(\frac{1}{x} - 1 \right) \\ &= -(1/x) - x - (1/x) + x = -(2/x) \neq 0, \end{aligned}$$

showing that $u(x)$ and $v(x)$ are independent solutions of (4).

$$\text{From (3), (14) and (15), we have} \quad p(t) = t, \quad u(t) = t, \quad v(t) = (1/t) - t. \quad \dots (16)$$

Substituting these values in (8), we have

$$\frac{A}{t} = t \frac{d}{dt} \left(\frac{1}{t} - t \right) - \left(\frac{1}{t} - 1 \right) \frac{dt}{dt} = t \left(-\frac{1}{t^2} - 1 \right) - \left(\frac{1}{t} - t \right)$$

$$\therefore A/t = -2/t \quad \text{so that} \quad A = -2. \quad \dots (17)$$

$$\text{Using (14), (15), (16) and (17),} \quad -\frac{u(x)v(t)}{A} = \frac{x}{2} \left(\frac{1}{t} - t \right) = \frac{x}{2t} (1 - t^2) \quad \dots (18)$$

$$\text{and} \quad -\frac{u(t)v(x)}{A} = \frac{t}{2} \left(\frac{1}{x} - x \right) = \frac{t}{2x} (1 - x^2). \quad \dots (19)$$

Using (18) and (19), (7) becomes

$$G(x, t) = \begin{cases} (x/2t) \times (1 - t^2), & 0 \leq x < t \\ (t/2x) \times (1 - x^2), & t < x \leq 1. \end{cases} \quad \dots (20)$$

Hence the boundary value problem $Ly + \lambda r(x)y(x) = f(x)$, $y(0) = y(1) = 0$
reduces to the integral equation (refer equation (12) in Art. 11.3)

$$y(x) = \lambda \int_0^1 G(x, t) r(t) y(t) dt - \int_0^1 G(x, t) f(t) dt$$

$$\text{or} \quad y(x) = \lambda \int_0^1 G(x, t) r(t) y(t) dt. \quad [\because f(x) = 0 \Rightarrow f(t) = 0, \text{ by (3)}]$$

Ex.6. (a) Show that the Green's function for the Bessel operator of order n ,

$$Ly = \frac{d}{dx} \left(x \frac{dy}{dx} \right) - \frac{n^2}{x} y, \quad (n \neq 0) \text{ relevant to the end conditions } y(0) = y(1) = 0, \text{ is of the form}$$

$$G(x, t) = \begin{cases} (x^n / 2nt^n) \times (1 - t^{2n}) & \text{when } x < t \\ (t^n / 2nx^n) \times (1 - x^{2n}) & \text{when } x > t, \end{cases}$$

(b) Use the result of part (a) to reduce the problem

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\lambda x^2 - n^2) y = 0, \quad y(0) = y(1) = 0 \text{ to an integral equation, when } n \neq 0.$$

Sol. Part (a) We shall find Green's function for the boundary-value problem

$$Ly = 0 \quad \text{or} \quad \left[\frac{d}{dx} \left(x \frac{d}{dx} \right) + \left(-\frac{n^2}{x} \right) \right] y = 0, \quad \dots (1)$$

$$\text{with the boundary conditions} \quad y(0) = 0 \quad \dots (2)$$

$$\text{and} \quad y(1) = 0. \quad \dots (3)$$

We know that the required Green's functions $G(x, t)$ is given by (refer Art. 11.4)

$$G(x, t) = \begin{cases} -(1/A) u(x) v(t), & 0 \leq x < t \\ -(1/A) u(t) v(x), & t < x \leq 1, \end{cases} \quad \dots (4)$$

$$\text{where } A \text{ is given by the Abel's formula} \quad A/p(t) = u(t)v'(t) - v(t)u'(t), \quad \dots (5)$$

$u(t)$ and $v(t)$ being two independent solutions of (1) satisfying boundary conditions (5) and (6) respectively.

$$\text{Comparing (1) with the differential operator} \quad L \equiv \frac{d}{dx} \left(p \frac{d}{dx} \right) + q, \quad \dots (6)$$

11.30 *Applications of integral equations and Green's function to ordinary differential equation*

we have $p(x) = x$ so that $p(t) = t$ (7)

Re-writing (1), we have

$$\frac{d}{dx} \left(x \frac{dy}{dx} \right) - \frac{n^2 y}{x} = 0 \quad \text{or} \quad x \frac{d^2 y}{dx^2} + \frac{dy}{dx} - \frac{n^2 y}{x} = 0$$

$$\text{or } x^2 (d^2 y / dx^2) + x (dy / dx) - n^2 y = 0 \quad \text{or} \quad (x^2 D^2 + xD - n^2) y = 0, \quad D \equiv d / dx \quad \dots (8)$$

which is homogeneous differential equation. To solve it, we proceed as follows :

$$\text{Let } x = e^z \quad \text{so that } z = \log x. \quad \dots (9)$$

Also, let $D_1 \equiv d / dz$. Then, we have

$$xD = D_1 \quad \text{and} \quad x^2 D^2 = D_1 (D_1 - 1). \quad \dots (10)$$

Using (10), (8) reduces to

$$[D_1 (D_1 - 1) + D_1 - n^2] y = 0 \quad \text{or} \quad (D_1^2 - n^2) y = 0. \quad \dots (11)$$

$$\text{The auxiliary equation of (11) is } D_1^2 - n^2 = 0 \quad \text{so that } D_1 = n, -n.$$

Hence the general solution of (11) is

$$y = c_1 e^{nz} + c_2 e^{-nz} = c_1 (e^z)^n + c_2 (e^z)^{-n}$$

$$\text{or } y(x) = c_1 x^n + c_2 x^{-n}, \text{ using (9)} \quad \dots (12)$$

As a solution for which boundary condition (2) is satisfied, we may take $y = u(x)$, where

$$u(x) = x^n, \quad \dots (13)$$

and as a solution for which boundary condition (3) is satisfied, we may take $y = v(x)$, where

$$v(x) = (1/x^n) - x^n. \quad \dots (14)$$

The Wronskian of $u(x)$ and $v(x)$ is given by

$$\begin{aligned} W[u(x), v(x)] &= \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} = \begin{vmatrix} x^n & (1/x^n) - x^n \\ nx^{n-1} & -(n/x^{n+1}) - nx^{n-1} \end{vmatrix} \\ &= -x^n \left(\frac{n}{x^{n+1}} + n x^{n-1} \right) - \frac{n}{x} + n x^{2n-1} = -\frac{n}{x} - nx^{2n-1} - \frac{n}{x} + nx^{2n-1} \neq 0, \end{aligned}$$

showing that $u(x)$ and $v(x)$ are two independent solutions of (1).

$$\text{From (13) and (14),} \quad u(t) = t^n \quad \text{and} \quad v(t) = (1/t^n) - t^n. \quad \dots (15)$$

Using (7) and (15), (5) becomes

$$\frac{A}{t} = t^n \left[-\frac{n}{t^{n+1}} - nt^{n-1} \right] - nt^{n-1} \left(\frac{1}{t^n} - t^n \right) \quad \text{or} \quad \frac{A}{t} = -\frac{n}{t} - nt^{2n-1} - \frac{n}{t} + nt^{2n-1}$$

$$\text{or } A/t = (-2n)/t \quad \text{so that } A = -2n. \quad \dots (16)$$

Using (13), (14), (15) and (16), we have

$$-\frac{u(x)v(t)}{A} = \frac{x^n}{2n} \left(\frac{1}{t^n} - t^n \right) = (x^n / 2nt^n) (1 - t^{2n}) \quad \dots (17)$$

$$\text{and } -\frac{u(t)v(x)}{A} = \frac{t^n}{2n} \left(\frac{1}{x^n} - x^n \right) = (t^n / 2nx^n) (1 - x^{2n}). \quad \dots (18)$$

$$\text{Using (17) and (18), (4) becomes } G(x, t) = \begin{cases} (x^n / 2nt^n) (1 - t^{2n}), & x < t, \\ (t^n / 2nx^n) (1 - x^{2n}), & x > t. \end{cases} \quad \dots (19)$$

which gives the required Green's function.

Part (b). Given differential equation is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\lambda x^2 - n^2) y = 0 \quad \text{or} \quad x \frac{dy}{dx} + \frac{dy}{dx} - \frac{n^2}{x} y + \lambda xy = 0$$

or

$$Ly + \lambda xy(x) = 0. \quad \dots (20)$$

Comparing (20) with

$$Ly + \lambda r(x) y(x) = f(x), \quad \dots (21)$$

here $r(x) = x$ and $f(x) = 0$ so that $r(t) = t$ and $f(t) = 0 \quad \dots (22)$

We know that, if $G(x, t)$ is Green's function of the associated boundary value problem $Ly = 0$, $y(0) = 0$, $y(1) = 0$, then the boundary value problem $Ly + \lambda r(x) y(x) = f(x)$ can be reduced to the integral equation (refer equation (12) in Art. 11.3)

$$y(x) = \lambda \int_0^1 G(x, t) r(t) dt - \int_0^1 G(x, t) f(t) dt$$

or

$$y(x) = \lambda \int_0^1 G(x, t) t y(t) dt, \quad [\because f(t) = 0, \text{ by (22)}].$$

which is the required integral equation.

EXERCISE

Reduce the following boundary-value problems to integral equations :

1. $y'' = \lambda y + x^2$; $y(0) = y(\pi/2) = 0$. **Ans.** $y(x) = \frac{x^4}{12} - \frac{\pi^3 x}{90} - \lambda \int_0^{\pi/2} G(x, t) y(t) dt$,

$$\text{where } G(x, t) = \begin{cases} (1 - 2t/\pi)x, & 0 \leq x < t \\ (1 - 2x/\pi)t, & t < x \leq \pi/2. \end{cases}$$

2. $y'' = \lambda y + e^x$; $y(0) = y(1) = 0$. **Ans.** $y(x) = e^x - ex + x - 1 - \lambda \int_0^1 G(x, t) y(t) dt$,

$$\text{where } G(x, t) = \begin{cases} (1 - t)x, & 0 \leq x < t \\ (1 - x)t, & t < x \leq 1. \end{cases}$$

3. $y'' + \lambda y = e^x$; $y(0) = y'(0)$, $y(1) = y'(1)$.

Ans. $y(x) = e^x + \lambda \int_0^1 G(x, t) y(t) dt$, where $G(x, t) = \begin{cases} -(1 + x)t, & 0 \leq x < t \\ -(1 + t)x, & t < x \leq 1. \end{cases}$

4. $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) + ky(1) = 0$. **Ans.** $y(x) = \frac{x}{1+k} + \lambda \int_0^1 G(x, t) y(t) dt$,

$$\text{where } G(x, t) = \begin{cases} \frac{1+k(1-t)}{1+k}x, & 0 \leq x < t, \\ \frac{1+k(1-x)}{1+k}t, & t < x \leq 1. \end{cases}$$

5. $y'' + \lambda y = 2x + 1$; $y(0) = y'(1)$, $y'(0) = y(1)$. **Ans.** $y(x) = (1/6) \times (2x^3 + 3x^2 - 17x - 5)$

$$+ \lambda \int_0^1 G(x, t) y(t) dt, \quad \text{where } G(x, t) = \begin{cases} (2-t)x + 1 - t, & 0 \leq x < t, \\ (1-t)x + 1, & t < x \leq 1. \end{cases}$$

6. $y'' + (\pi^2/4)y = \lambda y + \cos(\pi x/2)$; $y(-1) = y(1)$, $y'(-1) = y'(1)$. [Meerut 2005, 2008]

Ans. $y(x) = \frac{x}{\pi} \sin \frac{\pi x}{2} + \frac{2}{\pi^2} \cos \frac{\pi x}{2} - \lambda \int_{-1}^1 G(x,t) y(t) dt$, where $G(x,t) = \begin{cases} \frac{1}{\pi} \sin \frac{\pi}{2}(x-t), & -1 \leq x < t, \\ \frac{1}{\pi} \sin \frac{\pi}{2}(t-x), & t < x \leq 1. \end{cases}$

7. $y''' + \lambda y = 2x$; $y(0) = y(1) = 0$, $y'(0) = y'(1)$. Ans. $y(x) = (1/12) \times (x-1)(x^2 + x - 1) + \lambda \int_0^1 G(x,t) y(t) dt$, where $G(x,t) = \begin{cases} 1/2 \times x(t-x)(t-1), & 0 \leq x < t, \\ 1/2 \times t(t-x)(x-1), & t < x \leq 1. \end{cases}$

8. $dy^4/dx^4 = \lambda y + 1$; $y(0) = y'(0) = 0$, $y''(1) = y'''(1) = 0$.

Ans. $y(x) = \frac{x^2}{24}(x^2 - 4x + 6) - \lambda \int_0^1 G(x,t) y(t) dt$, where $G(x,t) = \begin{cases} x^2(x-3t)/6, & 0 \leq x < t \\ t^2(t-3x)/6, & t < x \leq 1. \end{cases}$

11.8 SOLVED EXAMPLES BASED ON RESULT 3 OF ART. 11.3. THE CASE OF NONHOMOGENEOUS END CONDITIONS.

Ex.1. Reduce the following boundary value problem into an integral equation

$$y'' + xy = 1, y(0) = 0, y(l) = 1.$$

Sol. Given boundary value problem is $y'' + xy = 1$, $y(0) = 0$, $y(l) = 1$ (1)

Notice that the prescribed end conditions $y(0) = 0$ and $y(l) = 1$ are not homogeneous because the boundary conditions are respectively 0 and 1, which are unequal. We, therefore, proceed as follows.

We first consider the associated boundary value problem with homogeneous boundary condition, that is,

$$y'' = 0 \quad \text{or} \quad D^2 y = 0, \quad D \equiv d/dx \quad \dots (2)$$

with boundary conditions $y(0) = 0$... (3)

and $y(l) = 0$ (4)

We now determine Green's function for the above associated boundary value problem.

The general solution of (2) is $y(x) = Ax + B$ (5)

Putting $x = 0$ and $x = l$ by turn in (5) and using B.C. (3) and (4), we have

$$0 = B \quad \text{and} \quad 0 = Al + B \quad \dots (6)$$

From (6), $A = B = 0$. Hence (5) yields only the trivial solution $y(x) \equiv 0$. Therefore, Green's function $G(x, t)$ exists and is given by

$$G(x,t) = \begin{cases} a_1 x + a_2, & 0 \leq x < t, \\ b_1 x + b_2, & t < x \leq l. \end{cases} \quad \dots (7)$$

In addition to the above property (7), the proposed Green's function must also satisfy the following properties :

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 t + b_2 = a_1 t + a_2 \quad \text{or} \quad (b_1 - a_1)t + b_2 - a_2 = 0. \quad \dots (8)$$

(ii) The derivative of G has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x) =$ coefficient of the highest order derivative in (2) = 1, that is,

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1 \quad \text{or} \quad b_1 - a_1 = -1 \quad \dots (9)$$

(iii) $G(x, t)$ must satisfy the boundary conditions (3) and (4), that is,

$$G(0, t) = 0 \quad \text{so that} \quad a_2 = 0. \quad \dots (10)$$

$$\text{and} \quad G(l, t) = 0 \quad \text{so that} \quad b_1 l + b_2 = 0. \quad \dots (11)$$

$$\text{Using (9), (8) gives} \quad b_2 - a_2 = t. \quad \dots (12)$$

Solving (9), (10), (11) and (12), we have $a_2 = 0$, $b_2 = t$, $b_1 = -t/l$, $a_1 = 1 - (t/l)$.

$$\therefore \quad a_1 x + a_2 = \left(1 - \frac{t}{l}\right)x = \frac{x}{l}(l - t) \quad \dots (13)$$

$$\text{and} \quad b_1 x + b_2 = -\frac{tx}{l} + t = \frac{t}{l}(l - x). \quad \dots (14)$$

Using (13) and (14), (7) reduces to

$$G(x, t) = \begin{cases} (x/l) \times (l - t), & 0 \leq x < t \\ (t/l) \times (l - x), & t < x \leq l. \end{cases} \quad \dots (15)$$

$$\text{Comparing (1) with} \quad y'' + \Phi(x) = 0, \quad y(0) = 0, \quad y(l) = 1, \quad \dots (16)$$

$$\text{here} \quad \Phi(x) = xy(x) - 1 \quad \text{so that} \quad \Phi(t) = ty(t) - 1. \quad \dots (17)$$

Then, we know that (16) can be reduced to the following integral equation (refer equation (13) in result 3 of Art. 11.3)

$$y(x) = P(x) + \int_0^l G(x, t) \Phi(t) dt \quad \text{or} \quad y(x) = P(x) + \int_0^l G(x, t) [ty(t) - 1] dt$$

$$\text{or} \quad y(x) = P(x) + \int_0^l G(x, t) t y(t) dt - \int_0^l G(x, t) dt, \quad \dots (18)$$

where $G(x, t)$ is given by (15) and $P(x)$ is the solution of the following boundary value problem with nonhomogeneous end conditions :

$$P''(x) = 0 \quad \text{or} \quad D^2 P(x) = 0, \quad D \equiv d/dx \quad \dots (19)$$

$$\text{with the boundary conditions} \quad P(0) = 0 \quad \text{and} \quad P(l) = 1. \quad \dots (20)$$

$$\text{The general solution of (19) is} \quad P(x) = A'x + B'. \quad \dots (21)$$

Putting $x = 0$ and $x = l$ successively in (21) and using (20), we get

$$0 = B' \quad \text{and} \quad 1 = lA' + B',$$

so that $B' = 0$ and $A' = 1/l$. Putting these values in (21), we have

$$P(x) = (x/l). \quad \dots (22)$$

$$\text{Again,} \quad \int_0^l G(x, t) dt = \int_0^x G(x, t) dt + \int_x^l G(x, t) dt$$

$$\begin{aligned} &= \int_0^x \frac{t}{l}(l - x) dt + \int_x^l \frac{x}{l}(l - t) dt = \frac{l - x}{2} \left[\frac{t^2}{2} \right]_0^x + \frac{x}{l} \left[lt - \frac{t^2}{2} \right]_x^l, \text{ using (15)} \\ &= \frac{l - x}{2l} x^2 + \frac{x}{l} \left[l^2 - \frac{l^2}{2} - lx + \frac{x^2}{2} \right] = \frac{x}{l} \left(\frac{xl}{2} - \frac{x^2}{2} + \frac{l^2}{2} - lx + \frac{x^2}{2} \right) = \frac{x}{2}(l - x). \quad \dots (23) \end{aligned}$$

$$\text{Using (22) and (23) is (18), we get} \quad y(x) = \frac{x}{l} - \frac{x}{2}(l - x) + \int_0^l G(x, t) ty(t) dt,$$

which is the required integral equation, where $G(x, t)$ is given by (15).

Ex.2. Transform the problem $d^2 y/dx^2 + y = x$, $y(0) = 1$, $y'(l) = 0$ to a Fredholm integral equation.

11.34 *Applications of integral equations and Green's function to ordinary differential equation*

Sol. Given $y'' + y = x$, $y(0) = 1$, $y'(1) = 0$, ... (1)
which possesses nonhomogeneous end conditions.

We consider the associated boundary value problem with homogeneous boundary conditions,

$$y'' = 0 \quad \text{or} \quad D^2 y = 0, \quad d \equiv d/dx. \quad \dots (2)$$

$$\text{with boundary conditions} \quad y(0) = 0 \quad \dots (3)$$

$$\text{and} \quad y'(1) = 0. \quad \dots (4)$$

We now determine Green's function for the above associated boundary-value problem.

$$\text{The general solution of (2) is} \quad y(x) = Ax + B. \quad \dots (5)$$

$$\text{From (5),} \quad y'(x) = A. \quad \dots (5)'$$

$$\text{Putting } x = 0 \text{ in (5) and using B.C. (3), we get} \quad 0 = B. \quad \dots (6)$$

$$\text{Next, putting } x = 1 \text{ in (5)' and using B.C. (4), we get} \quad 0 = A. \quad \dots (6)'$$

From (6) and (6)', $A = B = 0$. Hence (5) yields only the trivial solution $y(x) \equiv 0$. Therefore, Green's functions $G(x, t)$ exists and is given by

$$G(x, t) = \begin{cases} a_1 x + a_2, & 0 \leq x < t \\ b_1 x + b_2, & t < x \leq 1. \end{cases} \quad \dots (7)$$

In addition to the above property (7), the proposed Green's function must also satisfy the following properties.

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 t + b_2 = a_1 t + a_2 \quad \text{or} \quad (b_1 - a_1)t + b_2 - a_2 = 0. \quad \dots (8)$$

(ii) The derivative of G has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x) = \text{coefficient of the highest order derivative in (2)} = 1$, that is

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1 \quad \text{or} \quad b_1 - a_1 = -1. \quad \dots (9)$$

(iii) $G(x, t)$ must satisfy the boundary conditions (3) and (4), that is,

$$G(0, t) = 0 \quad \text{so that} \quad a_2 = 0 \quad \dots (10)$$

$$\text{and} \quad G'(1, t) = 0 \quad \text{so that} \quad b_1 = 0. \quad \dots (11)$$

$$\text{Using (9), (8) reduces to} \quad b_2 - a_2 = t. \quad \dots (12)$$

$$\text{Solving (9), (10), (11) and (12), we have} \quad a_2 = 0, b_1 = 0, a_1 = 1, b_2 = t.$$

$$\therefore \quad a_1 x + a_2 = x \quad \dots (13)$$

$$\text{and} \quad b_1 x + b_2 = t. \quad \dots (14)$$

$$\text{Using (13) and (14), (7) reduces to} \quad G(x, t) = \begin{cases} x, & 0 \leq x < t \\ t, & t < x \leq 1. \end{cases} \quad \dots (15)$$

$$\text{Comparing (1) with} \quad y'' + \Phi(x) = 0, \quad y(0) = 1, \quad y'(1) = 0, \quad \dots (16)$$

$$\text{here} \quad \Phi(x) = y(x) - x \quad \text{so that} \quad \Phi(t) = y(t) - t. \quad \dots (17)$$

Then, we know that (16) can be reduced to the following integral equation

$$y(x) = P(x) + \int_0^1 G(x, t) \Phi(t) dt$$

$$\text{or} \quad y(x) = P(x) + \int_0^1 G(x, t) \{y(t) - t\} dt$$

$$\text{or} \quad y(x) = P(x) + \int_0^1 G(x, t) y(t) dt - \int_0^1 t G(x, t) dt, \quad \dots (18)$$

where $G(x, t)$ is given by (15) and $P(x)$ is the solution of the following boundary value problem with nonhomogeneous end conditions :

$$P''(x) = 0 \quad \text{or} \quad D^2P(x) = 0, \quad D \equiv d/dx. \quad \dots (19)$$

$$\text{with the boundary conditions} \quad P(0) = 1 \text{ and} \quad P'(1) = 0. \quad \dots (20)$$

$$\text{The general solution of (19) is} \quad P(x) = A'x + B' \quad \dots (21)$$

$$\text{From (21),} \quad P'(x) = A'. \quad \dots (22)$$

$$\text{Putting } x = 0 \text{ in (21) and using B.C. } P(0) = 1, \text{ we get } 1 = B'. \quad \dots (23)$$

$$\text{Putting } x = 1 \text{ in (22) and using B.C. } P'(1) = 0, \text{ we get} \quad 0 = A'. \quad \dots (24)$$

$$\text{Using (23) and (24), (21) gives} \quad P(x) = 1. \quad \dots (25)$$

$$\begin{aligned} \text{Now,} \quad \int_0^1 t G(x, t) dt &= \int_0^x t G(x, t) dt + \int_x^1 t G(x, t) dt \\ &= \int_0^x t^2 dt + \int_x^1 xt dt = \left[\frac{t^3}{3} \right]_0^x + x \left[\frac{t^2}{2} \right]_x^1 = \frac{x^3}{3} + \frac{x}{2} (1 - x^2). \end{aligned} \quad \dots (26)$$

Using (25) and (26) in (18), we get

$$y(x) = 1 + \int_0^1 G(x, t) y(t) dt - \frac{x^3}{3} - \frac{x}{2} (1 - x^2)$$

$$\text{or} \quad y(x) = \frac{1}{6} (x^3 - 3x + 6) + \int_0^1 G(x, t) y(t) dt,$$

which is the required integral equation.

Ex.4. Reduce the following boundary value problem to a Fredholm integral equation :

$$y'' + \lambda xy = 1, \quad y(0) = 0, \quad y(l) = 0.$$

$$\text{Sol. Given boundary value problem is} \quad y'' + \lambda xy = 1, \quad y(0) = 0, \quad y(l) = 0. \quad \dots (1)$$

Since the prescribed end conditions $y(0) = 0, y(l) = 1$ are not homogeneous, we proceed as explained in result 3 of Art 11.3.

Consider the associated boundary value problem with homogeneous boundary conditions, namely,

$$y'' = 0 \quad \text{or} \quad D^2y = 0, \text{ where} \quad D \equiv d/dx \quad \dots (2)$$

$$\text{with boundary conditions} \quad y(0) = 0 \quad \dots (3)$$

$$\text{and} \quad y(l) = 0 \quad \dots (4)$$

We now proceed to find the Green's function of the above associated boundary value problem.

$$\text{The general solution of (2) is} \quad y = Ax + B. \quad \dots (5)$$

Putting $x = 0$ and $x = l$ by turn in (5) and using (3) and (4), we have

$$0 = B \quad \text{and} \quad 0 = Al + B \quad \dots (6)$$

Solving (6), $A = B = 0$. Hence (5) yields only the trivial solution $y(x) = 0$. Therefore, Green's function $G(x, t)$ exists and is given by

$$G(x, t) = \begin{cases} a_1 x + a_2, & 0 \leq x < t \\ b_1 x + b_2, & t < x \leq l \end{cases} \quad \dots (7)$$

In addition to the above property (7), the proposed Green's function must also satisfy the following properties :

(i) $G(x, t)$ is continuous at $x = t$, that is,

$$b_1 t + b_2 = a_1 t + a_2 \quad \text{or} \quad (b_1 - a_1) t + b_2 - a_2 = 0. \quad \dots (8)$$

(ii) The derivative of G has a discontinuity of magnitude $-1/p_0(t)$ at the point $x = t$, where $p_0(x)$ = coefficient of the highest order in (2) = 1, that is,

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1 \quad \text{or} \quad b_1 - a_1 = -1 \quad \dots (9)$$

(iii) $G(x, t)$ must satisfy the boundary conditions (3) and (4), that is

$$G(0, t) = 0 \quad \text{so that} \quad a_2 = 0 \quad \dots (10)$$

$$\text{and} \quad G(l, t) = 0 \quad \text{so that} \quad b_1 l + b_2 = 0 \quad \dots (11)$$

$$\text{Using (9), (8) gives} \quad b_2 - a_2 = t. \quad \dots (12)$$

$$\text{Solving (9), (10), (11) and (12), we have} \quad a_2 = 0, b_2 = t, b_1 = -t/l \text{ and } a_1 = 1 - (t/l)$$

$$\text{Hence} \quad a_1 x + a_2 = (1 - t/l)x = (x/l)(l - t) \quad \dots (13)$$

$$\text{and} \quad b_1 x + b_2 = -(t/l)x + t = (t/l)(l - x) \quad \dots (14)$$

Using (13) and (14), (7) reduces to

$$G(x, t) = \begin{cases} (x/l) \times (l - t), & 0 \leq x < t \\ (t/l) \times (l - x), & t \leq x \leq l \end{cases} \quad \dots (15)$$

$$\text{Comparing (1) with} \quad y'' + \phi(x) = 0, \quad y(0) = 0, \quad y(l) = 1 \quad \dots (16)$$

$$\text{here} \quad \phi(x) = \lambda xy(x) - 1 \quad \text{so that} \quad \phi(t) = \lambda t y(t) - 1 \quad \dots (17)$$

Then (16) can be reduced to the following integral equation (refer equation (13) in result 3 of Art. 11.3)

$$y(x) = P(x) + \int_0^l G(x, t) \phi(t) dt \quad \text{or} \quad y(x) = P(x) + \int_0^l G(x, t) \{\lambda t y(t) - 1\} dt$$

$$\text{or} \quad y(x) = P(x) + \lambda \int_0^l t G(x, t) y(t) dt - \int_0^l G(x, t) dt \quad \dots (18)$$

where $P(x)$ is the solution of the following boundary value problem with non-homogeneous boundary conditions :

$$P''(x) = 0 \quad \text{or} \quad D^2 P(x) = 0 \quad \dots (19)$$

$$\text{with the boundary conditions} \quad P(0) = 0 \text{ and } P(l) = 1 \quad \dots (20)$$

$$\text{The general solution of (19) is} \quad P(x) = A'x + B'. \quad \dots (21)$$

Putting $x = 0$ and $x = l$ successively in (21) and using (20), we have

$$0 = B' \quad \text{and} \quad 1 = lA' + B' \quad \text{giving} \quad B' = 0 \quad \text{and} \quad A' = 1/l.$$

$$\text{Substituting these values in (21),} \quad P(x) = (x/l) \quad \dots (22)$$

$$\text{Now, } \int_0^l G(x, t) dt = \int_0^x G(x, t) dt + \int_x^l G(x, t) dt = \int_0^x \frac{t}{l} (l - x) dt + \int_x^l \frac{x}{l} (l - t) dt, \text{ by (15)}$$

$$\begin{aligned} &= \frac{l-x}{l} \left[\frac{t^2}{2} \right]_0^x + \frac{x}{l} \left[lt - \frac{t^2}{2} \right]_x^l = \frac{l-x}{2l} x^2 + \frac{x}{l} \left(l^2 - \frac{l^2}{2} - lx + \frac{x^2}{2} \right) \\ &= \frac{x}{l} \left(\frac{xl}{2} - \frac{x^2}{2} + \frac{l^2}{2} - lx + \frac{x^2}{2} \right) = \frac{x}{2} (l - x) \end{aligned} \quad \dots (23)$$

Using (22) and (23) in (18), we have

$$y(x) = \frac{x}{l} + \lambda \int_0^l t G(x, t) y(t) dt - \frac{x}{2} (l - x)$$

$$\text{or} \quad y(x) = (x/2l) \times (2 - l^2 + xl) + \lambda \int_0^l t G(x, t) y(t) dt$$

which is the required Fredholm integral equation.

11.9 LINEAR INTEGRAL EQUATIONS IN CAUSE AND EFFECT. THE INFLUENCE FUNCTION.

Linear integral equations arise most frequently in physical problems as a result of the possibility of superimposing the effects due to several causes. To understand the procedure, let x and t be variables. Suppose each of these variables takes on all values in a certain common interval or region Ω . For example, x and t may be regarded as each representing position (in space of one, two or three dimensions) and time. Furthermore, let a distribution of causes be active over the region Ω , and let us suppose that we wish to study the resultant distribution of effects in Ω .

If the *effect* at x due to a *unit* cause concentrated at t is denoted by the function $G(x, t)$, then the differential effect at x due to a uniform distribution of cause of *intensity* $c(t)$ over an elementary region $(t, t + dt)$ is given by $c(t) G(x, t) dt$. It follows that the effect $e(x)$ at x , due to a distribution of causes $c(t)$ over the entire region Ω is given by the integral

$$e(x) = \int_{\Omega} G(x, t) c(t) dt, \quad \dots (1)$$

provided that superposition is valid, that is, if the effect due to the sum of two different causes is (exactly or approximately) the sum of the effects due to each of the causes.

The function $G(x, t)$, representing the effect at x due to a unit concentrated cause at t , is said to be the *influence function* of the problem. Clearly, influence function is either identical with or proportional to the Green's function defined in Art. 11.2, when the definition is applicable.

If the distribution of causes is prescribed, and if the influence function is known, (1) may be used to find the effect by direct integration. On the other hand suppose we wish to find a distribution of causes which can produce a known or required effect distribution then for this purpose (1) can be treated as a Fredholm integral equation of the first kind to find out c . Also the kernel of the integral equation is the influence function $G(x, t)$ of the problem.

Now, let us consider another type of physical problem. Suppose that the problem under consideration prescribes neither the cause nor the effect separately, but required that they satisfy a certain linear relation of the form

$$c(x) = f(x) + \lambda e(x), \quad \dots (2)$$

where f is a given function or zero and λ is a constant.

Eliminating the effect e from (1) and (2), we obtain the following Fredholm integral equation of second kind for finding the cause distribution :

$$c(x) = f(x) + \lambda \int_{\Omega} G(x, t) c(t) dt. \quad \dots (3)$$

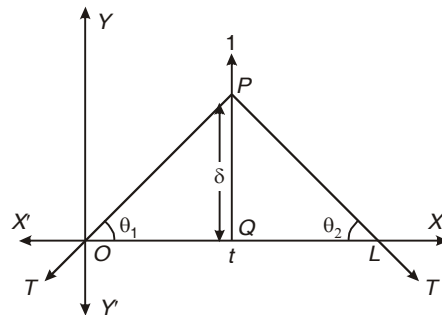
Similarly, by eliminating the cause c from (1) and (2), we obtain the following Fredholm integral equation of the second kind for finding the effect distribution.

$$e(x) = \int_{\Omega} G(x, t) f(t) dt + \lambda \int_{\Omega} G(x, t) e(t) dt. \quad \dots (4)$$

Both cause and effect can be obtained by solving either (3) or (4), and using (2).

In order to understand such derivations, consider the following simple example :

Let us examine small deflections of a string fixed at a point O ($x = 0$) and L ($x = l$), under a *loading distribution* of *intensity* $p(x)$.



Assume that the string is initially so tightly stretched that nonuniformity of the tension, due to small *deflections*, can be neglected. If a unit concentrated load is applied in the y direction at an arbitrary point t , the string will then be deflected into two linear parts OP and LP intersecting at P ($x = t$). Assume that (approximately) uniform tension T acts along PO as well as PL . Then, considering equilibrium of string in y -direction, we obtain

$$T \sin \theta_1 + T \sin \theta_2 = 1. \quad \dots (5)$$

But for small deflections (and slopes), we have the following results (approximately).

$$\left. \begin{aligned} \sin \theta_1 &= \tan \theta_1 = \frac{PQ}{OQ} = \frac{\delta}{t} \\ \sin \theta_2 &= \tan \theta_2 = \frac{PQ}{LQ} = \frac{\delta}{l-t} \end{aligned} \right\}, \quad \dots (6)$$

and

where δ is the maximum deflection of the string, at the loaded point t . Substituting the above approximate values of $\sin \theta_1$ and $\sin \theta_2$ in (5), we obtain

$$T \left(\frac{\delta}{t} + \frac{\delta}{l-t} \right) = 1 \quad \text{so that} \quad \delta = \frac{1}{Tl} t (l-t). \quad \dots (7)$$

The equation of linear part OP , joining $P(t, \delta)$ and $O(0, 0)$ is given by

$$y = \frac{\delta - 0}{t - 0} x \quad \text{or} \quad y = \delta \frac{x}{t}. \quad \dots (8)$$

Again, the equation of linear part LP , joining $P(t, \delta)$ and $L(l, 0)$ is given by

$$y - 0 = \frac{\delta - 0}{t - l} (x - l) \quad \text{or} \quad y = \delta \frac{l-x}{l-t}. \quad \dots (9)$$

Using (8) and (9), the equation of the corresponding deflection curve OPL is given by

$$y = \begin{cases} \delta (x/t), & \text{when } x < t. \\ \delta [(l-x)/(l-t)], & \text{when } x > t. \end{cases} \quad \dots (10)$$

Using (7), (10) shows that the influence function $G(x, t)$ (for small deflection) is given by

$$G(x, t) = \begin{cases} (x/Tl)(l-t), & \text{when } x < t, \\ (t/Tl)(l-x), & \text{when } x > t. \end{cases} \quad \dots (11)$$

Hence, by superposition, the deflection $y(x)$ due to a loading distribution $p(x)$ is given by

$$y(x) = \int_0^l G(x, t) p(t) dt. \quad \dots (12)$$

If the deflection is prescribed, (12) is an integral equation of the first kind for finding the necessary loading distribution.

Again, assume that the string is rotating uniformly about the x -axis, with angular velocity ω , and that in addition a continuous distribution of loading $g(x)$ is imposed in the direction radially outward from the axis of revolution. Let the linear mass density of the string be denoted by $\rho(x)$. Then the total effective load intensity is given by

$$p(x) = \omega^2 \rho(x) y(x) + g(x). \quad \dots (13)$$

Using (13), (12) reduces to

$$y(x) = \omega^2 \int_0^l G(x, t) \rho(t) y(t) dt + \int_0^l G(x, t) g(t) dt. \quad \dots (14)$$

Re-writing (14), we have

$$y(x) = \omega^2 \left[\int_0^x G(x, t) \rho(t) y(t) dt + \int_x^l G(x, t) \rho(t) y(t) dt \right] + \int_0^x G(x, t) g(t) dt + \int_x^l G(x, t) g(t) dt$$

$$\text{or } y(x) = \omega^2 \left[\int_0^x \frac{t}{Tl} (l-x) \rho(t) y(t) dt + \int_x^l \frac{x}{Tl} (l-t) \rho(t) y(t) dt \right] \\ + \int_0^x \frac{t}{Tl} (l-x) g(t) dt + \int_x^l \frac{x}{Tl} (l-t) g(t) dt, \text{ using (11)} \quad \dots (15)$$

$$\text{From (15), } y(0) = 0 \quad \text{and} \quad y(l) = 0 \quad \dots (16)$$

Differentiating both sides of (15) w.r.t. 'x' and using Leibnitz's rule, we have

$$y'(x) = \omega^2 \left[\int_0^x \left(-\frac{t}{Tl} \right) \rho(t) y(t) dt + \frac{x(l-x)}{Tl} \rho(x) y(x) \right] + \int_x^l \frac{l-t}{Tl} \rho(t) y(t) dt - \frac{x(l-x)}{Tl} \rho(x) y(x) \\ + \int_0^x \left(-\frac{t}{Tl} \right) g(t) dt + \frac{x(l-x)}{Tl} g(x) + \int_x^l \frac{l-t}{Tl} g(t) dt - \frac{x(l-x)}{Tl} g(x)$$

$$\text{or } y'(x) = \omega^2 \left[\int_0^x \left(-\frac{t}{Tl} \right) \rho(t) y(t) dt + \int_x^l \frac{l-t}{Tl} \rho(t) y(t) dt \right] \\ + \int_0^x \left(-\frac{t}{Tl} \right) g(t) dt + \int_x^l \frac{l-t}{Tl} g(t) dt. \quad \dots (17)$$

Differentiating both sides of (17) w.r.t. 'x' and using Leibnitz's rule as before, we have

$$y''(x) = \omega^2 \left[\left(-\frac{x}{Tl} \right) \rho(x) y(x) - \frac{l-x}{Tl} \rho(x) y(x) \right] + \left(-\frac{x}{Tl} \right) g(x) - \frac{l-x}{Tl} g(x).$$

$$\text{or } y''(x) = -(\omega^2 / T) \rho(x) y(x) - (1/T) g(x). \quad \dots (18)$$

From (16) and (18), we find that the integral equation (14) can be reduced the following boundary value problem

$$T(d^2y/dx^2) + \rho \omega^2 y + g = 0, \quad y(0) = 0, \quad y(l) = 0. \quad \dots (19)$$

$$\text{Re-writing (19), we have} \quad Ly + \Phi = 0, \quad \dots (20)$$

$$\text{where } L = T(d^2/dx^2) \quad \text{and} \quad \Phi = \rho \omega^2 y + g \quad \dots (21)$$

We known that the Green's function of the problem would then be that function which satisfies $T(d^2y/dx^2) = 0$ except at $x = t$, which vanishes when $x = 0$ and $x = l$, which is continuous at $x = t$, and for which the radical force resultant $T(dy/dx)$ decreases abruptly by unity at $x = t$. These are exactly the conditions which determines the *influence function*.

EXERCISE

1. Define 'influence function.' Prove that the deflection $\phi(x)$ due to a loading distribution of intensity $p(x)$ is given as $\phi(x) = \int_0^t G(x, t) p(t) dt$, where $G(x, t)$ is the influence function.

2. Define influence function and determine the deflection $\phi(x)$ of a string due to a loading distribution of intensity $p(x)$.

3. Define influence function.

[Meerut 2004, 05]

11.10 GREEN'S FUNCTION APPROACH FOR CONVERTING AN INITIAL VALUE PROBLEM INTO AN INTEGRAL EQUATION

Consider the following initial value problem

$$\frac{d}{dx} \left(p \frac{dy}{dx} \right) + qy = f(x) \quad \dots (1)$$

$$y(a) = 0, \quad y'(a) = 0 \quad \dots (2)$$

Let
$$L \equiv \frac{d}{dx} \left(p \frac{d}{dx} \right) + q = p \frac{d^2}{dx^2} + \frac{dp}{dx} \frac{d}{dx} + q, \quad \dots (3)$$

which is a self-adjoint differential operator. Here the function $p(x)$ is continuously differentiable and positive and $q(x)$ and $f(x)$ are continuous in a given interval (a, b) .

The associated homogeneous second-order equation

$$Ly = 0, \quad \text{i.e.,} \quad \frac{d}{dx} \left(p \frac{dy}{dx} \right) + qy = 0 \quad \dots (4)$$

has exactly two linearly independent solutions $u(x)$ and $v(x)$ which are twice differentiable in the interval $a < x < b$. Any other solution of (4) is a linear combination of $u(x)$ and $v(x)$, i.e., $y(x) = c_1 u(x) + c_2 v(x)$, where c_1 and c_2 are constants.

For the self adjoint operator L , the Green's formula is given by (refer Art. 10.5, Chapter 10)

$$\int_a^b (vLu - uLv) dx = [p(x)(vu' - uv')]_a^b \quad \dots (5)$$

In order to convert the initial value problem (1)—(2) into an integral equation, we consider the function $w(x)$ given by

$$w(x) = u(x) \int_a^x v(t) f(t) dt - v(x) \int_a^x u(t) f(t) dt. \quad \dots (6)$$

Differentiating both sides of (6) w.r.t. 'x' we have,

$$w'(x) = u'(x) \int_a^x v(t) f(t) dt + u(x) \frac{d}{dx} \int_a^x v(t) f(t) dt - v'(x) \int_a^x u(t) f(t) dt - v(x) \frac{d}{dx} \int_a^x u(t) f(t) dt$$

or
$$w'(x) = u'(x) \int_a^x v(t) f(t) dt + u(x) v(x) f(x) - v'(x) \int_a^x u(t) f(t) dt - v(x) u(x) f(x), \text{ by Leibnitz's rule (see Art. 1.13)}$$

or
$$w'(x) = u'(x) \int_a^x v(t) f(t) dt - v'(x) \int_a^x u(t) f(t) dt \quad \dots (7)$$

From (6) and (7), we have
$$w(a) = w'(a) = 0 \quad \dots (8)$$

Now, u and v are solutions of (4)

$$\begin{aligned} \Rightarrow \quad \frac{d}{dx} (pu') + qu &= 0 & \text{and} & \quad \frac{d}{dx} (pv') + qv = 0 \\ \Rightarrow \quad \frac{d}{dx} (pu') &= -qu & \text{and} & \quad \frac{d}{dx} (pv') = -qv \end{aligned} \quad \dots (9)$$

Using the value of $w'(x)$ given by (7), we have

$$\frac{d}{dx} (pw') = \frac{d}{dx} \left[p(x) u'(x) \int_a^x v(t) f(t) dt - p(x) v'(x) \int_a^x u(t) f(t) dt \right]$$

$$\begin{aligned}
&= \frac{d}{dx} (pu') \int_a^x v(t) f(t) dt + pu' \frac{d}{dx} \int_a^x v(t) f(t) dt \\
&\quad - \frac{d}{dx} (pv') \int_a^x u(t) f(t) dt - pv' \frac{d}{dx} \int_a^x u(t) f(t) dt \\
&= \frac{d}{dx} (pu') \int_a^x v(t) f(t) dt + pu'(x) v(x) f(x) - \frac{d}{dx} (pv') \int_a^x u(t) f(t) dt \\
&\quad - pv'(x) u(x) f(x), \text{ by Leibnitz's rule} \\
&= -qu \int_a^x v(t) f(t) dt + qv \int_a^x u(t) f(t) dt + p(u'v - uv') f(x)
\end{aligned}$$

$$\therefore \frac{d}{dx} (pw') = -q \left[u(x) \int_a^x v(t) f(t) dt - v(x) \int_a^x u(t) f(t) dt \right] + p(u'v - uv') f(x)$$

$$\text{or} \quad \frac{d}{dx} (pw') = -q(x) w(x) + p(u'v - uv') f(x), \text{ using (6)} \quad \dots (10)$$

Now,

$$\begin{aligned}
\frac{d}{dx} [p(uv' - u'v)] &= \frac{d}{dx} [(pv')u - (pu')v] = \frac{d}{dx} (pv') \cdot u + pv' \cdot u' - \frac{d}{dx} (pu') v - pu' \cdot v' \\
&= (-qv) \times u - (-qu) \times v = 0, \text{ using (9)}
\end{aligned}$$

$$\text{Thus,} \quad \frac{d}{dx} \{p(uv' - u'v)\} = 0 \quad \text{so that} \quad p(uv' - u'v) = A, \quad \dots (11)$$

where A is a constant. (11) is Abel's formula (refer page 11.7)

$$\text{Also,} \quad uv' - u'v = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = W(u, v), \quad \dots (12)$$

where $W(u, v)$ is the Wronskian of u and v . Since u and v are linearly independent solutions of (4), we have

$$W(u, v) = uv' - u'v \neq 0 \quad \dots (13)$$

Using (11), (10) may be re-written as

$$\frac{d}{dx} \left(p \frac{dw}{dx} \right) + qw = -A f(x) \quad \text{or} \quad \frac{d}{dx} \left(p \frac{d(-w/A)}{dx} \right) + q \left(-\frac{w}{A} \right) = f(x) \quad \dots (14)$$

$$\text{where} \quad w(a) = w'(a) = 0, \text{ by (8)} \quad \dots (15)$$

Comparing (14) with (1), we have $y = -w/A$ so that $w = -Ay$. Substituting this value of w in (6), we obtain

$$-Ay = \int_a^x \{u(x)v(t) - v(x)u(t)\} f(t) dt \quad \text{or} \quad y(x) = \int_a^x \frac{u(t)v(x) - v(t)u(x)}{A} f(t) dt$$

$$\text{or} \quad y(x) = \int_a^x R(x, t) f(t) dt \quad \dots (16)$$

$$\text{where} \quad R(x, t) = (1/A) \{v(x)u(t) - u(x)v(t)\} \quad \dots (17)$$

$$\text{From (17), we find that} \quad R(x, t) = -R(t, x) \quad \dots (18)$$

One can easily verify that, for a fixed value of t , the function $R(x, t)$ is completely characterized as the solution of the initial value problem

$$L R = \frac{d}{dx} \left\{ p(x) \frac{dR}{dx} \right\} + q(x) R = \delta(x - t), \quad \dots (19)$$

$$[R]_{x=t} = 0, \quad (dR/dx)_{x=t} = 1/p(t), \quad \dots (20)$$

where $\delta(x - t)$ is the Dirac delta function (refer Art. 10.6 in chapter 10)

This function describes the influence on the value of y at x due to a concentrated disturbance at t . It is called the *influence function*. The function $G(x, t)$ defined by

$$G(x, t) = \begin{cases} 0, & x < t \\ R(x, t), & x > t \end{cases} \quad \dots (21)$$

is known as the *causal Green's function*.

Remark. When the values of $y(a)$ and $y'(a)$ are prescribed to be other than zero, then we simply add a suitable solution $Au(x) + Bv(x)$ of (4) to the integral equation (16) to get volterra integral equation of the second kind of the form

$$y(x) = Au(x) + Bv(x) + \int_a^x R(x, t) f(t) dt. \quad \dots (22)$$

The constants A and B are evaluated by using the prescribed initial conditions.

Example 1. Convert the initial value problem

$$y'' + y = f(x), \quad 0 < x < 1, \quad y(0) = y'(0) = 0$$

into an integral equation.

Sol. Given

$$y'' + y = f(x), \quad 0 < x < 1 \quad \dots (1)$$

with initial conditions :

$$y(0) = 0, \quad y'(0) = 0 \quad \dots (2)$$

$$\text{Comparing (1) with } \frac{d}{dx} \left(p \frac{dy}{dx} \right) + qy = f(x), \quad \text{here } p = 1 = q$$

The associated homogeneous equation of (1) is

$$y'' + y = 0 \quad \text{or} \quad (D^2 + 1)y = 0, \quad \text{where } D \equiv d/dx \quad \dots (3)$$

Its general solution is

$$y = A \cos x + B \sin x.$$

Let

$$u = \cos x$$

and

$$v = \sin x,$$

$$\dots (4)$$

where u and v are linearly independent solution of (3).

Now,

$$A = p(uv' - u'v) = p[\cos^2 x + \sin^2 x] = p = 1$$

$$\therefore R(x, t) = (1/A) \{v(x)u(t) - u(x)v(t)\} = \sin x \cos t - \cos x \sin t = \sin(x - t)$$

Hence the given initial value problem reduces to the integral equation

$$y(x) = \int_0^x R(x, t) f(t) dt, \quad \text{i.e.,} \quad y(x) = \int_0^x \sin(x - t) f(t) dt \quad \dots (5)$$

Example 2. Convert the initial value problem

$$y'' + y = f(x), \quad 0 < x < 1, \quad y(0) = 1, \quad y'(0) = -1$$

into a Volterra integral equation of the second kind.

Sol. Refer remark at the end of Art. 11.10.

Here the values of $y(0)$ and $y'(0)$ are prescribed to be other than zero, hence the given initial value problem will transform into Volterra integral equation of the second kind of the form

$$y(x) = Au(x) + Bv(x) + \int_0^x R(x, t) f(t) dt \quad \dots (1)$$

Proceed as in Ex. 1 and show that $u(x) = \cos x$, $v(x) = \sin x$ and $R(x, t) = \sin(x - t)$. So

$$(1) \text{ reduces to } y(x) = A \cos x + B \sin x + \int_0^x \sin(x - t) f(t) dt \quad \dots (2)$$

Putting $y = 0$ in (2) and using the given condition $y(0) = 1$, we get $A = 1$.

Now, differentiating both sides of (2) w.r.t. 'x' and using Leibnitz's rule (see Art. 1.13), we

$$\text{obtain } y'(x) = -A \sin x + B \cos x + \int_0^x \cos(x - t) f(t) dt. \quad \dots (3)$$

Putting $x = 0$ in (3) and using the given condition $y'(0) = -1$, we get $B = -1$.

Putting $A = 1$ and $B = -1$ in (2), the required Volterra integral equation is given by

$$y(x) = \cos x - \sin x + \int_0^x \sin(x - t) f(t) dt$$

11.11(a) GREEN'S FUNCTION APPROACH FOR CONVERTING A BOUNDARY VALUE PROBLEM INTO AN INTEGRAL EQUATION. AN ALTERNATIVE PROCEDURE. (COMPARE WITH ART. 11.3)

Consider the following boundary value problem

$$\frac{d}{dx} \left(p \frac{dy}{dx} \right) + qy = f(x), \quad a \leq x \leq b \quad \dots (1)$$

$$y(a) = 0, \quad y(b) = 0 \quad \dots (2)$$

$$\text{Let } L \equiv \frac{d}{dx} \left(p \frac{d}{dx} \right) + q = p \frac{d^2}{dx^2} + \frac{dp}{dx} \frac{d}{dx} + q, \quad \dots (3)$$

which is a self-adjoint differential operator. Here the function $p(x)$ is continuously differentiable and positive and $q(x)$ and $f(x)$ are continuous in a given interval (a, b) .

The associated homogeneous second order equation

$$Ly = 0, \quad \text{i.e.,} \quad \frac{d}{dx} \left(p \frac{dy}{dx} \right) + qy = 0 \quad \dots (4)$$

has exactly two linearly independent solution $u(x)$ and $v(x)$ which are twice differentiable in the interval $a \leq x \leq b$. Any other solution of (4) is a linear combination of $u(x)$ and $v(x)$, i.e., $y(x) = C_1 u(x) + C_2 v(x)$ where C_1 and C_2 are constants.

Suppose that the given boundary value problem (1)—(2) possess its general solution as an integral equation of the form

$$y(x) = \int_a^x R(x, t) f(t) dt + C_1 u(x) + C_2 v(x), \quad \dots (5)$$

$$\text{where } R(x, t) = (1/A) \{v(x) u(t) - u(x) v(t)\} \quad \dots (6)$$

$$\text{and } A = p(x) \{u(x) v'(x) - u'(x) v(x)\} \quad \dots (7)$$

[Refer equations (11) and (17) of Art. 11.10]

Putting $x = a$ and $x = b$ in succession in (5) and using the boundary conditions (2), we have

$$C_1 u(a) + C_2 v(a) = 0 \quad \dots (8)$$

$$\text{and } C_1 u(b) + C_2 v(b) = - \int_a^b R(b, t) f(t) dt \quad \dots (9)$$

Let $D = u(a) v(b) - v(a) u(b) \neq 0$... (10)

Then the system of equations (8) and (9) give a unique pair of constants C_1 and C_2 given by

$$C_1 = [v(a)/D] \int_a^b R(b, t) f(t) dt = [v(a)/D] \int_a^x R(b, t) f(t) dt + [v(a)/D] \int_x^b R(b, t) f(t) dt \quad \dots (11)$$

$$C_2 = -[u(a)/D] \int_a^b R(b, t) f(t) dt = -[u(a)/D] \int_a^x R(b, t) f(t) dt - [u(a)/D] \int_x^b R(b, t) f(t) dt \quad \dots (12)$$

Substituting the values of C_1 and C_2 given by (11) and (12) in (5), we obtain

$$y(x) = \int_a^x [R(x, t) + (1/D) \{v(a) u(x) - u(a) v(x)\} R(b, t)] f(t) dt \\ + \int_x^b (1/D) \{v(a) u(x) - u(a) v(x)\} R(b, t) f(t) dt. \quad \dots (13)$$

From (6), $R(b, t) = (1/A) \{v(b) u(t) - u(b) v(t)\}$... (14)

Now, $(1/D) \{v(a) u(x) - u(a) v(x)\} R(b, t)$
 $= (1/AD) \{v(a) u(x) - u(a) v(x)\} \{v(b) u(t) - u(b) v(t)\}$, by (14) ... (15)

and $R(x, t) + (1/D) \{v(a) u(x) - u(a) v(x)\} R(b, t)$
 $= (1/A) \{v(x) u(t) - u(x) v(t)\} + (1/AD) \{v(a) u(x) - u(a) v(x)\} \{v(b) u(t) - u(b) v(t)\}$
[using (6) and (14)]

$$= (1/AD) [\{v(x) u(t) - u(x) v(t)\} \{u(a) v(b) - v(a) u(b)\} \\ + \{v(a) u(x) - u(a) v(x)\} \{v(b) u(t) - u(b) v(t)\}], \text{ using (10)} \\ = (1/AD) \{u(t) v(a) - v(t) u(a)\} \{u(x) v(b) - v(x) u(b)\} \quad \dots (16)$$

[on simplification]

Using (15) and (16), we define the so called Green's function $G(x, t)$ as follows :

$$G(x, t) = \begin{cases} (1/AD) \{u(t) v(a) - v(t) u(a)\} \{u(x) v(b) - v(x) u(b)\}, & t < x \\ (1/AD) \{u(x) v(a) - v(x) u(a)\} \{u(t) v(b) - v(t) u(b)\}, & t > x \end{cases} \quad \dots (17)$$

Using (17), (13) may be re-written as

$$y(x) = \int_a^b G(x, t) f(t) dt \quad \dots (18)$$

From (17), we see that $G(x, t)$ is symmetric, i.e., $G(x, t) = G(t, x)$... (19)

Again, $G(x, t)$ satisfies, for all t , the following *auxiliary problem*:

$$LG = \frac{d}{dx} \left\{ p(x) \frac{dG}{dx} \right\} + q(x) G = -\delta(x - t) \quad \dots (20)$$

$$[G(x, t)]_{x=a} = 0, \quad [G(x, t)]_{x=b} = 0 \quad \dots (21)$$

$$[G(x, t)]_{x=t+0} = [G(x, t)]_{x=t-0} \quad \dots (22)$$

$$(\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -\{1/p(t)\} \quad \dots (23)$$

where $\delta(x - t)$ is the Dirac delta function (refer Art. 10.6 of chapter 10). Here $[G(x, t)]_{x=t+0}$ stands for the limit of $G(x, t)$ as x approaches t from the right and there are similar meanings for the rest of such expressions occurring in (22) and (23).

The condition (22) states that the Green's function is continuous at $x = t$. Again, the condition (23) implies that the derivative of G has a discontinuity of magnitude $-1/p(t)$ at $x = t$. The condition (22) and (23) are known as the *matching conditions*.

Remark 1. The relation (23) can be deduced from the relations (19) and (20) as follows.

The value of the jump in the derivative of $G(x, t)$ can be obtained by integrating (20) over a small interval $(t - \epsilon, x)$ and by remembering that the indefinite integral of $\delta(x - t)$ is the Heaviside function $H(x - t)$, 1 (See Print header also Art. 10.9 of chapter 10). Thus, we obtain

$$p(t) \frac{\partial G(x, t)}{\partial x} + \int_{t-\epsilon}^x q(x) G(x, t) dx = p(t - \epsilon) \frac{\partial G(t - \epsilon, x)}{\partial x} - H(x - t)$$

When x traverses the source point t , then on the right side Heaviside function has a unit jump discontinuity. Since other terms are continuous functions of x , it follows that $\partial G / \partial x$ has at t , a jump discontinuity as given by (23).

Remark 2. From the properties (20)–(23) of the Green's function $G(x, t)$, it follows that $\partial G(x, a) / \partial t$ satisfies the following system of equations :

$$\left. \begin{aligned} \frac{d}{dx} \left[p(x) \frac{d}{dx} \left\{ \frac{\partial G(x, a)}{\partial t} \right\} \right] + q(x) \frac{\partial G(x, a)}{\partial t} &= 0, \quad a < x < b \\ \partial G(a, a) / \partial t &= 1 / p(a), \quad \partial G(b, a) / \partial t = 0 \end{aligned} \right\} \quad \dots (24)$$

Similarly, $\partial G(x, b) / \partial t$ satisfies the following system of equation

$$\left. \begin{aligned} \frac{d}{dx} \left[p(x) \frac{d}{dx} \left\{ \frac{\partial G(x, b)}{\partial t} \right\} \right] + q(x) \frac{\partial G(x, b)}{\partial t} &= 0, \quad a < x < b, \\ \partial G(a, b) / \partial t &= 0, \quad \partial G(b, b) / \partial t = -1 / p(b) \end{aligned} \right\} \quad \dots (25)$$

Hence, it can be easily shown that the boundary value problem

$$\frac{d}{dx} \left\{ p(x) \frac{dy}{dx} \right\} + q(x) y = f(x), \quad y(a) = \alpha, \quad y(b) = \beta \quad \dots (26)$$

has the solution

$$y(x) = - \int_a^b G(x, t) f(t) dt + \alpha p(a) \frac{\partial G(x, a)}{\partial t} - \beta p(b) \frac{\partial G(x, b)}{\partial t}. \quad \dots (27)$$

11.11 (b) INTEGRAL- EQUATION FORMULATION FOR THE BOUNDARY VALUE PROBLEM WITH MORE GENERAL AND INHOMOGENEOUS BOUNDARY CONDITIONS. WORKING RULE

Consider the following boundary value problem

$$\left. \begin{aligned} \frac{d}{dx} \left\{ p(x) \frac{dy}{dx} \right\} + qy &= f(x), \\ -\mu_1 y'(a) + \nu_1 y(a) &= \alpha, \quad \mu_2 y'(b) + \nu_2 y(b) = \beta \end{aligned} \right\} \quad \dots (28)$$

The above boundary value problem is solved in exactly the same way as the system (1)–(2). We can easily show that the Green's function for the system (28) can also be derived provided the determinant

$$D = \left\{ -\mu_1 u'(a) + \nu_1 u(a) \right\} \left\{ \mu_2 v'(b) + \nu_2 v(b) \right\} - \left\{ -\mu_1 v'(a) + \nu_1 v(a) \right\} \left\{ \mu_2 u'(b) + \nu_2 u(b) \right\} \neq 0$$

where $u(x)$ and $v(x)$ are the solutions of the homogeneous equation

$$\frac{d}{dx} \left\{ p(x) \frac{dy}{dx} \right\} + qy = 0 \quad \dots (29)$$

Green's function has the following five properties :

$$(i) \quad LG = \frac{d}{dx} \left\{ p(x) \frac{\partial G}{\partial x} \right\} + q(x) G = \delta(x - t) \quad \dots (30)$$

$$(ii) \quad -\mu_1 \left[\frac{\partial G}{\partial x} \right]_{x=a} + \nu_1 [G]_{x=a} = \mu_2 \left[\frac{\partial G}{\partial x} \right]_{x=b} + \nu_2 [G]_{x=b} = 0 \quad \dots (31)$$

$$(iii) \quad [G]_{x=t+0} = [G]_{x=t-0} \quad \dots (32)$$

$$(iv) \quad (\partial G / \partial x)_{x=t+0} - (\partial G / \partial x)_{x=t-0} = -1 / p(t) \quad \dots (33)$$

$$(v) \quad G(x, t) = G(t, x), \text{ i.e., } G(x, t) \text{ is symmetric} \quad \dots (34)$$

With help of the Green's function, the boundary value problem (28) has the unique solution.

$$y(x) = \int_a^b G(x, t) f(t) dt + \frac{p(a)}{\mu_1} \alpha G(x, a) + \frac{p(b)}{\mu_2} \beta G(x, b) \quad \dots (35)$$

provided μ_1 and μ_2 do not vanish. If $\mu_1 = 0$, then the factor $(1/\mu_1) G(x, a)$ is replaced by $(1/\nu_1) (\partial G(x, a) / \partial t)$. Similarly, if $\mu_2 = 0$, then $(1/\mu_2) G(x, b)$ is replaced by $-(1/\nu_2) (\partial G(x, b) / \partial t)$.

Since $D \neq 0$, we cannot have both μ_1 and ν_1 or both μ_2 and ν_2 equal to zero. When $\alpha = \beta = 0$, the relation (35) reduces to relation (18).

An important deduction. The well known *Sturm-Liouville problem* is the following boundary value problem involving a parameter λ

$$\left. \begin{aligned} \frac{d}{dx} \left\{ p(x) \frac{dy}{dx} \right\} + q(x) y + \lambda r(x) y &= f(x) \\ -\mu_1 y'(a) + \nu_1 y(a) &= 0, \quad \mu_2 y'(b) + \nu_2 y(b) = 0 \end{aligned} \right\} \quad \dots (36)$$

The values of λ for which (36) has a nontrivial solution are known as the eigenvalues and the corresponding solutions are known as the eigenfunctions. If $p(a) = p(b)$, the boundary condition in (36) are replaced by the periodic boundary conditions $y(a) = y(b)$, $y'(a) = y'(b)$.

Using the relation (35), the solution of (36) is given by

$$y(x) = \lambda \int_a^b r(t) G(x, t) y(t) dt - \int_a^b G(x, t) f(t) dt, \quad \dots (37)$$

which is a Fredholm integral equation of the second kind.

We observe that the kernel $K(x, t)$ of (37) is the product $r(t) G(x, t)$. While the condition (v) of the Green's function shows that $G(x, t)$ is symmetric, the product $r(t) K(x, t)$ is not symmetric unless $r(t)$ is a constant. However, if we write $\{r(x)\}^{1/2} y(x) = Y(x)$, under the assumption that $r(x)$ is nonnegative over (a, b) as is usually the case in practice, equation (37) can be re-written in

$$\text{the form} \quad Y(x) = \lambda \int_a^b \bar{K}(x, t) Y(t) dt - \int_a^b \bar{K}(x, t) \frac{f(t)}{\{r(t)\}^{1/2}} dt$$

where

$$\bar{K}(x, t) = \{r(x) r(t)\}^{1/2} G(x, t).$$

and hence $\bar{K}(x, t)$ possesses the same symmetry as $G(x, t)$.

Remark. We have seen that the discussion so far regarding the boundary value problem (1)—(2), (26) or (28) is based on the assumption that

$$D = u(a) v(b) - v(a) u(b) \neq 0.$$

However, if $D = 0$, then the system of homogeneous equation

$$C_1 u(a) + C_2 v(a) = 0, \quad C_1 u(b) + C_2 v(b) = 0$$

will yield nonzero values of C_1 and C_2 and the function $w(x) = C_1 u(x) + C_2 v(x)$ satisfies the completely homogeneous boundary value problem.

$$\frac{d}{dx} \left\{ p(x) \frac{dw}{dx} \right\} + q(x) w = 0, \quad w(a) = 0, \quad w(b) = 0 \quad \dots (38)$$

It follows that if y is a solution of (1), (26), or (28), then so is $y + cw$ for any constant c . This implies that these systems will not possess a unique solution. This is not all. There is an additional *consistency condition* which must be satisfied for these systems to possess a solution. To this end, we proceed with system (26), i.e.,

$$f(x) = (py')' + qy \quad \dots (39)$$

$$\text{with boundary condition} \quad y(a) = \alpha \quad \text{and} \quad y(b) = \beta \quad \dots (40)$$

Multiplying (39) by $w(x)$ and then integrating w.r.t. 'x' from a to b , we obtain

$$\begin{aligned} \int_a^b w(x) f(x) dx &= \int_a^b w(x) \{ (py')' + qy \} dx = \int_a^b w(x) (py')' dx + \int_a^b w q y dx \\ &= [w(py')]_a^b - \int_a^b w'(py') dx + \int_a^b w q y dx, \text{ integrating by parts} \\ &= w(b) p(b) y'(b) - w(a) p(a) y'(a) - \int_a^b (w'p) y' dx + \int_a^b w q y dx \\ &= -[(w'p) y]_a^b + \int_a^b (w'p)' y dx + \int_a^b w q y dx, \text{ using (38)} \\ &= -w'(b) p(b) y(b) + w'(a) p(a) y(a) + \int_a^b [(w'p)' + wq] y dx \\ &= \alpha w'(a) p(a) - \beta w'(b) p(b), \text{ using (38)} \end{aligned}$$

$$\text{Thus,} \quad \int_a^b w(x) f(x) dx = \alpha w'(a) p(a) - \beta w'(b) p(b) \quad \dots (41)$$

Hence, in order that (26) may possess a solution, the given function $f(x)$ must satisfy the consistency condition (41).

As a particular case $\alpha = \beta = 0$, we see that the system (1)—(2) will possess a solution if the

$$\text{consistency condition} \quad \int_a^b w(x) f(x) dx = 0 \quad \dots (42)$$

is satisfied.

We have thus shown that if $D = 0$, then the boundary value problems either have no solution or many solutions; but never a unique solution.

Illustration Solved Example Reduce the boundary value problem $y'' + \lambda xy = 1$, $y(0) = 0$, $y(l) = 1$ to Fredholm integral equation.

Sol. Given $y'' = 1 - \lambda xy$, ... (1)

subject to the boundary conditions : $y(0) = 0$, $y(l) = 1$ (2)

[We have already dealt with the present problem in Ex. 4 base 11.35. However, we shall again give an alternative method of dealing with boundary value problem (1)—(2), with help of results (24)—(27) of remark 2 of Art. 11.11.(a)]

For the Green's function $G(x, t)$ proceed yourself upto result (15) of Ex. 4, base 11.35.

Thus,
$$G(x, t) = \begin{cases} (x/l)(l-t), & 0 \leq x < t \\ (t/l)(l-x), & t \leq x \leq l \end{cases} \quad \dots (3)$$

Comparing (1) with (26) of Art. 11.11(a), here

$p(x) = 1$, $f(x) = 1 - \lambda xy$, $a = 0$, $\alpha = 0$, $b = l$, $\beta = 1$ (4)

Using result (27) of Art. 11.11(a), we see that the given boundary value problem transforms to

$$y(x) = - \int_0^l (1 - \lambda xy) G(x, t) dt - \partial G(x, l) / \partial t$$

or
$$y(x) = - \int_0^l G(x, t) dt + \lambda \int_0^l t y(t) G(x, t) dt - \partial G(x, l) / \partial t \quad \dots (5)$$

Now, $\partial G(x, l) / \partial t$ must satisfy the system of equations (25) of Art. 11.11(a), which for the present problem reduce to

$$\frac{d^2}{dx^2} \left\{ \frac{\partial G(x, l)}{\partial t} \right\} = 0, \quad \frac{\partial G(0, l)}{\partial t} = 0, \quad \frac{\partial G(l, l)}{\partial t} = -1 \quad \dots (6)$$

whose solution is
$$\partial G(x, l) / \partial t = -x / l \quad \dots (7)$$

Now,
$$\begin{aligned} \int_0^l G(x, t) dt &= \int_0^x G(x, t) dt + \int_x^l G(x, t) dt = \int_0^x \frac{t}{l} (l-x) dt + \int_x^l \frac{x}{l} (l-t) dt, \text{ by (3)} \\ &= \frac{l-x}{l} \left[\frac{t^2}{2} \right]_0^x + \frac{x}{l} \left[lt - \frac{t^2}{2} \right]_x^l = \frac{x^2(l-x)}{2l} + \frac{x}{l} \left[l^2 - \frac{l^2}{2} - \left(lx - \frac{x^2}{2} \right) \right] \\ &= \frac{x}{l} \left(\frac{xl}{2} - \frac{x^2}{2} + \frac{l^2}{2} - lx + \frac{x^2}{2} \right) = \frac{x}{2} (l-x) \end{aligned} \quad \dots (8)$$

With help of (7) and (8), (5) reduces to

$$y(x) = -\frac{x}{2} (l-x) + \lambda \int_0^l t y(t) G(x, t) dt + \frac{x}{l}$$

or
$$y(x) = (x/2l) \times (2 - l^2 + xl) + \lambda \int_0^l G(x, t) t y(t) dt,$$

which is the desired Fredholm integral equation.

11.12 MODIFIED (OR GENERALIZED) GREEN'S FUNCTION

Consider the inhomogeneous equation $Ly = \phi(x)$... (1')

where L is the self-adjoint operator defined by

$$L \equiv \frac{d}{dx} \left\{ p(x) \frac{d}{dx} \right\} + q(x), \quad a \leq x \leq b. \quad \dots (1)$$

We have discussed in remark of Art. 11.11 (b) that if the homogeneous equation $L y = 0$ with the prescribed boundary conditions at $x = a$ and $x = b$ has a nontrivial solution $w(x)$, then the corresponding inhomogeneous equation either has no solution or many solutions depending on the consistency condition. Accordingly, the Green's function, as defined in Art. 11.11 (b), does not

exist, because
$$\int_a^b \delta(x-t) w(x) dx \neq 0, \quad a < t < b$$

In what follows, we shall discuss a method of constructing the Green's function for this type of problem. Such a function is known as the *modified Green's function* and it will be denoted by $G_M(x, t)$. We start by selecting a normalized solution of the completely homogeneous system

so that
$$\int_a^b w^2(x) dx = 1 \quad \dots (2)$$

By a generalized Green's function, we mean a function $G_M(x, t)$ satisfying the following properties :

(i) $G_M(x, t)$ satisfies the differential equation

$$L G_M(x, t) = \delta(x-t) - w(x) w(t) \quad \dots (3)$$

This amounts to introducing an addition *source density* so that the consistency condition is satisfied, as shown below
$$\int_a^b \{\delta(x-t) - w(x) w(t)\} w(x) dx = 0.$$

(ii) $G_M(x, t)$ satisfies the prescribed homogeneous boundary conditions.

(iii) $G_M(x, t)$ is continuous at $x = t$.

(iv) The x derivative of $G_M(x, t)$ possesses a jump of magnitude $1/p(t)$ as the point $x = t$ is crossed in the positive x -direction, that is,

$$(\partial G_M / \partial x)_{x=t+0} - (\partial G_M / \partial x)_{x=t-0} = 1/p(t)$$

It follows that the construction of modified Green's function is entirely similar to that for the ordinary Green's function described in Art. 11.4, but the modified Green's function is not uniquely determined. We can add $c w(x)$ to a Green's function without violating any of the above four properties. It is often convenient to choose a particular modified Green's function which is symmetric function of x and t . To this end, we consider two functions $G_M(x, t_1)$ and $G_M(x, t_2)$ which satisfy the equations

$$L G_M(x, t_1) = \delta(x-t_1) - w(t_1) w(x) \quad \dots (4)$$

and

$$L G_M(x, t_2) = \delta(x-t_2) - w(t_2) w(x) \quad \dots (5)$$

respectively along with prescribed (same for both) homogeneous boundary condition.

Multiply (4) by $G_M(x, t_2)$ and (5) by $G_M(x, t_1)$, subtract, and integrate with respect to x from a to b . Thus, we obtain

$$\begin{aligned} & \int_a^b \{G_M(x, t_2) L G_M(x, t_1) - G_M(x, t_1) L G_M(x, t_2)\} dx \\ &= \int_a^b G_M(x, t_2) \delta(x-t_1) dx - \int_a^b G_M(x, t_1) \delta(x-t_2) dx \\ & \quad - w(t_1) \int_a^b G_M(x, t_2) w(x) dx + w(t_2) \int_a^b G_M(x, t_1) w(x) dx \end{aligned}$$

Now, using the Green's formula (refer relation (5) in Art. 10.5) and shifting property of Dirac delta function (refer Art. 10.7), we arrive at

$$G_M(t_1, t_2) - G_M(t_2, t_1) - w(t_1) \int_a^b G_M(x, t_2) w(x) dx + w(t_2) \int_a^b G_M(x, t_1) w(x) dx = 0 \quad \dots (5')$$

If we impose the condition :

(v) $G_M(x, t)$ satisfies the property

$$\int_a^b G_M(x, t) w(x) dx = 0, \text{ for every } t.$$

then, (5)' reduces to

$$G_M(t_1, t_2) - G_M(t_2, t_1) = 0 \quad \text{so that} \quad G_M(t_1, t_2) = G_M(t_2, t_1),$$

showing $G_M(x, t)$ will be symmetric. It then follows that the Green's function is uniquely determined.

Remark. The Green's function can be defined in an equivalent form which does not involve the Dirac delta function $\delta(x - t)$ as follows.

Suppose that a problem, consisting of the differential equation $L y = (py')' + qy = 0$ and homogeneous conditions at the ends of an interval (a, b) , is satisfied by a normalized non-trivial

solution $y = w(x)$ so that $\int_a^b \{w(x)\}^2 dx = 1$. Then the generalized Green's function is defined as a

function $G_M(x, t)$ which, when considered as a function of x for a fixed number t , possesses the following properties :

(i) G_M satisfies the differential equation $L G_M = -w(x) w(t)$ in the sub-intervals (a, t) and (t, b) .

(ii) G_M satisfies the prescribed end conditions.

(iii) G_M is continuous at $x = t$.

(iv) $(\partial G_M / \partial x)_{x=t+0} - (\partial G_M / \partial x)_{x=t-0} = 1 / p(t)$

(v) G_M satisfies the condition $\int_a^b G_M(x, t) w(x) dx = 0$

Method of reducing the inhomogeneous equation

$$L y = \phi(x) \quad \dots (6)$$

with prescribed homogeneous end conditions into an integral equation when the following consistency condition is satisfied

$$\int_a^b \phi(x) w(x) dx = 0. \quad \dots (7)$$

Multiplying (3) by $y(x)$ and (6) by $G_M(x, t)$ and then subtracting, we get

$$G_M L y - y L G_M = G_M(x, t) \phi(x) - \delta(x - t) y(x) + w(x) w(t) y(x).$$

Now, integrating both sides of the above equation w.r.t. 'x' from a to b , we obtain

$$\int_a^b (G_M L y - y L G_M) dx = \int_a^b G_M(x, t) \phi(x) dx - \int_a^b \delta(x - t) y(x) dx + \int_a^b w(x) w(t) y(x) dx$$

$$\text{or} \quad \int_a^b (G_M L y - y L G_M) dx = \int_a^b G_M(x, t) \phi(x) dx - y(t) + c w(t) \quad \dots (8)$$

$$\text{where} \quad c = \int_a^b w(x) y(x) dx = \text{an unknown constant} \quad \dots (9)$$

and $\int_a^b \delta(x - t) y(x) dx = y(t)$, by shifting property (refer Art. 10.7, Chapter 10)

Now, the L.H.S. of (8) vanishes because of the Green's formula (refer result (5) of Art. 10.5, chapter 10) and the boundary conditions. Hence (8) reduces to

$$y(t) = \int_a^b G_M(x, t) \phi(x) dx + c w(t) \quad \dots (10)$$

By interchanging x and t in (10), we obtain

$$y(x) = \int_a^b G_M(t, x) \phi(t) dt + c w(x) \quad \dots (11)$$

For a symmetric Green's function, we have $G_M(t, x) = G_M(x, t)$

Therefore, (11) reduces to

$$y(x) = \int_a^b G_M(x, t) \phi(t) dt + c w(x) \quad \dots (12a)$$

Using (9), (12a) can be re-written as

$$y(x) = \int_a^b G_M(x, t) \phi(t) dt + w(x) \int_a^b w(x) y(x) dx \quad \dots (12b)$$

Remark. When $w(x) = 0$, (12) reduces to (18) of Art. 11.11(a).

11.13 WORKING RULE FOR CONSTRUCTION OF MODIFIED GREEN'S FUNCTION

Given an inhomogeneous equation with boundary conditions :

$$Ly = \phi(x); \quad \alpha_1 y(a) + \beta_1 y'(a) = 0, \quad \alpha_2 y(b) + \beta_2 y'(b) = 0 \quad \dots (1)'$$

Consider a linear homogeneous equation of order two

$$Ly = 0, \quad \text{where} \quad L \equiv p(x) \frac{d^2}{dx^2} + \frac{dp}{dx} \frac{d}{dx} + q(x) \quad \dots (1)$$

together with homogeneous boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = 0 \quad \dots (2a)$$

$$\alpha_2 y(b) + \beta_2 y'(b) = 0 \quad \dots (2b)$$

with usual assumption that at least one of α_1 and β_1 and one of α_2 and β_2 are non-zero.

Suppose that the homogeneous boundary value problem given by (1), (2a) and (2b) has a non-trivial solution $y(x)$.

$$\text{Then,} \quad \|y(x)\| = \text{norm of } y(x) = \left\{ \int_a^b [y(x)]^2 dx \right\}^{1/2} \quad \dots (3)$$

$$\text{Let} \quad w(x) = y(x) / \|y(x)\|$$

so that $w(x)$ is non-trivial normalized solution of the boundary value problem given by (1), (2a) and (2b). Clearly, by definition, we have

$$\|w(x)\| = 1 \quad \text{so that} \quad \int_a^b \{w(x)\}^2 dx = 1 \quad \dots (4)$$

Then, by definition $G_M(x, t)$ is called the modified Green's function of the given boundary value problem if it satisfies the differential equation

$$L G_M = \delta(x - t) - w(x) w(t) \quad \dots (5)'$$

For $x \neq t$, (5)' reduces to

$$L G_M = -w(x) w(t) \quad \dots (5)$$

$$\text{For a given } t, \text{ let} \quad G_M(x, t) = \begin{cases} G_1(x, t), & \text{if } a \leq x < t \\ G_2(x, t), & \text{if } t < x \leq b \end{cases} \quad \dots (6)$$

where G_1 and G_2 are such that

(i) The functions G_1 and G_2 satisfy the equation (5) in their respective intervals of definition,

$$\text{that is} \quad L G_1 = w(x) w(t), \quad a \leq x < t \quad \dots (7a)$$

$$L G_2 = w(x) w(t), \quad t \leq x \leq b \quad \dots (7b)$$

(ii) G_1 satisfies the boundary condition (2a) whereas G_2 satisfies the boundary condition (2b).

(iii) The function $G_M(x, t)$ is continuous at $x = t$, i.e., $G_1(t, t) = G_2(t, t)$... (8)

(iv) The derivative of $G_M(x, t)$ with respect to x at the point $x = t$ has a discontinuity of the first kind, the jump being equal to $1/p(t)$. Here $p(x)$ is the coeff. of d^2y/dx^2 in (1).

$$\text{Thus,} \quad (\partial G_M / \partial x)_{x=t+0} - (\partial G_M / \partial x)_{x=t-0} = 1/p(t) \quad \dots (9)$$

(v) In order that $G_M(x, t)$ may be symmetric, we must have

$$\int_a^b G_M(x, t) w(x) dx = 0 \quad \dots (10)$$

Method of reducing the inhomogeneous differential equation (1)' with prescribed homogeneous boundary condition into an integral equation.

The required integral equation is given by

$$y(x) = \int_a^b G_M(x, t) \phi(t) dt + c w(x), \quad \text{where } c \text{ is an arbitrary constant.} \quad \dots (11a)$$

(11a) may also be re-written in the form

$$y(x) = \int_a^b G_M(x, t) \phi(t) dt + w(x) \int_a^b w(x) y(x) dx \quad \dots (11b)$$

Consistency condition for existence of the desired integral equation (11a) or (11b) is given by :

$$\int_a^b \phi(x) w(x) dx = 0 \quad \dots (12)$$

11.14 SOLVED EXAMPLES BASED ON ART. 11.13

Ex.1. Find the modified Green's function for the system

$$y'' + f(x) = 0, \quad y'(0) = y'(l) = 0, \quad 0 < x < l$$

and hence transform this boundary value problem into an integral equation.

$$\text{Sol. Given} \quad -y'' = f(x), \quad y'(0) = y'(l) = 0, \quad 0 \leq x \leq l \quad \dots (1)$$

Here $-(d^2/dx^2)$ is a self adjoint operator

$$\text{Consider the associated self adjoint system} \quad -y'' = 0, \quad 0 \leq x \leq l \quad \dots (2)$$

with boundary conditions :

$$y'(0) = 0 \quad \dots (3a)$$

$$y'(l) = 0 \quad \dots (3b)$$

The general solution of (2) is

$$y(x) = Ax + B \quad \dots (4)$$

From (4),

$$y'(x) = A \quad \dots (5)$$

Putting $x = 0$ and $x = l$ in (5) and using (3a) and (3b), we get $A = 0$. Hence the boundary value problem given by (2), (3a) and (3b) has a non trivial solution $y(x) = B$, where B is an arbitrary constant.

$$\text{Here,} \quad \|y(x)\| = \text{norm of } y(x) = \left[\int_0^l [y(x)]^2 dx \right]^{1/2} = \left[\int_0^l B^2 dx \right]^{1/2} = B \sqrt{l}$$

$$\text{Let} \quad w(x) = y(x) / \|y(x)\| = B / B\sqrt{l} = 1/\sqrt{l}, \quad \dots (6)$$

so that $w(x)$ is a non-zero normalized solution of the boundary value problem given by (2), (3a)

$$\text{and (3b). Clearly,} \quad \int_0^l [w(x)]^2 dx = 1 \quad \dots (7)$$

Then, for $x \neq t$ the required modified Green's function $G_M(x, t)$ must satisfy the equation

$$-d^2 G_M / dx^2 = -w(x) w(t) \quad \text{or} \quad d^2 G_M / dx^2 = 1/l, \text{ by (6)} \quad \dots (8)$$

The general solution of (8) is of the form $G_M(x, t) = Ax + B + x^2/2l$

Hence, we take
$$G_M(x, t) = \begin{cases} a_1 x + a_2 + x^2/2l, & 0 \leq x < t \\ b_1 x + b_2 + x^2/2l, & t < x \leq l \end{cases} \quad \dots (9)$$

From (9),
$$\partial G_M / \partial x = \begin{cases} a_1 + x/l, & 0 \leq x < t \\ b_1 + x/l, & t < x \leq l \end{cases} \quad \dots (10)$$

In addition to the above property (9), the proposed modified Green's function must satisfy the following properties :

(i) Since $G_M(x, t)$ must satisfy the boundary conditions (3a) and (3b), (10) gives

$$\begin{aligned} (\partial G_M / \partial x)_{x=0} &= 0 & \text{and} & & (\partial G_M / \partial x)_{x=l} &= 0 \\ \therefore a_1 &= 0 & \text{and} & & b_1 + 1 &= 0 \quad \text{so that} & & a_1 = 0 & \text{and} & & b_1 = -1 \quad \dots (11) \end{aligned}$$

(ii) $G_m(x, t)$ is continuous at $x = t$, that is

$$a_1 t + a_2 + t^2/2l = b_1 t + b_2 + t^2/2l \quad \text{so that} \quad (a_1 - b_1) t + a_2 - b_2 = 0 \quad \dots (12)$$

(iii) The derivative of $G_M(x, t)$ with respect to x at the point $x = t$ has a discontinuity of the first kind, the jump being $1/p(t)$, where $p(x)$ is the coefficient of y'' in (1), i.e., $p(x) = -1$. Thus,

$$\left(\partial G_M / \partial x \right)_{x=t+0} - \left(\partial G_M / \partial x \right)_{x=t-0} = 1/p(t)$$

$$\text{i.e.,} \quad b_1 + t/l - (a_1 + t/l) = -1 \quad \text{or} \quad a_1 - b_1 = 1 \quad \dots (13)$$

$$\text{From (12) and (13),} \quad t + a_2 - b_2 = 0 \quad \text{so that} \quad b_2 = a_2 + t \quad \dots (14)$$

Substituting values of a_1 , b_1 and b_2 from (11) and (14) in (9), we get

$$G_M(x, t) = \begin{cases} a_2 + x^2/2l, & \text{if } 0 \leq x < t \\ a_2 - x + t + x^2/2l, & \text{if } t \leq x \leq l. \end{cases} \quad \dots (15)$$

(iv) In order that $G_M(x, t)$ may be symmetric, we have

$$\int_0^l G_M(x, t) w(x) dx = 0 \quad \text{or} \quad \int_0^l G_M(x, t) dx = 0, \quad \text{as} \quad w(x) = \frac{1}{\sqrt{l}}$$

$$\text{or} \quad \int_0^t G_M(x, t) dx + \int_t^l G_M(x, t) dx = 0$$

$$\text{or} \quad \int_0^t \left(a_2 + \frac{x^2}{2l} \right) dx + \int_t^l \left(a_2 - x + t + \frac{x^2}{2l} \right) dx = 0, \text{ using (15)}$$

$$\text{or} \quad \left[a_2 x + x^3/6l \right]_0^t + \left[a_2 x - x^2/2 + t x + x^3/6l \right]_t^l = 0$$

$$\text{or} \quad a_2 t + t^3/6l + a_2 l - l^2/2 + t l + l^2/6 - (a_2 t - t^2/2 + t^2 + t^3/6l) = 0$$

$$\text{or} \quad a_2 l = l^2/3 - t l + t^2/2 \quad \text{or} \quad a_2 = l/3 - t + t^2/2l$$

Substituting the above value of a_2 in (15), the symmetric modified Green's function

$G_M(x, t)$ is given by

$$G_M(x, t) = \begin{cases} l/3 - t + (x^2 + t^2)/2l, & \text{if } 0 \leq x < t \\ l/3 - x + (x^2 + t^2)/2l, & \text{if } t < x \leq l \end{cases} \quad \dots (16)$$

11.54 Applications of integral equations and Green's function to ordinary differential equation
which may also be re-written as

$$G_M(x, t) = \frac{l}{3} + \frac{x^2 + t^2}{2l} - \begin{cases} t, & \text{if } 0 \leq x < t \\ x, & \text{if } t < x \leq l \end{cases} \quad \dots (17)$$

The above result (16) could have been obtained by inspecting (15) and making a judicious choice of a_2 .

Second part. Transformation of the given boundary value problem into an integral equation.

Refer result (11a) of Art. 11.13. The required integral equation is given by

$$y(x) = \int_M^l G_M(x, t) f(t) dt + \frac{c}{\sqrt{l}} \quad \text{or} \quad y(x) = c' + \int_0^l G_M(x, t) f(t) dt,$$

where $c' (= c/\sqrt{l})$ is an arbitrary constant and $G_M(x, t)$ is given by (16) or (17).

Ex.2. Find the modified Green's function for the system $y'' = 0$, $-1 < x < 1$ subject to the boundary conditions $y(-1) = y(1)$ and $y'(-1) = y'(1)$.

Sol. Given self adjoint system $y'' = 0$, $-1 \leq x \leq 1$... (1)

with boundary conditions : $y(-1) = y(1)$... (2a)

and $y'(-1) = y'(1)$... (2b)

The general solution of (1) is $y(x) = Ax + B$... (3)

From (3), $y'(x) = A$... (4)

From (3) and (4), $y(-1) = -A + B$, $y(1) = A + B$, $y'(-1) = y'(1) = A$... (5)

From (2a) and (5), we get $-A + B = A + B$ so that $A = 0$... (6)

Then, (2b), (5) and (6) $\Rightarrow A = 0$.

Hence the given boundary value problem has a non-trivial solution $y(x) = B$, where B is an arbitrary constant.

$$\text{Here } \|y(x)\| = \text{norm of } y(x) = \left[\int_{-1}^1 (y(x))^2 dx \right]^{1/2} = \left[\int_{-1}^1 B^2 dx \right]^{1/2} = B\sqrt{2}$$

$$\text{Let } w(x) = y(x) / \|y(x)\| = B / B\sqrt{2} = 1/\sqrt{2} \quad \dots (7)$$

so that $w(x)$ is a non-zero normalized solution of the given boundary value problem. Clearly,

$$\int_{-1}^1 [w(x)]^2 dx = 1.$$

Then for $x \neq t$ the required modified Green's function $G_M(x, t)$ must satisfy the equation

$$-d^2 G_M / dx^2 = -w(x), w(t) \quad \text{or} \quad d^2 G_M / dx^2 = 1/2, \text{ by (7)} \quad \dots (8)$$

The general solution of (8) is of the form $G_M(x, t) = Ax + B + x^2/4$

$$\text{Hence, we take } G_M(x, t) = \begin{cases} a_1 x + a_2 + x^2/4, & \text{if } -1 \leq x < t \\ b_1 x + b_2 + x^2/4, & \text{if } t < x \leq 1 \end{cases} \quad \dots (9)$$

$$\text{From (9), } \partial G_M / \partial x = \begin{cases} a_1 + x/2, & \text{if } -1 \leq x < t \\ b_1 + x/2, & \text{if } t < x \leq 1 \end{cases} \quad \dots (10)$$

In addition to the above property (9), the proposed modified Green's function $G_M(x, t)$ must satisfy the following properties :

(i) $G_M(x, t)$ is continuous at $x = t$, that is,

$$a_1 t + a_2 + t^2/4 = b_1 t + b_2 + t^2/4 \quad \text{so that} \quad a_2 - b_2 = t(b_1 - a_1) \quad \dots (11)$$

(ii) Since $G_M(x, t)$ must satisfy (2a), we must have

$$-a_1 + a_2 + 1/4 = b_1 + b_2 + 1/4 \quad \text{so that} \quad a_2 - b_2 = a_1 + b_1, \text{ by (9)} \quad \dots (12)$$

Again, $G_M(x, t)$ must satisfy (2b), hence we have

$$b_1 + 1/2 = a_1 - 1/2 \quad \text{or} \quad b_1 - a_1 = -1, \text{ by (10)} \quad \dots (13)$$

(iii) The derivative of $G_M(x, t)$ with respect to x at the point $x = t$ has a discontinuity of the first kind, the jump being $1/p(t)$, where $p(x)$ is the coefficient of y'' in (1), i.e., $p(x) = -1$. Thus,

$$\left(\partial G_M / \partial x \right)_{x=t+0} - \left(\partial G_M / \partial x \right)_{x=t-0} = 1/p(t)$$

$$\text{i.e.,} \quad b_1 + t/2 - (a_1 + t/2) = -1 \quad \text{or} \quad b_1 - a_1 = -1$$

which is the same relation as (13). Thus, we see that the jump condition on $\partial G_M / \partial x$ is automatically satisfied.

$$\text{From (11) and (13),} \quad a_2 - b_2 = -t \quad \text{so that} \quad b_2 = a_2 + t \quad \dots (14)$$

$$\text{Again, from (13),} \quad b_1 = a_1 - 1 \quad \dots (15)$$

Substituting the values of b_2 and b_1 given by (14) and (15) respectively in (12), we have

$$a_2 - (a_2 + t) = a_1 + a_1 - 1 \quad \text{so that} \quad a_1 = (1 - t)/2 \quad \dots (16)$$

Substituting the values of b_2 and b_1 given by (14) and (15) respectively in (9), we have

$$G_M(x, t) = \begin{cases} a_1 x + a_2 + x^2/4, & \text{if } -1 \leq x < t \\ (a_1 - 1) + a_2 + t + x^2/4, & \text{if } t < x \leq 1 \end{cases} \quad \dots (17)$$

(iv) In order that $G_M(x, t)$ may be symmetric, we have

$$\int_{-1}^1 G_M(x, t) w(x) dx = 0 \quad \text{or} \quad \int_{-1}^1 G_M(x, t) dx = 0, \text{ as } w(x) = \frac{1}{\sqrt{2}}$$

$$\text{or} \quad \int_{-1}^t G_M(x, t) dx + \int_t^1 G_M(x, t) dx = 0$$

$$\text{or} \quad \int_{-1}^t \left(a_1 x + a_2 + \frac{x^2}{4} \right) dx + \int_t^1 \left(a_2 + t + a_1 x - x + \frac{x^2}{4} \right) dx = 0, \text{ by (17)}$$

$$\text{or} \quad \left[a_1 (x^2/2) + a_2 x + x^3/12 \right]_{-1}^t + \left[a_2 x + t x + a_1 (x^2/2) - x^2/2 + x^3/12 \right]_t^1 = 0$$

$$\text{or} \quad a_1 (t^2/2) + a_2 t + t^3/12 - (a_1/2 - a_2 - 1/12) + a_2 + t + a_1/2 - 1/2 + 1/12 - (a_2 t + t^2 + a_1 (t^2/2) - t^2/2 + t^3/12) = 0$$

$$\text{or} \quad 2a_2 = t^2/2 - t + 1/3 \quad \text{or} \quad a_2 = t^2/4 - t/2 + 1/6 \quad \dots (18)$$

Substituting the value of a_1 and a_2 given by (16) and (18) respectively in (17), the symmetric Green's function is given by

$$G_M(x, t) = \begin{cases} (1-t)x/2 + t^2/4 - t/2 + 1/6 + x^2/4, & \text{if } -1 \leq x < t \\ -(1+t)x/2 + t^2/4 - t/2 + 1/6 + t + x^2/4, & \text{if } t < x \leq 1 \end{cases}$$

$$\left[\because (a_1 - 1)x = \left(\frac{1-t}{2} - 1 \right) x = -\frac{1+t}{2} \right]$$

or
$$G_M(x, t) = \begin{cases} t^2/4 + x^2/4 - xt/2 + (x-t)/2 + 1/6, & \text{if } -1 \leq x < t \\ t^2/4 + x^2/4 - xt/2 - (x-t)/2 + 1/6, & \text{if } t < x \leq 1 \end{cases}$$

or
$$G_M(x, t) = \begin{cases} (x-t)^2/4 + (x-t)/2 + 1/6, & \text{if } -1 \leq x < t \\ (x-t)^2/4 - (x-t)/2 + 1/6, & \text{if } t < x \leq 1 \end{cases}$$

which can also be re-written as

$$G_m(x, t) = (1/4) \times (x-t)^2 - (1/2) \times |x-t| + 1/6.$$

Ex.3. Transform the following boundary value problem into an integral equation

$$\frac{d}{dx} \left\{ (1-x^2) \frac{dy}{dx} \right\} + \lambda y = 0, \quad \text{where } y(-1) \text{ and } y(1) \text{ are both finite}$$

Sol. Given boundary value problem is $-\frac{d}{dx} \left\{ (1-x^2) \frac{dy}{dx} \right\} = \lambda y, \quad -1 \leq x \leq 1 \quad \dots (1)$

with boundary conditions: $y(1) = \text{finite} \quad \text{and} \quad y(-1) = \text{finite} \quad \dots (2)$

Here the operator $-(d/dx)\{(1-x^2)(d/dx)\}$ is a self adjoint operator.

Consider the associated self adjoint system :

$$-\frac{d}{dx} \left\{ (1-x^2) \frac{dy}{dx} \right\} = 0, \quad -1 \leq x \leq 1 \quad \dots (3)$$

and boundary conditions $y(-1) = \text{finite} \quad \dots (4a)$

and $y(1) = \text{finite} \quad \dots (4b)$

Integrating (3), we have $(1-x^2) (dy/dx) = A$

Separating variables, $dy = \{A / (1-x^2)\} dx.$

Integrating it, $y(x) = A \log \frac{1+x}{1-x} + B, \quad -1 \leq x \leq 1 \quad \dots (4)$

Since (4) has to satisfy the boundary condition (4a) and (4b), we must have $A = 0$.

Hence the boundary value problem given by (3), (4a) and (4b) has a non-trivial solution $y(x) = B$, where B is an arbitrary constant

Here, $\|y(x)\| = \text{norm of } y(x) = \left[\int_{-1}^1 (y(x))^2 dx \right]^{1/2} = \left[\int_{-1}^1 B^2 dx \right]^{1/2} = B\sqrt{2}$

Let $w(x) = y(x) / \|y(x)\| = B / B\sqrt{2} = 1/\sqrt{2} \quad \dots (5)$

so that $w(x)$ is a non-zero normalized solution of the given boundary value problem. Clearly,

$$\int_{-1}^1 [w(x)]^2 dx = 1.$$

Then, for $x \neq t$, the required modified Green's function $G_M(x, t)$ must satisfy the equation

$$-\frac{d}{dx} \left\{ (1-x^2) \frac{dG_M}{dx} \right\} = -w(x) w(t) = -\frac{1}{2}, \quad \text{using (5)} \quad \dots (6)$$

Integrating (6), $(1-x^2) (dG_M/dx) = x/2 + A$

or
$$\frac{dG_M}{dx} = \frac{x}{2(1-x^2)} + \frac{A}{1-x^2} \quad \text{or} \quad dG_M = \left[\frac{1}{4} \times \frac{(-2x)dx}{1-x^2} + \frac{A}{1-x^2} \right] dx$$

Integrating it,
$$G_M(x, t) = -\frac{1}{4} \log(1-x^2) + \frac{A}{2} \log \frac{1+x}{1-x} + B$$

or $G_M(x, t) = -(1/4) \times \{\log(1+x) + \log(1-x)\} + (A/2) \times \{\log(1+x) - \log(1-x)\} + B$
or $G_M(x, t) = (A/2 - 1/4) \log(1+x) - (A/2 + 1/4) \log(1-x) + B$

Hence, we take

$$G_M(x, t) = \begin{cases} (a_1/2 - 1/4) \log(1+x) - (a_1/2 + 1/4) \log(1-x) + a_2 & \text{if } -1 \leq x < t \\ (b_1/2 - 1/4) \log(1+x) - (b_1/2 + 1/4) \log(1-x) + b_2 & \text{if } t < x \leq 1 \end{cases} \quad \dots (7)$$

In addition to the above property (7), the proposed modified Green's function $G_M(x, t)$ must satisfy the following conditions.

(i) By virtue of boundary conditions (4a) and (4b), $G_M(x, t)$ must be finite at $x = -1$ and $x = 1$. Accordingly, we must have $a_1 = 1/2$ and $b_1 = -1/2$. Hence (7) reduces to

$$G_M(x, t) = \begin{cases} a_2 - (1/2) \times \log(1-x), & \text{if } -1 \leq x < t \\ b_2 - (1/2) \times \log(1+x), & \text{if } t < x \leq 1 \end{cases} \quad \dots (8)$$

(ii) $G_M(x, t)$ is continuous at $x = t$, that is,

$$a_2 - (1/2) \times \log(1-t) = b_2 - (1/2) \times \log(1+t), \text{ using (8)}$$

so that $a_2 - b_2 = (1/2) \times \log(1-t) - (1/2) \times \log(1+t) \quad \dots (9)$

(iii) The derivative of $G_M(x, t)$ with respect to x at the point $x = t$ has a discontinuity of the first kind, the jump being $1/p(t)$, where $p(x)$ is the coefficient of y'' in (3), i.e., $p(x) = -(1-x^2)$. [Note that (3) may be re-written as $-(1-x^2)y'' + 2xy' = 0$]

i.e.,
$$\left(\partial G_M / \partial x \right)_{x=t+0} - \left(\partial G_M / \partial x \right)_{x=t-0} = 1/p(t) \quad \dots (10)$$

But from (8),
$$\frac{\partial G_M}{\partial x} = \begin{cases} 1/2(1-x), & -1 \leq x < t \\ (-1)/2(1+x), & t < x \leq 1 \end{cases}$$

Hence, (10) reduces to

$$-\frac{1}{2(1+t)} - \frac{1}{2(1-t)} = -\frac{1}{1-t^2} \quad \text{or} \quad \frac{1}{1-t^2} = \frac{1}{1-t^2},$$

showing that the jump condition at $x = t$ is automatically satisfied.

(iv) In order that $G_M(x, t)$ may be symmetrical, we have

$$\int_{-1}^1 G_M(x, t) w(x) dx = 0 \quad \text{or} \quad \int_{-1}^1 G_M(x, t) dx = 0, \text{ as } w(x) = \frac{1}{\sqrt{2}}$$

or
$$\int_{-1}^t G_M(x, t) dx + \int_t^1 G_M(x, t) dx = 0$$

or
$$\int_{-1}^t \left\{ a_2 - \frac{1}{2} \log(1-x) \right\} dx + \int_t^1 \left\{ b_2 - \frac{1}{2} \log(1+x) \right\} dx = 0, \text{ using (8)} \quad \dots (11)$$

Now,
$$\begin{aligned} \int_{-1}^t \left\{ a_2 - \frac{1}{2} \log(1-x) \right\} dx &= \int_{-1}^t a_2 dx - \frac{1}{2} \int_{-1}^t 1 \cdot \log(1-x) dx \\ &= \left[a_2 x \right]_{-1}^t - \frac{1}{2} \left\{ \left[x \log(1-x) \right]_{-1}^t - \int_{-1}^t \frac{(-x)}{1-x} dx \right\}, \text{ integrating by parts} \end{aligned}$$

$$\begin{aligned}
&= a_2 (t+1) - \frac{1}{2} \{t \log (1-t) + \log 2\} + \frac{1}{2} \int_{-1}^t \frac{(-x) dx}{1-x} \\
&= a_2 (t+1) - \frac{t}{2} \log (1-t) - \frac{1}{2} \log 2 + \frac{1}{2} \int_{-1}^t \left(1 - \frac{1}{1-x}\right) dx \\
&= a_2 (t+1) - \frac{t}{2} \log (1-t) - \frac{1}{2} \log 2 + \frac{1}{2} [x - \log (1-x)]_{-1}^t \\
&= a_2 (t+1) - \frac{t}{2} \log (1-t) - \frac{1}{2} \log 2 + \frac{t}{2} - \frac{1}{2} \log (1-t) + \frac{1}{2} + \frac{1}{2} \log 2 \\
&= a_2 (t+1) - (t/2) \times \log (1-t) - (1/2) \times \log (1-t) + t/2 + 1/2 \quad \dots (12)
\end{aligned}$$

and $\int_t^1 \left\{ b_2 - \frac{1}{2} \log (1+x) \right\} dx = \int_t^1 b_2 dx - \frac{1}{2} \int_t^1 1 \cdot \log (1+x) dx$

$$\begin{aligned}
&= [b_2 x]_t^1 - \frac{1}{2} \left\{ [x \log (1+x)]_t^1 - \int_t^1 \frac{x}{1+x} dx \right\}, \text{ integrating by parts} \\
&= b_2 (1-t) - \frac{1}{2} \{ \log 2 - t \log (1+t) \} + \frac{1}{2} \int_t^1 \frac{(1+x)-1}{1+x} dx \\
&= b_2 (1-t) - \frac{1}{2} \log 2 + \frac{t}{2} \log (1+t) + \frac{1}{2} \int_t^1 \left(1 - \frac{1}{1+x}\right) dx \\
&= b_2 (1-t) - \frac{1}{2} \log 2 + \frac{t}{2} \log (1+t) + \frac{1}{2} [x - \log (1+x)]_t^1 \\
&= b_2 (1-t) - \frac{1}{2} \log 2 + \frac{t}{2} \log (1+t) + \frac{1}{2} - \frac{1}{2} \log 2 - \frac{t}{2} + \frac{1}{2} \log (1+t) \\
&= b_2 (1-t) + (t/2) \times \log (1+t) + (1/2) \times \log (1+t) - t/2 + 1/2 - \log 2 \quad \dots (13)
\end{aligned}$$

Using (12) and (13), (11) reduces to

$$\begin{aligned}
&a_2 (t+1) - (t/2) \times \log (1-t) - (1/2) \times \log (1-t) + t/2 + 1/2 \\
&+ b_2 (1-t) + (t/2) \times \log (1+t) + (1/2) \times \log (1+t) - t/2 + 1/2 - \log 2 = 0
\end{aligned}$$

or $a_2 (t+1) + b_2 (1-t) = \log 2 - 1 + \frac{t+1}{2} \log (1+t) - \frac{t+1}{2} \log (1-t)$

or $a_2 + b_2 = 2 \log 2 - 1 - (1/2) \times \{ \log (1+t) - \log (1-t) \} \quad \dots (14)$

Solving (9) and (14), we obtain

$$a_2 = \log 2 - 1/2 - (1/2) \times \log (1+t), \quad b_2 = \log 2 - 1/2 - (1/2) \times \log (1-t)$$

Substituting these values in (8), the symmetric modified Green's function is given by

$$\begin{aligned}
G_M(x, t) &= \begin{cases} \log 2 - 1/2 - (1/2) \times \{ \log (1-x) + \log (1+t) \}, & -1 \leq x < t \\ \log 2 - 1/2 - (1/2) \times \{ \log (1+x) + \log (1-t) \}, & t < x \leq 1 \end{cases} \\
\text{or } G_M(x, t) &= \begin{cases} \log 2 - 1/2 - (1/2) \times \log \{ (1-x)(1+t) \}, & -1 \leq x < t \\ \log 2 - 1/2 - (1/2) \times \log \{ (1+x)(1-t) \}, & t < x \leq 1 \end{cases} \quad \dots (15)
\end{aligned}$$

which may also be re-written as

$$G_M(x, t) = \log 2 - \frac{1}{2} - \begin{cases} (1/2) \times \log \{(1-x)(1+t)\}, & -1 < x < t \\ (1/2) \times \log \{(1+x)(1-t)\}, & t < x \leq 1 \end{cases} \quad \dots(15)$$

Transformation of given boundary value problem given by (1)—(2) into an integral equation.

Comparing (1) with $Ly = \phi(x)$, here $\phi(x) = \lambda y$.

Refer result (11b) of Art. 11.13. The required integral equation is given by

$$y(x) = \int_{-1}^1 G_M(x, t) \phi(t) dt + w(x) \int_{-1}^1 w(x) y(x) dx$$

$$i.e., \quad y(x) = \lambda \int_{-1}^1 G_M(x, t) y(t) dt + \frac{1}{2} \int_{-1}^1 y(x) dx, \text{ as } w(x) = \frac{1}{\sqrt{2}}$$

$$\text{which can also be re-written as} \quad y(x) = \lambda \int_{-1}^1 G_M(x, t) y(t) dt + c,$$

$$\text{where } c \text{ is an arbitrary constant given by } c = \frac{1}{2} \int_{-1}^1 y(x) dx$$

Ex.4. Show that the boundary value problem

$(1-x^2) y'' - 2xy' + \lambda y = f(x)$, $y(1), y(-1)$ finite, transforms into the integral equation

$$y(x) = \lambda \int_{-1}^1 G_M(x, t) y(t) dt - \int_{-1}^1 G_M(x, t) f(t) dt + \frac{1}{2} \int_{-1}^1 y(t) dt.$$

$$\text{Also show that} \quad \int_{-1}^1 y dt = \frac{1}{\lambda} \int_{-1}^1 f dt,$$

where $G_M(x, t)$ is Green's function of associated boundary value problem.

Sol. Given boundary value problem is

$$-\frac{d}{dx} \left\{ (1-x^2) \frac{dy}{dx} \right\} = \lambda y - f(x), \quad -1 \leq x \leq 1, \quad y(1) \text{ finite and } y(-1) \text{ finite} \quad \dots (1)$$

Here the operator $-(d/dx)\{(1-x^2)(d/dx)\}$ is a self adjoint operator.

Consider the associated self adjoint system is

$$-\frac{d}{dx} \left\{ (1-x^2) \frac{dy}{dx} \right\} = 0, \quad -1 \leq x \leq 1, \quad y(1) \text{ finite and } y(-1) \text{ finite} \quad \dots (2)$$

Proceed as in Ex. 3. to show that modified Green's function of (2) is given by relation (15) of Ex.

3. Comparing (1) with $Ly = \phi(x)$, here $\phi(x) = \lambda y(x) - f(x)$

Refer result (11b) of Art. 11.13. The required integral equation is given by

$$y(x) = \int_{-1}^1 G_M(x, t) \phi(t) dt + w(x) \int_{-1}^1 w(x) y(x) dx$$

$$\text{or} \quad y(x) = \int_{-1}^1 G_M(x, t) \{\lambda y(t) - f(t)\} dt + w(x) \int_{-1}^1 w(t) y(t) dt$$

$$\text{or} \quad y(x) = \lambda \int_{-1}^1 G_M(x, t) y(t) dt - \int_{-1}^1 G_M(x, t) f(t) dt + \frac{1}{2} \int_{-1}^1 y(t) dt$$

$$\left[\because w(x) = 1/\sqrt{2} \Rightarrow w(t) = 1/\sqrt{2} \right]$$

MISCELLANEOUS EXERCISE ON CHAPTER 11

1. Find the boundary value problem that is equivalent to the integral equation

$$y(x) = \lambda \int_{-1}^1 (1 - |x - t|) y(t) dt$$

2. Transform the boundary value problem $y'' + (y'/2x) + \lambda x^{1/2} y = 0$, $y(0) = y(1) = 0$ to integral equation.

3. Convert the initial value problem $y'' + \lambda x^2 y = 0$, $y'(0) = y(0) = 0$ into a Volterra integral equation.

4. Reduce the differential equation $y'' + \{P(x) - \eta^2\} y = 0$, η is a known positive constant, with end condition $y(0) = 0$, $(y'/y) = -\eta$ at $x = x_0$ into a Fredholm integral equation.

5. Show that the Green's function for $y'' = 0$, $y(0) + y(1) = 0$, $y'(0) + y'(1) = 0$ is

$$G(x, t) = \begin{cases} 1 - t, & 0 \leq x < t \\ 1 - x, & t < x \leq 1 \end{cases}$$

Hence solve the B.V.P. $y'' = f(x)$, $y(0) + y(1) = 0$, $y'(0) + y'(1) = 0$ where

$$(i) f(x) = \sin \pi x \quad (ii) f(x) = e^x, \quad 0 \leq x \leq 1 \quad (iii) f(x) = x$$

6. Find the modified Green's function for the system $y'' - \lambda y = 0$, $y(0) = y(1)$, $y'(0) = y'(1)$.

7. Transform the boundary value problem $y'' + y = f(x)$, $y(0) = y(\pi) = 0$, into an integral equation.

8. Reduce the boundary value problem

$$\frac{d^2}{dx^2} \left\{ (4 + x)^3 \frac{d^2 y}{dx^2} \right\} - \lambda (4 + x) y = 0, \quad y(0) = 0 = y(1), \quad y''(0) = 0 = y''(1) \text{ to a Fredholm}$$

integral equation.

9. Find Green's function for the initial value problem $m(d^2 y / dx^2) = f(x)$, with initial conditions

$$y(0) = y'(0) = 0 \quad \text{Ans. } G(x, t) = \begin{cases} 0, & 0 \leq x < t \\ (x - t)/m, & t < x < \infty \end{cases}$$

10. Find the solution of the initial value problem $y'' + w^2 y = f(x)$, $y(0) = y_1$ and $y'(0) = y_2$.

$$\text{Ans. Green's function} = G(x, t) = \frac{1}{w} \sin w(x - t) H(x - t) = \begin{cases} 0, & 0 \leq x < t \\ (1/w) \times \sin w(x - t), & t < x < \infty \end{cases}$$

$$\text{and solution is } y(x) = \frac{1}{w} \int_0^x \sin w(x - t) H(x - t) f(t) dt + y_1 \cos wx + y_2 \sin wx,$$

where $H(x - t)$ is the well known Heaviside unit function. (refer chapter (10)).

11. Solve the boundary value problem $y'' - y = -2e^x$ with boundary conditions $y(0) = y'(0)$ and $y(l) + y'(l) = 0$.

$$\text{Ans. } y(x) = (l - x)e^x + \sinh x \quad \text{where} \quad G(x, t) = \begin{cases} (1/2) \times e^{x-t}, & 0 \leq x < t \\ (1/2) \times e^{-(x-t)}, & t < x \leq l \end{cases}$$

12. Reduce the boundary value problem $y'' + (\pi^2/4)y = \lambda y + \cos(\pi x/2)$, $y(-1) = y(1)$ and $y'(-1) = y'(1)$ to an integral equation.

Ans. $y(x) = \lambda \int_{-1}^1 G(x, t) y(t) dt - \left(\frac{x}{\pi} \sin \frac{\pi x}{2} + \frac{2}{\pi^2} \cos \frac{\pi x}{2} \right),$

where $G(x, t) = \begin{cases} (1/\pi) \sin \{(x-t) \pi/2\}, & -1 \leq x < t \\ (1/\pi) \sin \{(t-x) \pi/2\}, & t < x \leq 1 \end{cases}$

13. Reduce the following boundary value problem to an integral equation

$y'' + \lambda y = 2x + 1$ with $y(0) = y'(1), \quad y'(0) = y(1).$

Ans. $y(x) = -\lambda \int_0^1 G(x, t) y(t) dt - \frac{1}{6} (2x^3 + 3x^2 - 17x - 5),$

where $G(x, t) = \begin{cases} -\{(t-2)x + (t-1)\}, & 0 \leq x < t \\ -\{(t-1)x - 1\}, & t < x \leq 1 \end{cases}$

14. Construct the Green's function for the boundary value problem

$y'''(x) = 0, \quad y(0) = y'(1) = 0 \quad \text{and} \quad y'(0) = y(1)$

Ans. $G(x, t) = \begin{cases} (1/2) \times x(t-1)(x-xt+2t), & 0 \leq x < t \\ (1/2) \times [x(2-x)(t-2)+t], & t < x \leq 1 \end{cases}$

15. Find the modified Green's function for the system $L = d^2/dx^2$ with $y(0) = -y(1)$. This system is not self-adjoint.

16. (a) Show that a sufficient condition for the modified Green's function to be symmetric is

$$\int_a^b G_M(x, t) w(x) dx = 0,$$

where $G_M(x, t)$ and $w(x)$ have the same meaning as explained in Art. 11.12.

Show that there are problems where this condition is not necessary.

- (b) Apply the result of part (a) to find the modified Green's function for the system

$L = (d^2/dx^2) + 1$ with $y(0) = y(2\pi), \quad y'(0) = y'(2\pi).$

17. Find the consistency condition for the following systems,

(i) $-(d^2 y/dx^2) = f(x); \quad y'(0) = \alpha, \quad y'(1) = \beta$

(ii) $d^2 y/dx^2 + y = f(x); \quad y(0) - y(2\pi) = \alpha, \quad y'(0) - y'(2\pi) = \beta$

18. Obtain the modified Green's function for the systems

(i) $ky'' + f(x) = 0, \quad y'(0) = y'(l) = 0, \quad 0 \leq x \leq l$

(ii) $ky'' + f(x) = 0, \quad y(-l) = y(l); \quad y'(-l) = y'(l); \quad -l \leq x \leq l$

and hence transform these boundary value problems into respective integral equations.

Ans. (i) $G_M(x, t) = \frac{l}{3k} + \frac{x^2 + t^2}{2kl} - \begin{cases} t/k, & \text{if } 0 \leq x < t \\ x/k, & \text{if } t < x \leq l \end{cases}; \quad y(x) = c + \int_0^l G_M(x, t) f(t) dt$

(ii) $G_M(x, t) = 1/6k + (x-t)^2/4kl + |x-t|/2k; \quad y(x) = c + \int_{-l}^l G_M(x, t) f(t) dt$

19. Obtain the modified Green's function for the following boundary value problems:

(i) $y'' + ky = f(x); \quad y(0) = y(\pi) = 0, \quad 0 \leq x \leq \pi$

(ii) $y'' + f(x), \quad y(-\pi) = y(\pi); \quad y'(-\pi) = y'(\pi), \quad -\pi \leq x \leq \pi$

11.62 *Applications of integral equations and Green's function to ordinary differential equation*

(iii) $y'' = f(x), \quad y(0) = 0, \quad y(1) = y'(1), \quad 0 \leq x \leq 1$

(vi) $y'' + k^2 y = f(x), \quad y(0) = y(\pi) = 0; \quad 0 \leq x \leq \pi$

(v) $y'' + \pi^2 y = f(x), \quad y(0) = y(1) = 0; \quad 0 \leq x \leq 1$

20. Develop the theory of modified Green's function in the case of a self-adjoint system where the completely homogeneous system has two linearly independent solutions $w_1(x)$ and $w_2(x)$ (hence every solution of the homogeneous equation satisfies the boundary conditions.)

21. Obtain the modified Green's function for

$$y'' + k^2 y = f(x); \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi); \quad 0 \leq x \leq 2\pi \quad (\text{Kanpur 2011})$$

Hint. For solution of the above exercise 21 and other problems based on modified (or generalized) Green's function, refer Appendeix D of this book.

Applications of integral equations to partial differential equations

12.1 INTRODUCTION

We have already discussed the applications of integral equations to ordinary differential equations. But the most important applications of integral equations arise in finding the solutions of boundary value problems associated with partial differential equations of the second order. It has been shown that the boundary value problems for *equations of elliptic type* can be reduced to Fredholm integral equations, while the boundary value problems for *equations of parabolic and hyperbolic types* can be reduced to Volterra integral equations. However in the present chapter we shall deal with the linear partial differential equations of the elliptic type. In particular, we shall focus our attention to the *Laplace, Poisson and Helmholtz* equations because some important achievements have been arrived from their study.

Occurrence of the Laplace, Poisson and Helmholtz equations.

Various physical phenomena are governed by the well known Laplace, Poisson and Helmholtz equations. A few of them, frequently encountered in applications are : steady heat conduction, irrotational flow of an ideal fluid, distribution of gravitational potential, electrostatics, dielectrics, magnetostatics, steady currents, surface waves on a fluid etc.

While dealing with elliptic partial differential equations, the following three types of boundary value problems arise :

- (i) *The Dirichlet problem.* In this case we prescribe the value of the solution on the boundary.
- (ii) *The Neumann problem.* In this case we prescribe the normal derivative of the solution on the boundary.
- (iii) *Mixed boundary value problem.* In this case we prescribe the Dirichlet condition on some parts and the Neumann condition on the other parts of the boundary.

Types of boundary conditions while dealing with an elliptic equation

- (i) *Dirichlet condition* : When the value of solution is prescribed on the boundary surface.
- (ii) *Neumann condition* : When the value of the normal derivative of the solution is prescribed on the boundary surface.
- (iii) *Mixed boundary condition* : When we prescribe the Dirichlet condition on some parts and Neumann condition on the other parts of the body. Sometimes a linear combination of solution and its normal derivative is prescribed on the boundary surface.

Some important mathematical tools.

Suppose that $\mathbf{A}$ is a continuously differentiable vector field and u and v have partial derivatives of the second order which are continuous in the bounded region R . Let S denote the boundary of R and let $\mathbf{n}$ denote the unit normal outward to S . Then, we have

$$(i) \text{ Divergence theorem.} \quad \int_R \operatorname{div} \mathbf{A} \, dV = \int_S \mathbf{A} \cdot \mathbf{n} \, dS$$

$$(ii) \text{ Green's first identity} \quad \int_R u \nabla^2 v \, dV = - \int_R (\text{grad } u \cdot \text{grad } v) \, dV + \int_S u \frac{dv}{dn} \, dS$$

$$(iii) \text{ Green's second identity} \quad \int_R (u \nabla^2 v - v \nabla^2 u) \, dV = \int_S \left(u \frac{dv}{dn} - v \frac{du}{dn} \right) dS,$$

where the Laplacian operator ∇^2 , in cartesian coordinates x, y, z , is defined as

$$\nabla^2 \equiv \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$$

and $\partial / \partial n$ denotes differentiation along the outward-drawn normal on S .

The equation $\nabla^2 u = 0$ is called *Laplace's equation* and a function satisfying Laplace's equation is called a *harmonic function*.

Poisson equation is given by $\nabla^2 u = -4\pi\rho$ where ρ is given function of the position.

Some notations to be used in further study

Boldface letter such as $\mathbf{x}$ shall stand for the triplet (x_1, x_2, x_3) . The quantities R_i and R_e will stand for the regions interior and exterior to S , respectively. In what follows we shall not write $dS_{\mathbf{x}}$ or dS_{ξ} to signify the integration with respect to the variable $\mathbf{x}$ or ξ . In place of this, we shall merely write dS and it shall be clear from the context as to what the variable of integration is.

12.2 INTEGRAL REPRESENTATION FORMULAS FOR THE SOLUTIONS OF THE LAPLACE AND POISSON EQUATIONS.

We begin with the *fundamental solution* (or *free-space solution*) $E(\mathbf{x}; \xi)$ which satisfies

$$-\nabla^2 E = \delta(\mathbf{x} - \xi) \quad \dots (1)$$

and vanishes at infinity. The function $E(\mathbf{x}; \xi)$ can be interpreted as the electrostatic potential at an arbitrary field point $\mathbf{x}$ due to a unit charge at the source point ξ . We know that $E(\mathbf{x}; \xi)$ is given by

$$E(\mathbf{x}; \xi) = 1/4\pi r = 1/4\pi |\mathbf{x} - \xi| \quad \dots (2)$$

For the two-dimensional case, we have

$$E(\mathbf{x}; \xi) = (1/2\pi) \times \log(1/r) = (1/2\pi) \times \log(1/|\mathbf{x} - \xi|), \quad \dots (3)$$

where $\mathbf{x} = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2)$

The fundamental solution is employed to find the solution of the Poisson equation

$$-\nabla^2 u = 4\pi\rho. \quad \dots (4)$$

as follows. Multiply (1) by $u(\mathbf{x})$, (4) by $E(\mathbf{x}, \xi)$, subtract, integrate over the region R_p , and use Green's second identity and the shifting property of the Dirac delta function (refer Art 10.7 of chapter 10). After relabeling $\mathbf{x}$ and ξ , we finally obtain

$$u(\mathbf{x}) = \int_R \frac{\rho}{r} \, dV - \frac{1}{4\pi} \int_S u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS + \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial u}{\partial n} \, dS \quad \dots (5)$$

If the values of ρ, u and $\partial u / \partial n$ involved in formula (5) are known, for example, let

$$[u]_S = \tau \quad \text{and} \quad [\partial u / \partial n]_S = \sigma \quad \dots (6)$$

then (5) can be re-written as

$$u(P) = \int_R \frac{\rho}{r} \, dV + \frac{1}{4\pi} \int_S \tau(Q) \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS - \frac{1}{4\pi} \int_S \frac{\sigma(Q)}{r} \, dS \quad \dots (7)$$

where P is the field point $\mathbf{x}$ and Q is a point $\boldsymbol{\xi}$ on S . The formulas (5) and (7) can be used to find various important properties of the harmonic function.

With reference to (7), we define some important terms as follows :

The Newtonian or the volume potential. The integral $\int_R (\rho / r) dV$, is known the *Newtonian potential*. For the Newtonian potential $u = \int_R (\rho / r) dV$, we have the following properties :

- (i) $\nabla^2 u = 0$, for points P in R_e .
- (ii) For points P within R_i , the integral is improper but it converges and admits two differentials under the integral sign if the function ρ is sufficiently smooth; the result is $\nabla^2 u = -4 \pi \rho(P)$.

The simple or single-layer potential. The integral $\int_S (\sigma / r) dS$ is known as the *single-layer potential* with charge (or source) density σ . The single-layer potential $u = \int_S (\sigma / r) dS$ has the following properties :

- (i) $\nabla^2 u = 0$, outside S .
- (ii) The integral becomes improper at the surface S but converges uniformly if S is singular. Also, this integral remains continuous as we pass through S .
- (iii) Consider the derivative of u taken in the direction of a line normal to the surface S in the outward direction from S . Then, we have

$$\left(\frac{\partial u}{\partial n} \right)_{P_+} = -2 \pi \sigma(P) + \int_S \sigma(Q) \frac{\cos(\boldsymbol{\xi} - \mathbf{x}, \mathbf{n})}{|\mathbf{x} - \boldsymbol{\xi}|^2} dS \quad \dots (8)$$

and

$$\left(\frac{\partial u}{\partial n} \right)_{P_-} = 2 \pi \sigma(P) + \int_S \sigma(Q) \frac{\cos(\boldsymbol{\xi} - \mathbf{x}, \mathbf{n})}{|\mathbf{x} - \boldsymbol{\xi}|^2} dS, \quad \dots (9)$$

where P_+ and P_- signify that we approach S from R_i and R_e respectively, and where both $\mathbf{x}$ and $\boldsymbol{\xi}$ are on S . Subtracting (8) from (9), we obtain

$$\sigma = \frac{1}{4\pi} \left\{ \left(\frac{\partial u}{\partial n} \right)_{P_-} - \left(\frac{\partial u}{\partial n} \right)_{P_+} \right\}, \quad \dots (10)$$

giving the jump of the normal derivative of u across S .

The double-layer potential. The integral $\int_S \tau (\partial / \partial n) (1 / r) dS$ is known as the *double layer potential* with dipole of density τ . The double-layer potential $u = \int_S \tau (\partial / \partial n) (1 / r) dS$ has the following properties.

- (i) $\nabla^2 u = 0$ outside S .
- (ii) The integral becomes improper at the surface but it converges if the surface S is regular.

(iii) While passing through S , the double-layer potential undergoes a discontinuity such that

$$[u]_{P_+} = 2\pi\tau(P) + \int_S \tau(Q) \frac{\cos(\mathbf{x}-\boldsymbol{\xi}, \mathbf{n})}{|\mathbf{x}-\boldsymbol{\xi}|^2} dS \quad \dots (11)$$

and

$$[u]_{P_-} = -2\pi\tau(P) + \int_S \tau(Q) \frac{\cos(\mathbf{x}-\boldsymbol{\xi}, \mathbf{n})}{|\mathbf{x}-\boldsymbol{\xi}|^2} dS, \quad \dots (12)$$

where P_+ and P_- signify that we approach S from R_i and R_e , respectively, and where both $\mathbf{x}$ and $\boldsymbol{\xi}$ are on S . Subtracting (12), from (11), we have

$$\tau = (1/4\pi) \times \{[u]_{P_+} - [u]_{P_-}\}, \quad \dots (13)$$

giving the jump of u across S .

(iv) The normal derivative remains continuous as S is crossed.

Interior and exterior Dirichlet problems.

For the solution of a boundary value problem for an elliptic equation, we cannot prescribe u and $\partial u / \partial n$ arbitrarily on S . Hence, equation (7) does not allow us to construct a solution of (4) such that u shall itself have arbitrary values of S and also arbitrary values of its normal derivative there. It follows that there exist the following two types of boundary value problems for elliptic equations.

(i) *The Dirichlet problem.* In such a problem the value of the solution is prescribed on S .

(ii) *The Neumann problem.* In such a problem the value of the normal derivative of the solution is prescribed on S .

We begin with the Dirichlet problem. Consider first the Dirichlet problem for the region exterior to the unit sphere in three dimensions. In addition to prescribing boundary values on the surface of the unit sphere, it will be necessary to impose some sort of boundary conditions at infinity in order to get a unique solution. To illustrate the need for a boundary condition at infinity, we observe that the functions $u_1(\mathbf{x}) = 1$ and $u_2(\mathbf{x}) = 1/r$ are both harmonic for $r > 1$ and assume the same value 1 on the surface of the unit sphere. Hence if we require that the solution vanishes at infinity, then $u_2(\mathbf{x})$ is the required solution. Indeed, it has been proved that, when we solve the Dirichlet problem for the exterior of the unit sphere (by expansions in *spherical harmonics*) such that the potential vanishes at infinity, then we arrive at the following nature of the solution

$$[u]_{\infty} = O(1/r) \quad \text{and} \quad (\partial u / \partial r)_{\infty} = O(1/r^2) \quad \dots (14)$$

With help of these considerations and the value of the fundamental solution, it has been shown that

$$\lim_{r \rightarrow \infty} \int_S \left(E \frac{\partial u}{\partial r} - u \frac{\partial E}{\partial r} \right) dS = 0, \quad \dots (15)$$

on the surface of the sphere of radius r .

We now proceed to define and analyze the exterior and interior Dirichlet problems for an arbitrary surface S .

The exterior Dirichlet problem. Definition.

Let R_e be the exterior of a bounded region R_i in three dimensions, where the boundary of R_i is denoted by S . The exterior Dirichlet problem is the boundary value problem

$$\nabla^2 u_e = 0, \quad \mathbf{x} \in R_e; \quad [u_e]_S = f, \quad [u_e]_{\infty} = O(1/r), \quad (\partial u_e / \partial r)_{\infty} = O(1/r^2) \quad \dots (16)$$

where $f(\mathbf{x})$ is a given continuous function on S .

The interior Dirichlet problem. Definition.

Let R_i be a bounded interior region and let S denotes the boundary of R_i . The interior Dirichlet problem is the boundary value problem.

$$\nabla^2 u_i = 0, \quad \mathbf{x} \in R_i, \quad [u_i]_S = f, \quad \dots (17)$$

where $f(\mathbf{x})$ is a given continuous function on S .

Solution of the interior Dirichlet problem (17). Suppose that such a solution u is the potential of a double-layer with density τ (which is as yet unknown)

$$u_i(\mathbf{x}) = \int_S \frac{\tau(\boldsymbol{\xi}) \cos(\mathbf{x} - \boldsymbol{\xi}, \mathbf{n})}{r^2} dS \quad \dots (18)$$

In order that u_i may satisfy the boundary condition $[u_i]_S = f(\mathbf{x})$, we use relation (12) and obtain the following Fredholm integral equation of the second kind for $\tau(P)$:

$$\tau(P) = -\frac{1}{2\pi} f(P) + \int_S K(P, Q) \tau(Q) dS, \quad \dots (19)$$

where the kernel $K(P, Q)$ is given by

$$K(P, Q) = \{\cos(\mathbf{x} - \boldsymbol{\xi}, \mathbf{n})\} / 2\pi |\mathbf{x} - \boldsymbol{\xi}|^2, \quad \dots (20)$$

and $P (= \mathbf{x})$ and $Q (= \boldsymbol{\xi})$ are both on S .

Now, we solve (19) for τ and substitute this solution in (18) to obtain the desired solution of the Dirichlet problem given by (17).

Proceeding likewise, the Dirichlet problem for an external domain bounded internally by S can be reduced to the solution of a Fredholm integral equation of the second kind.

We now proceed to find an integral-equation formulation of the exterior and interior boundary value problems (16) and (17) in a composite medium when $f(\mathbf{x})$ is the same function in both these problems.

We know that the free space fundamental solution $E(\mathbf{x}; \boldsymbol{\xi})$ satisfies

$$-\nabla^2 E = \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \text{for all } \mathbf{x} \text{ and } \boldsymbol{\xi}. \quad \dots (21)$$

Multiplying (17) by E , (21) by u_i , adding, integrating and using Green's second identity, we obtain

$$\int_S \left(E \frac{\partial u_i}{\partial n} - u_i \frac{\partial E}{\partial n} \right) dS = \begin{cases} u_i(\boldsymbol{\xi}), & \text{when } \boldsymbol{\xi} \in R_i \\ 0, & \text{when } \boldsymbol{\xi} \in R_e \end{cases} \quad \dots (22)$$

where $\mathbf{n}$ is the outward normal to R_i on S .

Similarly, we multiply (16) by E , (21) by u_e , add and integrate over the region bounded internally by S and externally by a sphere S_r and apply Green's second identity. Noting that the contribution from S_r vanishes as $r \rightarrow \infty$ in view of the boundary conditions at infinity, we arrive at

$$\int_S \left(-E \frac{\partial u_e}{\partial n} + u_e \frac{\partial E}{\partial n} \right) dS = \begin{cases} 0, & \text{when } \boldsymbol{\xi} \in R_i \\ u_e(\boldsymbol{\xi}), & \text{when } \boldsymbol{\xi} \in R_e \end{cases} \quad \dots (23)$$

where we have used the fact that the outward normal to R_e and S is now in the $-\mathbf{n}$ direction.

We now add (22) and (23). Since u_i and u_e are both equal to f on S , their contributions cancel. Thus, we obtain

$$\int_S E(\mathbf{x}; \boldsymbol{\xi}) \left(\frac{\partial u_i}{\partial n} - \frac{\partial u_e}{\partial n} \right) dS = \begin{cases} u_i(\boldsymbol{\xi}), & \text{if } \boldsymbol{\xi} \in R_i \\ u_e(\boldsymbol{\xi}), & \text{if } \boldsymbol{\xi} \in R_e \end{cases} \quad \dots (24)$$

where $\mathbf{x} \in S$. Using the relations (2) and (10) and relabelling $\mathbf{x}$ and ξ , we obtain

$$\int_S \frac{\sigma(\xi)}{|\mathbf{x} - \xi|} dS = \begin{cases} u_i(\mathbf{x}), & \text{if } \mathbf{x} \in R_i \\ u_e(\mathbf{x}), & \text{if } \mathbf{x} \in R_e \end{cases} \quad \dots (25)$$

that is, a single-layer potential with unknown charge density σ . Finally, using the boundary conditions

$$[u_i]_S = [u_e]_S = f, \quad \dots (26)$$

in (25), we arrive at the Fredholm integral equation of the first kind

$$f(\mathbf{x}) = \int_S \frac{\sigma(\xi)}{|\mathbf{x} - \xi|} dS, \quad \dots (27)$$

with both $\mathbf{x}$ and ξ on S . We now solve (27) for the unknown $\sigma(\xi)$ and substitute this value of $\sigma(\xi)$ in (25) to calculate $u_i(\mathbf{x})$ and $u_e(\mathbf{x})$.

Interior and exterior Neumann problems.

In this case we are required to find the solution of Laplace or Poisson equation when the normal derivative is prescribed on S .

The exterior Neumann problem. Definition

Let R_e be the exterior of a bounded region R_i in three dimensions, where the boundary of R_i is denoted by S . The exterior Neumann problem is the boundary value problem :

$$\nabla^2 u_e = 0, \quad \mathbf{x} \in R_e, \quad (\partial u_e / \partial n)_S = f, \quad [u_e]_\infty = 0 \quad \dots (28)$$

The interior Neumann problem, Definition.

Let R_i be a bounded interior region and let S denote the boundary of R_i . The interior Neumann problem is the boundary value problem

$$\nabla^2 u_i = 0, \quad \mathbf{x} \in R_i, \quad (\partial u_i / \partial n)_S = f, \quad \dots (29)$$

where $\mathbf{n}$ is the outward normal to R_i and f is a given continuous function on S .

We first observe that (29) cannot have a solution for every f . This is clear from the physical interpretation of (29) as a steady-state heat conduction problem. We have no sources in the region R_i and the heat flow is prescribed on S . These conditions are consistent with the steady state only if the total heat flow through S vanishes. Accordingly, a solution of (29) will exist only if

$$\int_S f(\xi) dS = 0, \quad \xi \in S. \quad \dots (30)$$

Alternatively, we now propose to show that (30) is a necessary condition for a solution of (29) to exist.

$$\text{Indeed, recall that } u_i \text{ is harmonic in } R_i, \text{ that is,} \quad \nabla^2 u_i = 0 \quad \dots (31)$$

$$(31) \text{ implies that} \quad \int_{R_i} (\nabla^2 u_i) dV = 0 \quad \dots (32)$$

$$\text{Since } \nabla^2 = \text{div grad}, (32) \text{ reduces to} \quad \int_{R_i} \text{div grad } u_i dV = 0 \quad \dots (33)$$

Using the divergence theorem, (33) reduces to

$$\int_S \text{grad } u_i \cdot \mathbf{n} dS = 0, \quad \text{i.e.,} \quad \int_S \frac{\partial u_i}{\partial n} dS = 0$$

or
$$\int_S f(\xi) dS = 0, \quad \xi \in S,$$

where we have used the boundary condition $(\partial u_i / \partial n)_S = f$, which is involved in (29).

If the condition (30) is satisfied, it can be shown that (29) has a solution, but that the solution is no longer unique. Clearly, to any particular solution of (29) we can add an arbitrary constant and the resulting function still satisfies (29).

Remark. For the exterior Neumann problem (28) in three dimensions no such restriction as (30) is needed. The problem (28) has unique solution.

The exterior and interior Neumann problems can be reduced to integral equations by using a method employed for the corresponding Dirichlet problem.

Now, we propose to find a solution of the interior Neumann problem (29) in the form of the potential of a simple layer

$$u_i = \int_S \frac{\sigma(Q)}{r} dS, \quad \dots (34)$$

which is a harmonic function in R_i . Clearly, (34) will be a solution of (29) if the density σ is so chosen that

$$(\partial u_i / \partial n)_{P_-} = f(P), \quad P \in S \quad \dots (35)$$

Using the relation (9), we obtain

$$f(P) = \left[\frac{\partial u_i}{\partial n} \right]_{P_-} = 2\pi \sigma(P) + \int_S \sigma(Q) \frac{\cos(\xi - \mathbf{x}, \mathbf{n})}{r^2} dS,$$

or
$$\sigma(P) = \frac{1}{2\pi} f(P) - \int_S \sigma(Q) \frac{\cos(\xi - \mathbf{x}, \mathbf{n})}{2\pi r^2} dS, \quad \dots (37)$$

showing that $\sigma(P)$ is a solution of the Fredholm integral equation of the second kind (37).

Exercise. Show that the solution of the exterior Neumann problem also leads to a similar integral equation. Also, give the integral equation formulation of the problem (28) and (29) in a composite medium when f is the same function in both these problems. Proceeding as for the corresponding Dirichlet problem, obtain a Fredholm integral equation of the first kind. Also, show that instead of a single layer potential, a double-layer potential is obtained.

Solution. Left as an exercise

12.3 SOLVED EXAMPLES BASED ON ART 12.2

Ex. 1. Obtain electrostatic potential due to a thin circular disc.

Sol. Take S to be a circular disc of radius a and let V be the potential prescribed on S . With origin on the centre of the disc and z axis normal to the plane of the disc, we shall use cylindrical polar coordinates (ρ, ϕ, z) . Then the given disc occupies the region $z = 0$, $0 \leq \rho \leq a$ for all ϕ . Without any loss of generality, let the potential V on the disc be $f^{(n)}(\rho) \cos n\phi$, where n is an arbitrary integer, because we can use the Fourier superposition principle. It follows that the charge density σ will also be of the form $\sigma^{(n)}(\rho) \cos n\phi$. Then equation (27) of Art. 12.2 reduces to

$$f^{(n)}(\rho) \cos n\phi = \iint_{\text{disc}} \frac{\sigma(\xi)}{|\mathbf{x} - \xi|} dS, \quad \dots (1)$$

where $\mathbf{x} = (\rho, \phi, 0)$ and $\xi = (t, \phi_1, 0)$. Re-writing (1) we have

$$f^{(n)}(\rho) \cos n\phi = \int_0^a \int_0^{2\pi} \frac{t \sigma^{(n)}(t) \cos n\phi_1 d\phi_1 dt}{\{\rho^2 + t^2 + 2\rho t \cos(\phi - \phi_1)\}^{1/2}} \quad \dots (2)$$

Let $\phi_1 - \phi = \psi$ so that $\phi_1 = \phi + \psi$ and $d\phi_1 = d\psi$. Then, we have

$$\begin{aligned} \int_0^{2\pi} \frac{\cos n \phi_1 d\phi_1}{\{\rho^2 + t^2 - 2\rho t \cos(\phi - \phi_1)\}^{1/2}} &= \int_{-\phi}^{2\pi-\phi} \frac{\cos n(\phi + \psi) d\psi}{\{\rho^2 + t^2 - 2\rho t \cos \psi\}^{1/2}} = \int_{-\phi}^{2\pi-\phi} \frac{\cos n \phi \cos n \psi d\psi}{(\rho^2 + t^2 - 2\rho t \cos \psi)^{1/2}} \\ &= \int_{-\phi}^0 \frac{\cos n \phi \cos n \psi d\psi}{(\rho^2 + t^2 - 2\rho t \cos \psi)^{1/2}} + \int_0^{2\pi} \frac{\cos n \phi \cos n \psi d\psi}{(\rho^2 + t^2 - 2\rho t \cos \psi)^{1/2}} + \int_{2\pi}^{2\pi-\phi} \frac{\cos n \phi \cos n \psi d\psi}{(\rho^2 + t^2 - 2\rho t \cos \psi)^{1/2}} \\ &= I_1 + I_2 + I_3, \text{ say} \end{aligned} \quad \dots (3)$$

Now,
$$I_1 = - \int_{-\phi}^0 \frac{\cos n \phi \cos n \psi d\psi}{(\rho^2 + t^2 + 2\rho t \cos \psi)^{1/2}} = \int_{2\pi-\phi}^{2\pi} \frac{\cos n \phi \cos n \psi' d\psi'}{(\rho^2 + t^2 - 2\rho t \cos \psi')^{1/2}}$$

[Putting $\psi = \psi' - 2\pi$ so that $d\psi = d\psi'$]

or
$$I_1 = - \int_{2\pi}^{2\pi-\phi} \frac{\cos n \phi \cos n \psi d\psi}{(\rho^2 + t^2 - 2\rho t \cos \psi)^{1/2}}, \text{ using standard properties of definite integrals.}$$

Thus, $I_1 = -I_3$ and hence (3) reduces to

$$\int_0^{2\pi} \frac{\cos n \phi_1 d\phi_1}{\{\rho^2 + t^2 - 2\rho t \cos(\phi - \phi_1)\}^{1/2}} = -I_3 + I_2 + I_3 = I_2.$$

Hence,
$$\int_0^{2\pi} \frac{\cos n \phi_1 d\phi_1}{\{\rho^2 + t^2 - 2\rho t \cos(\phi - \phi_1)\}^{1/2}} = \cos n \phi \int_0^{2\pi} \frac{\cos n \psi d\psi}{(\rho^2 + t^2 - 2\rho t \cos \psi)^{1/2}} \quad \dots (4)$$

Using (4), (2) yields

$$f^{(n)}(\rho) = \int_0^a \int_0^{2\pi} \frac{t \sigma^{(n)}(t) \cos n \psi d\psi dt}{(\rho^2 + t^2 - 2\rho t \cos \psi)^{1/2}} \quad \dots (5)$$

We now use the following expansion formula :

$$(\rho^2 + t^2 - 2\rho t \cos \psi)^{-1/2} = \sum_{k=0}^{\infty} \int_0^{\infty} (2 - \delta_{ok}) \cos(k\psi) J_k(\rho u) J_k(tu) du, \quad \dots (6)$$

where δ_{ok} is the well known *Kronecker delta* defined by

$$\delta_{ok} = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases} \quad \dots (7)$$

Using the expansion formula (6) in (5) and using the orthogonality of cosine functions, we arrive at the following Fredholm integral equation

$$f^{(n)}(\rho) = \int_0^a t \sigma^{(n)}(t) K_0(t, \rho) dt, \quad \dots (8)$$

where the kernel $K_0(t, \rho)$ is given by

$$K_0(t, \rho) = 2\pi \int_0^{\infty} J_n(\rho u) J_n(tu) du \quad \dots (9)$$

Remark. For an annular disc of radius b and outer radius a , the formula corresponding to (8)

takes the form
$$f^{(n)}(\rho) = \int_b^a t \sigma^{(n)}(t) K_0(t, \rho) dt \quad \dots (10)$$

Ex. 2. Solve the integral equation (27) of Art. 12.2 when S is a unit sphere and $f = \sin \theta \cos \phi$.

Sol. In terms of spherical coordinates (r, θ, ϕ) , we know that the elementary area on the surface of unit sphere is $\sin \theta d\theta d\phi$. Hence the integral equation (27) of Art. 12.2 may be re-written as

$$\sin \theta \cos \phi = \int_0^{2\pi} d\phi_1 \int_0^\pi \frac{\sin \theta_1 \sigma(\theta_1, \phi_1) d\theta_1}{|\mathbf{x} - \boldsymbol{\xi}|} \quad \dots (1)$$

In what follows we shall use some results of spherical harmonics*

$$\frac{1}{|\mathbf{x} - \boldsymbol{\xi}|} = \sum_{n=0}^{\infty} N_{0,n} \sum_{m=-n}^n \frac{Y_n^m(\theta, \phi) Y_n^m(\theta_1, \phi_1)}{N_{m,n}} \quad \dots (2)$$

where $Y_n^m(\theta, \phi)$ are the spherical harmonics (refer Art 10.11 in chapter 10) and

$$N_{m,n} = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta |Y_n^m(\theta, \phi)|^2 d\theta = \frac{4\pi}{2n+1} \frac{(n+|m|)!}{(n-|m|)!} \quad \dots (3)$$

Again, let us assume that

$$\sigma(\theta_1, \phi_1) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sigma_{m,n} Y_n^m(\theta_1, \phi_1), \quad \dots (4)$$

$$\text{and observe that} \quad \sin \theta \cos \phi = (1/2) \times [Y_1^1(\theta, \phi) + Y_1^{-1}(\theta, \phi)] \quad \dots (5)$$

Substituting the values given by (2), (3) and (4) in (1) and also using the orthogonality properties of the spherical harmonics, we have

$$\sigma_{1,1} = \sigma_{1,-1} = 3/8\pi. \quad \dots (6)$$

$$\text{and} \quad \sigma_{m,n} \equiv 0 \quad \text{for all other values of } m \text{ and } n. \quad \dots (7)$$

Using (6) and (7), (4) reduces to

$$\sigma(\theta, \phi) = (3/4\pi) \times P_1^1(\cos \theta) \cos \phi, \quad \dots (8)$$

where P_1^1 is an associated Legendre function.**

12.4 GREEN'S FUNCTION APPROACH

The Green's function is an arbitrary function which plays the same role in the integral formulation of partial differential equations as it plays in the case of ordinary differential equation (refer chapter 11). The Green's function depends on the form of the differential equation, the boundary conditions and the region.

Green's function for the negative Laplacian.

Let R be an open, bounded and three-dimensional space and let its boundary be denoted by S . The Green's function for the negative Laplacian in R is the solution $G(\mathbf{x}, \boldsymbol{\xi})$ of the boundary value problem

$$-\nabla^2 G = \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \mathbf{x} \text{ and } \boldsymbol{\xi} \text{ in } R; \quad G = 0, \quad \mathbf{x} \text{ on } S \quad \dots (1)$$

Unless otherwise stated, all differentiations are with respect to the coordinates of $\mathbf{x}$.

The boundary value problem (1) has a simple interpretation in electrostatics or in steady heat conduction. We can view $G(\mathbf{x}, \boldsymbol{\xi})$ as the temperature at any point $\mathbf{x}$ in R due to a unit source located at $\boldsymbol{\xi}$, when the boundary temperature is required to vanish.

Alternatively, $G(\mathbf{x}, \boldsymbol{\xi})$ can be regarded as the *electrostatic potential* due to a unit charge at $\boldsymbol{\xi}$ when the boundary potential vanishes (which is the case, for instance, if S is a grounded metallic

* For details, refer Art 10.11 in chapter 10

**Refer Chapter 9, page 297 part II of "Advanced Differential Equations" by Dr. M.D. Raisinghania, published by S. Chand & Co. New Delhi.

shell). In view of this electrostatic interpretation of $G(\mathbf{x}, \boldsymbol{\xi})$, it follows that $G(\mathbf{x}, \boldsymbol{\xi})$ is the sum of the potential of the unit source at $\boldsymbol{\xi}$ in free space and of the potential due to the charge induced on S (this charge is present on S because of the requirement that the potential vanishes there). In any number of dimensions we can write

$$G(\mathbf{x}, \boldsymbol{\xi}) = E(\mathbf{x}, \boldsymbol{\xi}) + v(\mathbf{x}, \boldsymbol{\xi}) \quad \dots (2)$$

where E is the free-space fundamental solution satisfying

$$-\nabla^2 E = \delta(\mathbf{x} - \boldsymbol{\xi}), \text{ for all } \mathbf{x} \text{ and } \boldsymbol{\xi} \quad \dots (3)$$

and v is a harmonic function which satisfies the boundary value problem

$$\nabla^2 v = 0, \quad \mathbf{x} \in R; \quad v = -E \quad \text{for } \mathbf{x} \text{ on } S \quad \dots (4)$$

Thus the problem of finding $G(\mathbf{x}, \boldsymbol{\xi})$ is reduced to that of finding a harmonic function v in R , assuming certain special boundary values on S . For ready reference, we have

$$E = \begin{cases} (1/2\pi) \times \log(1/|\mathbf{x} - \boldsymbol{\xi}|) & \text{in two dimensions} \\ 1/4\pi |\mathbf{x} - \boldsymbol{\xi}| & \text{in three dimensions} \end{cases}$$

Note also that the fundamental solution $E(\mathbf{x}; \boldsymbol{\xi})$ is the *free-space Green's function*.

Some properties of Green's function

Property 1. *The Green's function exists and is unique*

Proof. Recall that the Green's function is given by (2) where $E(\mathbf{x}; \boldsymbol{\xi})$ is given by (5). Also note that $E(\mathbf{x}; \boldsymbol{\xi})$ exists and is unique and so we need to show that v exists and is unique. But (4) is a Dirichlet problem and so the existence and uniqueness of v follows from the existence and uniqueness of the Dirichlet problem (4) for Laplace's equation with continuous boundary values.

Property 2 : *The Green's function is symmetric, i.e.,* $G(\boldsymbol{\xi}, \boldsymbol{\eta}) = G(\boldsymbol{\eta}, \boldsymbol{\xi})$

Proof. Let $G(\mathbf{x}, \boldsymbol{\xi})$ and $G(\mathbf{x}, \boldsymbol{\eta})$ be the Green's functions for the region R corresponding to region R corresponding to sources located at $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, respectively and S be the boundary of R . Then, by definition of the Green's function, we have

$$-\nabla^2 G(\mathbf{x}, \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \mathbf{x} \text{ and } \boldsymbol{\xi} \text{ in } R; \quad G(\mathbf{x}, \boldsymbol{\xi}) = 0, \quad \mathbf{x} \text{ on } S \quad \dots (6)$$

$$\text{and} \quad -\nabla^2 G(\mathbf{x}, \boldsymbol{\eta}) = \delta(\mathbf{x} - \boldsymbol{\eta}), \quad \mathbf{x} \text{ and } \boldsymbol{\eta} \text{ in } R; \quad G(\mathbf{x}, \boldsymbol{\eta}) = 0, \quad \mathbf{x} \text{ on } S \quad \dots (7)$$

Multiplying the differential equation in (6) by $G(\mathbf{x}, \boldsymbol{\eta})$, the differential equation in (7) by $G(\mathbf{x}, \boldsymbol{\xi})$, subtracting and then integrating over R , we finally obtain

$$\int_R \{G(\mathbf{x}, \boldsymbol{\xi}) \nabla^2 G(\mathbf{x}, \boldsymbol{\eta}) - G(\mathbf{x}, \boldsymbol{\eta}) \nabla^2 G(\mathbf{x}, \boldsymbol{\xi})\} dV = \int_R \{G(\mathbf{x}, \boldsymbol{\eta}) \delta(\mathbf{x} - \boldsymbol{\xi}) - G(\mathbf{x}, \boldsymbol{\xi}) \delta(\mathbf{x} - \boldsymbol{\eta})\} dV$$

Using Green's second identity (refer Art 12.1) on the L.H.S. and the shifting property of Dirac delta function (refer Art. 10.7) on R.H.S., we obtain

$$\int_S \left\{ G(\mathbf{x}, \boldsymbol{\xi}) \frac{\partial G(\mathbf{x}, \boldsymbol{\eta})}{\partial n} - G(\mathbf{x}, \boldsymbol{\eta}) \frac{\partial G(\mathbf{x}, \boldsymbol{\xi})}{\partial n} \right\} dS = G(\boldsymbol{\xi}, \boldsymbol{\eta}) - G(\boldsymbol{\eta}, \boldsymbol{\xi}), \quad \dots (8)$$

where $\mathbf{n}$ is the outward normal to S at $\mathbf{x}$ and dS is the surface element on S at $\mathbf{x}$.

Now, from (6) and (7), we see that $G(\mathbf{x}, \boldsymbol{\xi})$ and $G(\mathbf{x}, \boldsymbol{\eta})$ vanish when $\mathbf{x}$ is on S . Hence (8) reduces to

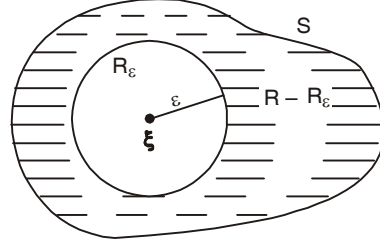
$$0 = G(\xi, \eta) - G(\eta, \xi) \quad \text{so that} \quad G(\xi, \eta) = G(\eta, \xi)$$

showing that $G(\xi, \eta)$ is a symmetric function of its arguments.

Physicists refer to the symmetric property $G(\xi, \eta) = G(\eta, \xi)$, ξ, η in R as the *reciprocity principle*. We now give its physical significance : the potential at η due to unit source at ξ is equal to the potential at ξ due to a unit source at η .

Property 3. The Green's function is positive

Proof. Draw a small circle R_ϵ of radius ϵ with centre at the source point ξ as shown in the adjoining figure. Let $R - R_\epsilon$ denote the part of R excluding R_ϵ . In $R - R_\epsilon$, $G(x, \xi)$ is harmonic at x . The boundary of $R - R_\epsilon$ consists of S and the spherical surface $|x - \xi| = \epsilon$. On S , $G(x, \xi)$ vanishes; since $G(x, \xi)$ is positively infinite at $x = \xi$, $G(x, \xi)$ will be positive on $|x - \xi| = \epsilon$ if ϵ is chosen sufficiently small.



In view of the *maximum principle for harmonic functions** $G(x, \xi)$ must be strictly positive in $R - R_\epsilon$. Since R_ϵ can be made as small as we please, $G(x, \xi)$ is strictly positive in R .

Property 4. In three or more dimensions, we have

$$0 < G(x, \xi) < E(x, \xi) \quad x, \xi \in R$$

Proof. Left as an exercise.

Solution of the Dirichlet problem. The Green's function plays its principal role to find the solution of the Dirichlet problem for the Poisson equation

$$-\nabla^2 u(x) = 4\pi \rho(x), \quad x \in R; \quad u = f, \quad x \in S. \quad \dots (9)$$

Here $\rho(x)$ is given function defined in the given region R and f is a given function on the boundary S of R .

Again, we know that the Green's function $G(x, \xi)$ for the negative Laplacian in an open, bounded region R in three dimensional space with boundary S is the solution of the boundary value problem

$$-\nabla^2 G = \delta(x - \xi), \quad G = 0, \quad x \in S. \quad \dots (10)$$

Multiplying the differential equation in (9) by G , the differential equation in (10) by u , subtracting and then integration over R , we finally obtain

$$\int_R \left\{ u(x) \nabla^2 G - G(x, \xi) \nabla^2 u \right\} dV = 4\pi \int_R \rho(x) G(x, \xi) dV - \int_R \delta(x - \xi) u(x) dV$$

Using Green's second identity (refer Art. 12.1) on L.H.S. and the shifting property of Dirac delta function (refer Art. 10.7 of chapter 10) on R.H.S., we obtain

$$\int_S \left\{ u(x) \frac{\partial G(x, \xi)}{\partial n} - G(x, \xi) \frac{\partial u}{\partial n} \right\} dS = 4\pi \int_R \rho(x) G(x, \xi) dV - u(\xi),$$

where n is the outward normal to S at x .

***The maximum principle for harmonic function.** Let R be a bounded, open region; S its boundary and $\bar{R}$ is the closed region which is the union of R and S . If u is harmonic in R and continuous in $\bar{R}$, the maximum and minimum values of u in $\bar{R}$ are both attained on S . Moreover, if u is not constant in R , the extremal values are attained on S .

Since $u = f$ and $G(\mathbf{x}, \xi) = 0$ on S , we obtain

$$\int_S f(\mathbf{x}) \frac{\partial G(\mathbf{x}, \xi)}{\partial n} dS = 4\pi \int_R \rho(\mathbf{x}) G(\mathbf{x}, \xi) dV - u(\xi)$$

Interchanging $\mathbf{x}$ and ξ , the above equation reduces to

$$u(\mathbf{x}) = 4\pi \int_R G(\xi, \mathbf{x}) \rho(\xi) dV - \int_S \frac{\partial G(\xi, \mathbf{x})}{\partial n} f(\xi) dS$$

or

$$u(\mathbf{x}) = 4\pi \int_R G(\mathbf{x}, \xi) \rho(\xi) dV - \int_S \frac{\partial G(\mathbf{x}, \xi)}{\partial n} f(\xi) dS \quad \dots (11)$$

[$\because G(\mathbf{x}, \xi)$ is symmetric, i.e., $G(\mathbf{x}, \xi) = G(\xi, \mathbf{x})$]

For the particular-case $\rho = 0$, (11) reduces to

$$u(\mathbf{x}) = - \int_S \frac{\partial G(\mathbf{x}, \xi)}{\partial n} f(\xi) dS \quad \dots (12)$$

When $f = 1$ on S , then the solution u of the Laplace's equation is clearly $u = 1$ for the interior Dirichlet problem. Substituting this value of u in (12), we find

$$- \int_S \frac{\partial G(\mathbf{x}, \xi)}{\partial n} dS = 1, \text{ for every } \mathbf{x} \in R \quad \dots (13)$$

As an application of result (12), consider the following example.

Poisson integral formula

The Green's function for the Laplace equation, when the surface S is a sphere of radius a will be obtained by expressing it as source and image point combination. Let $P(= \mathbf{x})$ be any point within the sphere such that $OP = \alpha$, where O is the centre of the sphere. Let $P'(= \mathbf{x}')$ be the inverse point of P such that $OP' = \beta$. Then, by definition of the inverse point we have $\alpha\beta = a^2$. Let $Q(= \xi)$ be any point on S . Let $PQ = r$ and $P'Q = r'$.

$$\begin{aligned} \text{Now, } \alpha\beta = a^2 &\Rightarrow OP \times OP' = OQ^2 \\ \Rightarrow OP/OQ &= OQ/OP' \quad \dots (i) \end{aligned}$$

In triangles OPQ and $OP'Q$, $\angle QOP = \angle QOP'$ and condition (i) is satisfied. Hence, the triangles OPQ and $OP'Q$ are similar and therefore

$$r'/r = a/\alpha \quad \text{so that} \quad 1/r = a/\alpha r' \quad \dots (14)$$

Now, for three dimensional case (2) reduces to

$$G(\mathbf{x}, \xi) = 1/4\pi |\mathbf{x} - \xi| + v(\mathbf{x}; \xi) \quad \dots (15)$$

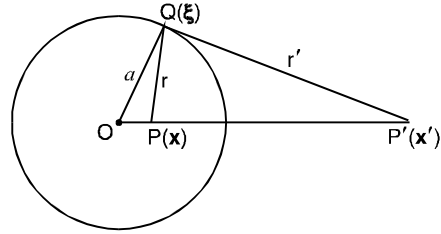
Hence in view of the relations (1) and (15), the value of the Green's function is given by

$$G(P, Q) = \frac{1}{4\pi} \left(\frac{1}{r} - \frac{a}{\alpha r'} \right) \quad \dots (16)$$

With help of this Green's function, we now proceed to solve the interior Dirichlet problem for the sphere

$$\nabla^2 u = 0, \quad r < a; \quad u = f(\theta, \phi) \quad \text{on} \quad r = a \quad \dots (17)$$

We now proceed to find the value of $\partial G / \partial n$ which is required in formula (12).



Using cosing formula of trigonometry, from triangles OPQ and $OP'Q$, we have

$$\alpha^2 = a^2 + r^2 - 2ar \cos(\mathbf{x} - \boldsymbol{\xi}, \boldsymbol{\eta}) \quad \dots (18)$$

and

$$\beta^2 = a^2 + r'^2 - 2ar' \cos(\mathbf{x}' - \boldsymbol{\xi}, \boldsymbol{\eta}), \quad \dots (19)$$

From (16), we have

$$\begin{aligned} \frac{\partial G}{\partial n} &= \frac{1}{4\pi} \left(-\frac{1}{r^2} \frac{\partial r}{\partial n} + \frac{a}{\alpha r'^2} \frac{\partial r'}{\partial n} \right) = \frac{1}{4\pi} \left\{ \frac{\cos(\mathbf{x} - \boldsymbol{\xi}, \boldsymbol{\eta})}{r^2} - \frac{a \cos(\mathbf{x}' - \boldsymbol{\xi}, \boldsymbol{\eta})}{\alpha r'^2} \right\} \\ &= \frac{1}{4\pi} \left\{ \frac{\alpha^2 - a^2 - r^2}{2ar^3} - \frac{a(\beta^2 - a^2 - r'^2)}{2a\alpha r'^3} \right\}, \text{ using (18) and (19)} \\ &= \frac{1}{4\pi} \left\{ \frac{\alpha^2 - a^2 - r^2}{2ar^3} - \frac{(a^4/\alpha^2) - a^2 - (r^2 a^2/\alpha^2)}{2\alpha \times (a^3 r^3/\alpha^3)} \right\}, \\ &\quad \left[\because \frac{r'}{r} = \frac{a}{\alpha} \text{ and } \alpha\beta = a^2 \Rightarrow r' = \frac{ar}{\alpha} \text{ and } \beta = \frac{a^2}{r} \right] \\ &= \frac{1}{4\pi} \left(\frac{\alpha^2 - a^2 - r^2}{2ar^3} - \frac{a^2 - \alpha^2 - r^2}{2ar^3} \right) = -\frac{a^2 - \alpha^2}{4\pi a r^3} \end{aligned}$$

Substituting the above value of $\partial\sigma/\partial n$ in (12), we finally obtain

$$u(P) = \frac{a(a^2 - \alpha^2)}{4\pi} \int_v \int_v^{2\pi} \frac{f(\theta', \phi') \sin \theta' d\theta' d\phi'}{\{\alpha^2 + a^2 - 2a\alpha \cos(\mathbf{x}, \boldsymbol{\xi})\}^{1/2}} \quad \dots (20)$$

which is known as *Poisson integral formula*

Solution of the Neumann problem

In order to extend the above analysis to the Neumann problem

$$-\nabla^2 u = 0, \mathbf{x} \in R, \quad \left[\frac{\partial u}{\partial n} \right]_S = f(\mathbf{x}) \quad \dots (21)$$

we define the Green's function $G(\mathbf{x}, \boldsymbol{\xi})$ by the boundary value problem (compare it with (1) for Dirichlet problem)

$$-\nabla^2 G(\mathbf{x}, \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}); \quad \left[\frac{\partial G}{\partial n} \right]_S = 0, \quad \dots (22)$$

where R is an open, bounded and three dimensional space and let its boundary be denoted by S . $\mathbf{n}$ is the outward normal to S at $\mathbf{x}$ and f is a given function.

Multiplying the differential equation in (21) by G , the differential equation in (22) by u , subtracting and then integrating over R , we finally obtain

$$\int_R \{u(\mathbf{x}) \nabla^2 G - G(\mathbf{x}, \boldsymbol{\xi}) \nabla^2 u\} dV = - \int_R \delta(\mathbf{x} - \boldsymbol{\xi}) u(\mathbf{x}) dV \quad \dots (23)$$

Using Green's second identity (refer Art. 12.1) on L.H.S. and the shifting property of Dirac delta function (refer Art. 10.7) on R.H.S., (23) reduces to

$$\int_S \left\{ u(\mathbf{x}) \frac{\partial G}{\partial n} - G(\mathbf{x}, \boldsymbol{\xi}) \frac{\partial u}{\partial n} \right\} dS = -u(\boldsymbol{\xi}). \quad \dots (24)$$

From (21) and (22), $\left[\frac{\partial G}{\partial n} \right]_S = 0$ and $(\partial u / \partial n)_S = f(\mathbf{x})$. Hence (24) yields

$$\int_S G(\mathbf{x}, \boldsymbol{\xi}) f(\mathbf{x}) dS = u(\boldsymbol{\xi}) \quad \dots (25)$$

Interchanging $\mathbf{x}$ and $\boldsymbol{\xi}$, (25) reduces to

$$u(\mathbf{x}) = \int_S G(\boldsymbol{\xi}, \mathbf{x}) f(\boldsymbol{\xi}) dS \quad \dots (26)$$

Since $G(\mathbf{x}, \boldsymbol{\xi})$ is symmetric, so $G(\boldsymbol{\xi}, \mathbf{x}) = G(\mathbf{x}, \boldsymbol{\xi})$.

Hence, from (26),
$$u(\mathbf{x}) = \int_S G(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi}) dS \quad \dots (27)$$

For existence of a solution of the Neumann problem, the prescribed function $f(\mathbf{x})$ must satisfy the consistency condition (refer result (30) of Art 12.2)

$$\int_S f(\boldsymbol{\xi}) dS = 0, \quad \boldsymbol{\xi} \in S \quad \dots (28)$$

Finally, we propose to discuss the interior and exterior Dirichlet problems for a body S enclosed within a surface S' :

$$\nabla^2 u_i = 0, \quad \mathbf{x} \in R_i, \quad [u_i]_S = f \quad \dots (29)$$

$$\nabla^2 u_e = 0, \quad \mathbf{x} \in R_e, \quad [u_e]_S = f, \quad [u_e]_{S'} = 0 \quad \dots (30)$$

The Green's function $G(\mathbf{x}, \boldsymbol{\xi})$ satisfies the auxiliary problem (on absorbing the factor 4π in G):

$$-\nabla^2 G = 4\pi \delta(\mathbf{x} - \boldsymbol{\xi}), \quad [G]_{S'} = 0 \quad \dots (34)$$

Now, we proceed exactly in the same manner as we did in Art. 12.2 while deriving the formula (27) of Art. 12.2. Then, the above the relations (29), (30) and (31) give rise to

$$f(\mathbf{x}) = \int_S G(\mathbf{x}, \boldsymbol{\xi}) \sigma(\boldsymbol{\xi}) dS, \quad \dots (32)$$

which reduces to (27) of Art. 12.2 for an unbounded medium.

12.4.A. THE METHOD OF IMAGES

Before solving the Dirichlet's problem with help of Green's function technique, we shall discuss the *method of images*. This method is frequently used in the development of electrostatics.

Working rule for the method of images : Investigate the effect of a certain type of singularity at some point P of a given region R together with the influence of another source type of singularity located at a point P' outside the region R under consideration. P' is chosen as the optical image of P in the boundary of the given region R .

Note. The method of images can be used only while dealing with very simple geometries.

12.5 SOLVED EXAMPLES BASED ON ART. 12.4 AND 12.4 A

Ex. 1. Discuss electrostatic potential problem of a conducting disc bounded by two parallel planes.

Sol. The given problem is an extension of the problem considered in Ex. 1 of Art. 12.3. In what follows we shall use the notations of Ex. 1 of Art. 12.3.

Let the given parallel planes be $z = b$ and $z = -c$, where $b > 0$ and $c > 0$. Then the boundary value problem takes the following form

$$\nabla^2 V(\rho, \phi, z) = 0 \text{ in } D, \quad \dots (1)$$

$$V(\rho, \phi, 0) = f^{(n)}(\rho) \cos n\phi, \quad 0 \leq \rho \leq a \quad \dots (2)$$

and
$$V(\rho, \phi, z) = 0, \quad z = b, \quad z = -c, \quad \dots (3)$$

where D is the region between the disc and the parallel planes.

Clearly, the Green's function $G(\mathbf{x}; \boldsymbol{\xi})$ corresponding to the boundary value problem (1) – (3) must satisfy the auxiliary system.

$$-\nabla^2 G(\mathbf{x}, \xi) = 4\pi\delta(\mathbf{x} - \xi), \quad G = 0 \quad \text{on} \quad z = b, \quad z = -c. \quad \dots (4)$$

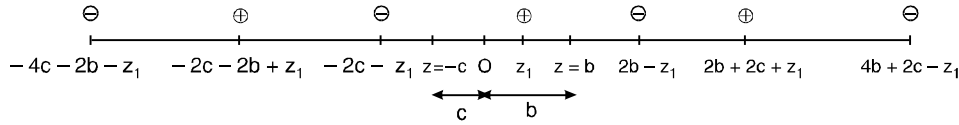
To compute $G(\mathbf{x}, \xi)$, we shall use the well known method of images (see Art. 12.4 A). Accordingly, for a positive unit charge at the source point $\xi = (t, \phi_1, z_1)$, the image system consists of positive unit charge at the points.

$$\xi_n^+ = [t, \phi_1, 2n(b+c) + z_1], \quad n = \pm 1, \pm 2, \pm 3, \dots \quad \dots (5)$$

and negative unit charge at points

$$\xi_n^- = [t, \phi_1, 2n(b+c) - 2c - z_1], \quad n = 0, \pm 1, \pm 2, \pm 3, \dots \quad \dots (6)$$

as shown in the following figure



Hence the value of the Green's function is given by

$$G(\mathbf{x}, \xi) = \frac{1}{|\mathbf{x} - \xi|} + \sum_{n=1}^{\infty} \frac{1}{|\mathbf{x} - \xi_n^+|} + \sum_{n=-1}^{\infty} \frac{1}{|\mathbf{x} - \xi_n^-|} - \sum_{n=-\infty}^{\infty} \frac{1}{|\mathbf{x} - \xi_n^-|} \quad \dots (7)$$

$$\text{Now, using the identity} \quad \frac{1}{|\mathbf{x} - \xi|} = \int_0^{\infty} J_0(u \bar{\omega}) e^{-u|z-z_1|} du, \quad \dots (8)$$

$$\text{where} \quad \bar{\omega} = \{\rho^2 + t^2 - 2\rho t \cos(\phi - \phi_1)\}^{1/2}, \quad \dots (9)$$

(7) reduces to

$$G(\mathbf{x}, \xi) = \frac{1}{|\mathbf{x} - \xi|} + \int_0^{\infty} J_0(u \bar{\omega}) \left\{ \sum_{n=1}^{\infty} e^{-u|z-2n(b+c)-z_1|} + \sum_{n=-1}^{-\infty} e^{-u|z-2n(b+c)-z_1|} - \sum_{n=0}^{\infty} e^{-u|z-2n(b+c)+2c+z_1|} - \sum_{n=-1}^{-\infty} e^{-u|z-2n(b+c)+2c+z_1|} \right\} du \quad \dots (10)$$

Summing the geometric series which are involved in (10) and simplifying, we finally get

$$G(\mathbf{x}, \xi) = \frac{1}{|\mathbf{x} - \xi|} + \int_0^{\infty} J_0(u \bar{\omega}) \frac{e^{-(c+z_1)u} \sinh u(z-b) - e^{-(n-z_1)u} \sinh u(z+c)}{\sinh u(b+c)} du \quad \dots (11)$$

We know that the expansion of $J_0(u \bar{\omega})$ is given by

$$J_0(u \bar{\omega}) = \sum_{k=0}^{\infty} (2 - \delta_{0k}) \cos k(\phi - \phi_1) J_k(u\rho) J_k(ut) \quad \dots (12)$$

$$\text{where} \quad \delta_{0k} = \text{Kronecker delta} = \begin{cases} 0, & \text{if } k \neq 0 \\ 1, & \text{if } k = 0 \end{cases} \quad \dots (13)$$

Using (12), (11) takes the form

$$G(\mathbf{x}, \xi) = 1/|\mathbf{x} - \xi| + \sum_{k=0}^{\infty} (2 - \delta_{0k}) \cos k(\phi - \phi_1) G^{(k)}(\rho, t, z, z_1), \text{ where} \quad \dots (14)$$

$$G^{(k)}(\rho, t, z, z_1) = \int_0^{\infty} \frac{e^{-u(c+z_1)} \sinh u(z-b) - e^{-u(b-z_1)} \sinh u(z+c)}{\sinh u(b+c)} J_k(u\rho) J_k(ut) du \quad \dots (15)$$

Multiplying (1) by G and (4) by V , subtracting, integrating and using Green's second identity, we obtain as usual

$$V(\rho, \phi, z) = \frac{1}{4\pi} \int_{S^+ + S^-} \left(G \frac{\partial V}{\partial n} - V \frac{\partial G}{\partial n} \right) dS, \quad \dots (16)$$

where S^+ and S^- are the upper and lower parts of the disc, respectively. Using the fact that on the surfaces S^+ and S^- , the value of the outward normal is $-(\partial/\partial z_1)$ and $\partial/\partial z_1$ respectively and also using the boundary conditions (2), (16) yields

$$f^{(n)}(\rho) \cos n\phi = \int_0^a \int_0^{2\pi} t \sigma^{(n)}(t) G(\rho, t, \phi, \phi_1, 0, 0) \cos n\phi_1 d\phi_1 dt, \quad \dots (17)$$

where
$$\sigma^{(n)}(t) \cos n\phi_1 = \frac{1}{4\pi} \left(\frac{\partial V}{\partial z_{1+}} - \frac{\partial V}{\partial z_{1-}} \right) \quad \dots (18)$$

Now, we assume that $\phi_1 - \phi = \psi$. Then, we proceed exactly as done in getting relation (5) from relation (2) in example 1 of Art. 12.3. This will give us the following relation :

$$f^{(n)}(\rho) = \int_0^a t \sigma^{(n)}(t) dt \int_0^{2\pi} \frac{\cos n\psi d\psi}{|\mathbf{x} - \boldsymbol{\xi}|_{z=z_1=0}} + \int_0^a t \sigma^{(n)}(t) \{2\pi G^{(n)}(\rho, t, 0, 0)\} dt \quad \dots (19)$$

where we have substituted the value of G given by (14).

Re-writing (19), we have

$$f^{(n)}(\rho) = \int_0^a t \sigma^{(n)}(t) K_1(t, \rho) dt, \quad \dots (20)$$

where
$$K_1(t, \rho) = \int_0^{2\pi} \frac{\cos n\psi d\psi}{(\rho^2 + t^2 + 2\rho t \cos \psi)^{1/2}} + 2\pi G^{(n)}(\rho, t, 0, 0) \quad \dots (21)$$

Remark 1. When $b \rightarrow \infty$ and $c \rightarrow \infty$, (21) reduces to the formula (8) of Art. 12.3.

Remark 2. For an annular disc of inner radius b and the outer radius a , the formula which corresponds to (21) is given by

$$f^{(n)}(\rho) = \int_b^a t \sigma^{(n)}(t) K_1(t, \rho) d\rho \quad \dots (22)$$

which contains the same kernel as in (21).

Ex. 2. Discuss a electrostatic potential problem of an axially symmetric conductor placed symmetrically inside a cylinder of radius b .

Sol. We take cylindrical polar co-ordinates (ρ, ϕ, z) with origin at the centre of the conductor and z axis along its axis of symmetry (which is also the axis of the cylinder). In order to simplify our calculations, we choose $V = 1$ on the surface of the conductor. Then, from relation (32), of Art. 12.4, we get Fredholm integral equation of the form

$$1 = \int_S G(\mathbf{x}, \boldsymbol{\xi}) \sigma(\boldsymbol{\xi}) dS, \quad \mathbf{x}, \boldsymbol{\xi} \in S \quad \dots (1)$$

where G satisfies the system

$$-\nabla^2 G = 4\pi \delta(\mathbf{x} - \boldsymbol{\xi}), \quad G = 0 \quad \text{on} \quad \rho = b \quad \dots (2)$$

Now, the differential equation for $G(\mathbf{x}, \boldsymbol{\xi})$ involving cylindrical polar coordinates is given by

$$\frac{1}{\rho} \left(\rho \frac{\partial G}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 G}{\partial \phi^2} + \frac{\partial^2 G}{\partial z^2} = -\frac{4\pi}{\rho} \delta(\rho - t) \delta(\psi) \delta(z - z_1) \quad \dots (3)$$

where $\phi_1 - \phi = \psi$. Using the definition of the Green's function given in Art. 12.4, we know that

$$G_1(\mathbf{x}, \boldsymbol{\xi}) = G(\mathbf{x}, \boldsymbol{\xi}) - (1/|\mathbf{x} - \boldsymbol{\xi}|) \quad \dots (4)$$

is finite in the limit as $\mathbf{x} \rightarrow \boldsymbol{\xi}$.

We now proceed to find the solution of (3) with help of the Fourier series expansions

$$G(\mathbf{x}; \boldsymbol{\xi}) = \sum_{k=1}^{\infty} (2 - \delta_{ok}) \cos k\psi \, g^{(k)}(\rho, t, z, z_1) \quad \dots (5)$$

$$\text{where } g^{(k)} = \frac{1}{2\pi} \int_0^{2\pi} G(\mathbf{x}; \boldsymbol{\xi}) \cos k\psi \, d\psi \quad \dots (6)$$

Multiplying (3) by $(1/2\pi) \cos k\psi$ and integrating with respect to ψ from 0 to 2π , we have

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} g^{(k)} \right) - \frac{k^2}{\rho^2} g^{(k)} + \frac{\partial^2 g^{(k)}}{\partial z^2} = -\frac{2}{\rho} \delta(\rho - \rho_1) \delta(z - z_1) \quad \dots (7)$$

Now, we take the Fourier transform of equation (7) by setting

$$F(g^{(k)}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iuz} g^{(k)} dz; \quad g^{(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iuz} F(g^{(k)}) du \quad \dots (8)$$

Then, the system (2) reduces to

$$\left. \begin{aligned} \rho^2 \frac{d^2}{d\rho^2} F(g^{(k)}) + \rho \frac{d}{d\rho} F(g^{(k)}) - (\rho^2 u^2 + k^2) F(g^{(k)}) &= -\frac{\sqrt{2}}{\rho\sqrt{\pi}} e^{iu z_1} \delta(\rho - \rho_1) \\ F(g^{(k)}) &= 0, \quad \rho = b \end{aligned} \right\} \quad \dots (9)$$

The boundary value problem (9) can be easily solved by the method and notations of chapter 11 and the solution so obtained is then inverted to give*

$$g^{(k)} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{iu(z_1 - z)} \{ K_k(u\rho_>) I_k(u\rho_<) - \{ K_k(ub)/I_k(ub) \} I_k(u\rho) I_k(ut) \} du \quad \dots (10)$$

where I_k and K_k are modified Bessel functions. Finally, from (5) and (10), value of G is given by

$$G(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{\pi} \sum_{k=0}^{\infty} (2 - \delta_{ok}) \cos k\psi \int_{-\infty}^{\infty} e^{iu(z_1 - z)} [K_k(u\rho_>) I_k(u\rho_<) - \{ K_k(ub)/I_k(ub) \} I_k(u\rho) I_k(ut)] du \quad \dots (11)$$

When $b \rightarrow \infty$, $G = 1/|\mathbf{x} - \boldsymbol{\xi}|$, and so (11) reduces to

$$\frac{1}{|\mathbf{x} - \boldsymbol{\xi}|} = \frac{1}{\pi} \sum_{k=0}^{\infty} (2 - \delta_{ok}) \cos k\psi \int_{-\infty}^{\infty} e^{iu(z_1 - z)} K_k(u\rho_>) I_k(u\rho_<) du \quad \dots (12)$$

Combining (11) and (12), we obtain

$$G(\mathbf{x}; \boldsymbol{\xi}) = 1/|\mathbf{x} - \boldsymbol{\xi}| + \sum_{k=0}^{\infty} (2 - \delta_{ok}) \cos k\psi \, G^{(k)}(\rho, t, z, z_1) \quad \dots (13)$$

$$\text{where } G^{(k)}(\rho, t, z, z_1) = -\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-iu(z - z_1)} I_k(u\rho) I_k(ut) \{ K_k(ub)/I_k(ub) \} du$$

$$= -\frac{1}{\pi} \left[\int_0^{\infty} e^{-iu(z - z_1)} I_k(u\rho) I_k(ut) \{ K_k(ub)/I_k(ub) \} du + \int_{-\infty}^{-0} e^{-iu(z - z_1)} I_k(u\rho) I_k(ut) \{ K_k(ub)/I_k(ub) \} du \right] \quad \dots (14)$$

* In equations (10), (11) and (12), we have used the following notations : $\rho_> = \max(\rho, t)$; $\rho_< = \min(\rho, t)$

Substituting $-u$ for u in the second integral on the R.H.S. of (14) and noting that

$$I_k(-z) = (-1)^k I_k(z), \quad K_k(-z) = (-1)^k K_k(z)$$

and hence $K_k(-ub)/I_k(-ub) = K_k(ub)/I_k(ub)$. Then (14) reduces to

$$G^{(k)}(\rho, t, z, z_1) = -\frac{2}{\pi} \int_0^\infty \frac{I_k(u\rho)I_k(ut)K_k(ub)}{I_k(ub)} \cos[u(z - z_1)] du \quad \dots (15)$$

Now, we use the fact that the conductor is axially symmetric. Then, since the Green's function is independent of ϕ for an axially symmetric body, so it leaves only one term in the series (13) :

$$\text{Thus,} \quad G(\mathbf{x}, \boldsymbol{\xi}) = 1/|\mathbf{x} - \boldsymbol{\xi}| + G^{(o)}(\rho, t, z, z_1) \quad \dots (16)$$

which is of the form (4). Substituting (16) in (1), we can easily obtain the required integral equation.

Remark. When $b \rightarrow \infty$, we obtain as a particular case equation (5) of solved example 1 of Art. 12.3 for $f^{(n)}(\rho) = 1$

12.6 THE HELMHOLTZ EQUATION

In this article we propose to extend the discussion the previous two articles to study the Helmholtz equation

$$(\nabla^2 + \lambda)u = 0. \quad \dots (1)$$

The free-space Green's function or the fundamental solution $E(\mathbf{x}; \boldsymbol{\xi})$ is the solution of the spherically symmetric differential equation

$$-\nabla^2 E - \lambda E = \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \dots (2)$$

and which vanishes at infinity. Such a solution in three dimensions is given by

$$E(\mathbf{x}; \boldsymbol{\xi}) = \frac{\exp(i|\mathbf{x} - \boldsymbol{\xi}|\sqrt{\lambda})}{4\pi|\mathbf{x} - \boldsymbol{\xi}|} = \frac{\exp(ir\sqrt{\lambda})}{4\pi r}, \quad \text{where } r = |\mathbf{x} - \boldsymbol{\xi}| \quad \dots (3)$$

We now discuss three situations :

Case (i) Let λ be a complex number. Then $\sqrt{\lambda}$ is selected to be that root of λ that has a positive imaginary part so that $E(\mathbf{x}; \boldsymbol{\xi})$ vanishes exponentially at infinity.

Case (ii) Let λ be real and positive, that is, $\lambda = \omega^2$, ω real, Then the solution

$$E(\mathbf{x}; \boldsymbol{\xi}) = \frac{\exp(i\omega|\mathbf{x} - \boldsymbol{\xi}|)}{4\pi|\mathbf{x} - \boldsymbol{\xi}|} = \frac{\exp(i\omega r)}{4\pi r}, \quad \dots (4)$$

is selected such that $\sqrt{\lambda} = \omega > 0$. This represents an outgoing wave if we adjoin the factor $e^{-i\omega t}$.

Case (iii). Let λ be real and negative. Then as in case (i), $\sqrt{\lambda}$ is selected to be that root of λ that has a positive imaginary part so that $E(\mathbf{x}; \boldsymbol{\xi})$ vanishes exponentially at infinity. For the particular case $\lambda = -k^2$, where k is real and positive, (3) reduces to

$$E(\mathbf{x}; \boldsymbol{\xi}) = e^{-kr}/(4\pi r) \quad \dots (5)$$

The solutions which corresponding to (3), (4) and (5) in two dimensions are given by $(i/4)H_0^{(1)}(|\mathbf{x} - \boldsymbol{\xi}|/\sqrt{\lambda})$, $(i/4)H_0^{(1)}(\omega|\mathbf{x} - \boldsymbol{\xi}|)$ and $(1/2\pi)K_0(k|\mathbf{x} - \boldsymbol{\xi}|)$, respectively $H_0^{(1)}$ and K_0 being well known Hankel and modified Bessel functions, respectively.

The integral representation formula for the solution of the inhomogeneous equation

$$(\nabla^2 - k^2)u = -4\pi\rho. \quad \dots (6)$$

can be obtained with help of (2) and (6) by using Green's identity and is given by

$$u(\mathbf{x}) = \int_R \frac{\rho e^{-kr}}{r} dV + \frac{1}{4\pi} \int_S u \frac{\partial}{\partial n} \left(\frac{e^{-kr}}{r} \right) dS - \frac{1}{4\pi} \int_S \frac{e^{-kr}}{r} \frac{\partial u}{\partial n} dS \quad \dots (7)$$

The interpretation of three integrals occurring in (7) as volume, single-layer, and double-layer potentials is the same as for the corresponding formulas in Art. 12.2. Again, the properties of these potentials are also same as given in Art. 12.2. For example, the formulas that correspond to (8) and (9) of Art. 12.2 are

$$\left(\frac{\partial u}{\partial n} \right)_{P\pm} = \mp 2\pi\sigma(\rho) + \int_S \sigma(Q) \frac{\partial}{\partial n} \left(\frac{e^{-kr}}{r} \right) dS, \quad \dots (8)$$

where
$$u = \int_S \sigma(Q) \left(\frac{e^{-kr}}{r} \right) dS \quad \dots (9)$$

Again, the formulas that correspond to (11) and (12) of Art. 12.2 are

$$[u]_{\pm P} = \pm 2\pi\tau(P) + \int_S \tau(Q) \frac{\partial}{\partial n} \left(\frac{e^{-kr}}{r} \right) dS, \quad \dots (10)$$

where
$$u = \int_S \tau(Q) \frac{\partial}{\partial n} \left(\frac{e^{-kr}}{r} \right) dS \quad \dots (11)$$

The remaining notations are same as in Art. 12.2.

The integral representation of solutions of the exterior and interior Dirichlet and Neumann problems can be obtained in an analogous way. This will be illustrated in solved examples of physical interest in next Art. 12.7.

12.7 SOLVED EXAMPLES BASED ON ART. 12.6

Example 1. Discuss steady Stokes flow in an unbounded medium.

Solution. The Stokes flow equations*

$$\nabla^2 \mathbf{q} = \nabla p, \quad \nabla \cdot \mathbf{q} = 0 \quad \dots (1)$$

govern the slow, steady flow of incompressible fluids. These have been made dimensionless with help of the free-stream velocity ' u ' and a characteristic length ' a ' inherent in the given physical problem. Here $\mathbf{q}$ and p denote the *velocity* vector and *pressure* respectively.

Let S be the surface of a given solid moving in the fluid; then the boundary conditions are

$$\mathbf{q}(\mathbf{x}) = \mathbf{e}_1, \quad \mathbf{x} \in S, \quad \mathbf{q}(\mathbf{x}) = \mathbf{0} \quad \text{as} \quad \mathbf{x} \rightarrow \infty \quad \dots (2)$$

where $\mathbf{e}_1$ is the direction of motion of the given solid B (say) taken to be in the x_1 direction.

The boundary value problem (1) – (2) can be converted into a Fredholm integral equation of the first kind by defining the *Green's tensor* $\mathbf{T}_1$ (or T_{1ik}) and *Green's vector* $\mathbf{p}_1$ (or p_{1j}), which satisfy the mathematical system

$$\nabla^2 \mathbf{T}_1 - \nabla \mathbf{p}_1 = -\mathbf{I} \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \dots (3)$$

*Refer Chapter 14 in "Fluid Dynamics" by Dr. M. D. Raisinghania, published by, S. Chand & Co. New Delhi

$$\nabla \cdot \mathbf{T}_1 = 0, \quad \mathbf{T}_1 \rightarrow \mathbf{0} \quad \text{as} \quad \mathbf{x} \rightarrow \infty, \quad \dots (4)$$

where

$$\mathbf{I} = \delta_{ij} = \text{the Kronecker delta.}$$

It follows by direct verification that the system (3) – (4) has the representation formulas

$$\mathbf{T}_1 = (1/8\pi) (\mathbf{I} \nabla^2 \phi - \text{grad grad } \phi), \quad \mathbf{p}_1 = -(1/8\pi) \text{grad } \nabla^2 \phi, \quad \dots (5)$$

$$\nabla^4 \phi = \nabla^2 \nabla^2 \phi = -8\pi \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \dots (6)$$

Clearly, an appropriate solution of the *biharmonic equation* (6) is given by $\phi = r = |\mathbf{x} - \boldsymbol{\xi}|$. Hence, we obtain

$$\mathbf{T}_1 = (1/8\pi) \{ \mathbf{I} \nabla^2 |\mathbf{x} - \boldsymbol{\xi}| - \text{grad grad } |\mathbf{x} - \boldsymbol{\xi}| \} \quad \dots (7)$$

and

$$\mathbf{p}_1 = - (1/8\pi) \text{grad } \nabla^2 |\mathbf{x} - \boldsymbol{\xi}|. \quad \dots (8)$$

We now proceed to find the integral equation equivalent to the boundary value problem given by (1) and (2). To this end, we take the scalar product of (1) by $\mathbf{T}_1$ and of (3) by $\mathbf{q}$ and use the usual steps of subtracting and integrating. The integral so obtained will contain terms $\mathbf{T}_1 \cdot \nabla p$ and $\mathbf{q} \cdot \nabla \mathbf{p}_1$ which can be re-written to using the following identities.

$$\nabla \cdot (\mathbf{q} \mathbf{p}_1) = \mathbf{q} \cdot \nabla \mathbf{p}_1, \quad \nabla \cdot (p \mathbf{T}_1) = \mathbf{T}_1 \cdot \nabla p, \quad \dots (9)$$

where we have made use of the following results

$$\nabla \cdot \mathbf{q} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{T}_1 = 0 \quad \dots (10)$$

Proceeding as indicated above, we finally obtain

$$\mathbf{q}(\mathbf{x}) = - \int_S \left\{ \left(\frac{\partial \mathbf{q}}{\partial n} - p \mathbf{n} \right) \cdot \mathbf{T}_1 - \mathbf{q} \cdot \left(\frac{\partial \mathbf{T}_1}{\partial n} - n \mathbf{p}_1 \right) \right\} dS \quad \dots (11)$$

In view of equation (3) and the divergence theorem, we conclude that, if $\mathbf{q}$ is constant on S , then (11) takes the following simple form

$$\mathbf{q}(\mathbf{x}) = - \int_S \mathbf{f} \cdot \mathbf{T}_1 dS \quad \dots (12)$$

where

$$\mathbf{f} = (\partial \mathbf{q} / \partial n) - p \mathbf{n} \quad \dots (13)$$

Now, making use of the boundary condition $\mathbf{q}(\mathbf{x}) = \mathbf{e}_1$ when $\mathbf{x} \in S$ as given by (2), in (12), the desired integral equation is given by

$$\mathbf{e}_1 = - \int_S \mathbf{f} \cdot \mathbf{T}_1 dS \quad \dots (14)$$

Example 2. By following the method and notations of example 1, prove that the integral representation formula for the velocity vector when the fluid is bounded by a vessel Σ is

$$\mathbf{q}(\mathbf{x}) = - \int \left\{ \left(\frac{\partial \mathbf{q}}{\partial n} - p \mathbf{n} \right) \cdot \mathbf{T} - \mathbf{q} \cdot \left(\frac{\partial \mathbf{T}}{\partial n} - n \mathbf{p} \right) \right\} dS$$

Substituting the boundary condition $\mathbf{q} = \mathbf{e}_1$ on S gives the Fredholm integral equation

$$\mathbf{e}_1 = - \int_S \mathbf{f} \cdot \mathbf{T} dS$$

Here $\mathbf{T}$ satisfies the boundary value problem

$$\nabla^2 \mathbf{T} - \text{grad } p = \mathbf{I} \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \nabla \cdot \mathbf{T} = 0, \quad \mathbf{T} = \mathbf{0} \quad \text{on} \quad \Sigma.$$

Solution. Left as an exercise.

Example 3. Discuss steady Oseen flow

Solution. The Oseen equations are governed by*

$$\mathbb{R}(\partial \mathbf{q} / \partial x_1) = -\nabla p + \nabla^2 \mathbf{q}, \quad \nabla \cdot \mathbf{q} = 0, \quad \dots (1)$$

which have been made dimensionless with the help of the free-stream velocity 'u' and a characteristic length 'a' inherent in the given problem. The quantities $\mathbb{R}$, ν , $\mathbf{q}$ and p stand for the Reynolds number, the coefficient of kinematic viscosity, velocity vector and pressure respectively. Also, we have

$$\mathbb{R} = ua / \nu \quad \dots (2)$$

Let S be the surface of a given solid moving in the fluid; then the boundary conditions are

$$\mathbf{q}(\mathbf{x}) = \mathbf{e}_1, \quad \mathbf{x} \in S, \quad \mathbf{q}(\mathbf{x}) \rightarrow \mathbf{0} \quad \text{as} \quad \mathbf{x} \rightarrow \infty \quad \dots (3)$$

where $\mathbf{e}_1$ is the direction of motion of the given solid B (say) taken to be in the x_1 -direction.

The boundary value problem (1) – (3) can be converted into a Fredholm integral equation of the first kind by defining the Green's tensor $\mathbf{T}$ and Green's vector $\mathbf{p}$ which satisfy the mathematical system

$$\mathbb{R}(\partial \mathbf{T} / \partial x_1) = -\nabla \mathbf{p} + \nabla^2 \mathbf{T} + \mathbf{I} \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \dots (4)$$

$$\nabla \cdot \mathbf{T} = 0, \quad \mathbf{T} \rightarrow \mathbf{0} \quad \text{as} \quad \mathbf{x} \rightarrow \infty, \quad \dots (5)$$

where $\mathbf{I} = \delta_{ij}$ is the Kronecker delta $\dots (6)$

It follows by direct verification that the system (4) – (5) has the representation formulas

$$\mathbf{T} = (1/8\pi) (\mathbf{I} \nabla^2 \phi - \text{grad grad } \phi) \quad \dots (7)$$

$$\mathbf{p} = -(1/8\pi) \text{grad} \left\{ \nabla^2 \phi - \mathbb{R}(\partial \phi / \partial x_1) \right\} \quad \dots (8)$$

$$\nabla^2 \left\{ \nabla^2 - \mathbb{R}(\partial / \partial x_1) \right\} \phi = -8\pi \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \dots (9)$$

We now proceed to solve (9) for ϕ . Using the following well known formula

$$\nabla^2 (1/|\mathbf{x} - \boldsymbol{\xi}|) = -4\pi \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \dots (10)$$

in (9), we obtain

$$\nabla^2 \left\{ \nabla^2 - \mathbb{R}(\partial / \partial x_1) \right\} \phi = 2 \nabla^2 (1/|\mathbf{x} - \boldsymbol{\xi}|), \quad \dots (11)$$

showing that if ϕ satisfies

$$\nabla^2 \phi - \mathbb{R}(\partial \phi / \partial x_1) = 2/|\mathbf{x} - \boldsymbol{\xi}|, \quad \dots (12)$$

then (9) is satisfied.

On the other hand, we can re-write (9) as

$$\left\{ \nabla^2 - \mathbb{R}(\partial / \partial x_1) \right\} \nabla^2 \phi = -8\pi \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \dots (13)$$

$$\text{Let} \quad \nabla^2 \phi = \psi \quad \dots (14)$$

Also, we have the following identity

$$(\nabla^2 - \sigma^2)[e^{-\sigma(x_1 - \xi_1)} \psi] = e^{-\sigma(x_1 - \xi_1)} [\nabla^2 - 2\sigma(\partial / \partial x_1)] \psi \quad \dots (15)$$

Using (14) and (15), (13) may be re-written as

$$(\nabla^2 - \sigma^2)[e^{-\sigma(x_1 - \xi_1)} \psi] = -8\pi e^{-\sigma(x_1 - \xi_1)} \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \dots (16)$$

$$\text{where} \quad \sigma = 2\mathbb{R} \quad \dots (17)$$

In view of the nature of the Dirac delta function, note that the factor $e^{-\sigma(x_1 - \xi_1)}$ can affect the equation (16) only at $x_1 = \xi_1$, where its value is unity. Hence, (16) reduces to

* Refer chapter 14 in "Fluid Dynamics" by Dr. M.D. Raisinghania, published by S.Chand & Co., New Delhi.

$$\psi(\mathbf{x}, \xi) = \nabla^2 \phi = \frac{2}{|\mathbf{x} - \xi|} \exp \left[-|\sigma| \left\{ |\mathbf{x} - \xi| - \frac{\sigma}{|\sigma|} (x_1 - \xi_1) \right\} \right] \quad \dots (18)$$

From (12) and (18), it follows that

$$\frac{\partial \phi}{\partial x_1} = -\frac{1}{\sigma |\mathbf{x} - \xi|} \left(1 - \exp \left[-|\sigma| \left\{ |\mathbf{x} - \xi| - \frac{\sigma}{|\sigma|} (x_1 - \xi_1) \right\} \right] \right) \quad \dots (19)$$

Let

$$s = |\mathbf{x} - \xi| - (\sigma / |\sigma|) (x_1 - \xi_1) \quad \dots (20)$$

$$\begin{aligned} \therefore \frac{\partial s}{\partial x_1} &= \frac{(x_1 - \xi_1)}{|\mathbf{x} - \xi|} - \frac{\sigma}{|\sigma|} = \frac{(x_1 - \xi_1)}{|\mathbf{x} - \xi|} - \frac{|\sigma|}{\sigma} \\ &= -\frac{|\sigma|}{\sigma |\mathbf{x} - \xi|} \left\{ |\mathbf{x} - \xi| - \frac{\sigma}{|\sigma|} (x_1 - \xi_1) \right\} = -\frac{|\sigma| s}{\sigma |\mathbf{x} - \xi|} \quad \dots (21) \end{aligned}$$

$$\text{We know that} \quad \frac{\partial \phi}{\partial x_1} = \frac{\partial \phi}{\partial s} \frac{\partial s}{\partial x_1} \quad \dots (22)$$

Then, from (19), (20), (21) and (22), we have

$$\partial \phi / \partial s = \left(1 - e^{-|\sigma|s} \right) / |\sigma| s$$

or

$$\phi = \frac{1}{|\sigma|} \int_0^{|\sigma|s} \frac{1 - e^{-t}}{t} dt \quad \dots (23)$$

Then, the Green's tensor $\mathbf{T}$ and Green's vector $\mathbf{p}$ can be determined.

In order to get the integral equation equivalent to the boundary value problem (1) – (3), we proceed exactly in the same way as we did to get equation (11) of example 1 on page 12.19. Then, for the present problem, the integral equation formula is given by

$$\mathbf{q}(\mathbf{x}) = -\int_S \left\{ \mathbf{T} \cdot \left(\frac{\partial \mathbf{q}}{\partial n} - \mathbf{p} \mathbf{n} \right) - \left(\frac{\partial \mathbf{T}}{\partial n} - \mathbf{p} \mathbf{n} \right) \cdot \mathbf{q} - R(\mathbf{T} \cdot \mathbf{q}) n_1 \right\} dS \quad \dots (24)$$

where n_1 is the x_1 component of the outward normal.

Using the boundary condition $\mathbf{q} = \mathbf{e}_1$ on S (as given by (3)), the relation (4) and the divergence theorem, (24) takes the following simple form

$$\mathbf{e}_1 = -\int_S \mathbf{T} \cdot \mathbf{f} dS, \quad \dots (25)$$

where

$$\mathbf{f} = (\partial \mathbf{q} / \partial n) - \mathbf{p} \mathbf{n} \quad \dots (26)$$

EXERCISE

1. (a) Discuss the single-layer and double-layer potentials in two-dimensional potential theory by starting with the formula $E(\mathbf{x}; \xi) = (1/2\pi) \log(1/|\mathbf{x} - \xi|)$

in place of $E(\mathbf{x}; \xi) = 1/4\pi |\mathbf{x} - \xi|$ as employed in Art. 12.2.

(b) With help of the results derived in part (a), prove that the solution of the interior Dirichlet problem in two dimensions can be written as

$$\omega(\underline{x}) = \int_C \frac{\cos \psi}{|\mathbf{x} - \xi|} \tau(\xi) dl,$$

where ψ is the angle between $(\mathbf{x} - \xi)$ and $\mathbf{n}$; and dl denotes the element of the arc length along the arc C .

2. Show that the solutions of the integral equations $\rho \cos \phi = \int_S \frac{\sigma(\rho_1, \xi_1)}{|\mathbf{x} - \xi|} \cos \phi_1 dS$

and
$$0 = \int_S \frac{\sigma'(\rho_1, \xi_1)}{|\mathbf{x} - \xi|} \cos \phi_1 dS + \frac{1}{2} \int_S \sigma(\rho_1, \xi_1) |\underline{X} - \underline{\xi}| \cos \phi_1 dS,$$

where S is the surface of a thin circular disc of unit radius, are

$$\sigma = 2\rho / \pi^2 (1 - \rho^2)^{1/2}, \quad \sigma' = \rho (2 - \rho^2) / 3\pi^2 (1 - \rho^2)^{1/2}.$$

3. Let S be the surface of a unit sphere. Then show that the solution of the integral equation

$$(i) \quad \rho z \cos \phi = \int_S \frac{\sigma(\rho_1, z_1)}{|\mathbf{x} - \xi|} \cos \phi_1 dS \quad \text{is} \quad \sigma = \left(\frac{5}{12} \right) P_2^1(\cos \theta)$$

$$(ii) \quad \frac{1}{2} \rho z^2 \cos \phi = \frac{3}{8\pi} \int_S P_1^1(\cos \theta_1) |\mathbf{x} - \xi| \cos \phi_1 dS + \int_S \frac{\sigma(\rho_1, z_1)}{|\mathbf{x} - \xi|} \cos \phi_1 dS$$

is
$$\sigma = (1/4\pi) \left[(3/2) P_1^1(\cos \theta) + (7/15) P_3^1(\cos \theta) \right]$$

In the above relations, (ρ, ϕ, z) are cylindrical polar coordinates.

Hints : (i) Use the formula $|\mathbf{x} - \xi| = r_> \sum_{n=1}^{\infty} \left(\frac{x^2}{2n+3} - \frac{1}{2n-1} \right) x^n P_n(\cos \gamma),$

$$x = r_< / r_>, r_< = \min(r, r_1), \quad r_> = \max(r, r_1)$$

where $\cos \gamma = \cos \theta \cos \theta_1 + \sin \theta \sin \theta_1 \cos(\phi - \phi_1)$

while (r, ϕ, θ) and (r_1, ϕ_1, θ_1) are the spherical polar coordinates of $\mathbf{x}$ and ξ , respectively.

(ii) On the surface of the unit sphere, $\rho = \sin \theta$ and $z = \cos \theta$

4. Starting with the Cauchy integral formula for an analytic function

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t - z} dt,$$

where C is the circumference $|z| = a$ and z is in the interior of C , and using the formula

$$0 = \frac{1}{2\pi i} \int_C \frac{f(t)}{t - z^*} dt, \quad z^* = \frac{a^2}{\bar{z}},$$

which is a result of Cauchy theorem because the image part z^* (of z) is exterior to C , derive the Poisson integral formula in a plane :

$$u(\rho, \phi) = \frac{a^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{f(a, \theta) d\theta}{\rho^2 - 2a\rho \cos(\theta - \phi) + a^2}$$

ADDITIONAL RESULTS ON GREEN'S FUNCTION AND ITS APPLICATIONS

12.8. Additional results about Green's function

In order to find analytical solution of the boundary value problem, the Green's function method is one of the convenient techniques. Consider the differential equation

$$L u(x) = f(x), \quad \dots (1)$$

where L is an ordinary linear differential operator, $f(x)$ is a known function, while $u(x)$ is an unknown function. In order to solve (1), one method is to get the operator L^{-1} in the form of an integral operator with a kernel $G(x, t)$ such that

$$u(x) = L^{-1} f(x) = \int G(x, t) f(t) dt \quad \dots (2)$$

Here the kernel $G(x, t)$ of this linear operator is known as *Green's function* for the differential operator. It follows that the solution of the non-homogeneous differential equation (1) can be easily obtained, once the Green's function for the problem is known.

Applying the differential operator L to both sides of equation (2), we have

$$f(x) = L \{L^{-1} f(x)\} = \int L \{G(x, t)\} f(t) dt \quad \dots (3)$$

From the shifting property of Dirac delta function (refer Art. 10.7), we know that

$$f(x) = \int \delta(x-t) f(t) dt \quad \dots (4)$$

$$\text{Comparing (3) and (4), we have} \quad L G(x, t) = \delta(x-t), \quad \dots (5)$$

where $\delta(x-t)$ is a Dirac delta function. The solution of (5) is known as a *singularity solution* of (1).

In what follows, we present an example for explaining the inversion of a differential operator. Consider the boundary value problem :

$$d^2 u / dx^2 = f(x), \quad u(0) = 0 \quad \text{and} \quad u(1) = 0 \quad \dots (6)$$

For this problem, equation (5) reduces to

$$L G = d^2 G / dx^2 = \delta(x-t) \quad \dots (7)$$

$$\text{We know that (refer Art. 10.10),} \quad \delta(x-t) = H'(x-t) \quad \dots (8)$$

$$\text{Then, from (7) and (8),} \quad d^2 G / dx^2 = H'(x-t) \quad \dots (9)$$

$$\text{Integrating (9),} \quad dG / dx = H(x-t) + c_1(t), \quad \dots (10)$$

where $c_1(t)$ is an arbitrary function. Integrating (10), we have

$$G(x, t) = \int H(x-t) dx + c_1(t) x + c_2(t), \text{ where } c_2(t) \text{ is another arbitrary function.}$$

$$\text{or} \quad G(x, t) = (x-t) H(x-t) + c_1(t) x + c_2(t), \quad \dots (11)$$

$$\left[\because \int H(x-t) dx = (x-t) H(x-t), \text{ by Art. 10.10} \right]$$

Now, from (2) and (11), we have

$$u(x) = \int_0^x (x-t) H(x-t) f(t) dt + x \int_{-\infty}^{\infty} c_1(t) f(t) dt + \int_{-\infty}^{\infty} c_2(t) f(t) dt \quad \dots (12)$$

Putting $x = 0$ in (12) and using the boundary condition $u(0) = 0$, we have

$$0 = 0 + 0 + \int_{-\infty}^{\infty} c_2(t) f(t) dt \quad \text{so that} \quad c_2(t) = 0$$

Putting $x = 1$ and $c_2(t) = 0$ in (12) and using the boundary condition $u(1) = 0$, we have

$$0 = \int_0^1 (1-t) f(t) dt + \int_{-\infty}^{\infty} c_1(t) f(t) dt \quad \Rightarrow \quad c_1(t) = \begin{cases} -(1-t), & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Substituting the values of $c_1(t)$ and $c_2(t)$ in (12), we have

$$u(x) = \int_0^x (x-t) H(x-t) f(t) dt - x \int_0^1 (1-t) f(t) dt \quad \dots (13)$$

Comparing (13) with the equation (2), we get the kernel of the integral operator, which is called *Green's function* or *source function*, given by

$$G(x, t) = (x - t) H(x - t) - x(1 - t), \quad 0 \leq t < 1, \quad \dots (14)$$

$$\text{satisfying the boundary conditions :} \quad G(0, t) = G(1, t) = 0 \quad \dots (15)$$

Extension of the above concepts to partial differential equations.

$$\text{Let us consider} \quad L\{u(X)\} = f(X), \quad \dots (16)$$

where L is some linear partial differential operator in three independent variables x, y, z and X is a vector in three dimensional space given by $X = (x, y, z)$. Let $X' = (x', y', z')$ be another vector in three-dimensional space. Then the Green's function may be denoted by $G(X, X')$ which satisfies the equation*

$$L\{G(X; X')\} = \delta(X - X') \quad \dots (17)$$

On expansion, (17) may be re-written as

$$L\{G(x, y, z; x', y', z')\} = \delta(x - x') \delta(y - y') \delta(z - z') \quad \dots (18)$$

Here the expression $\delta(X - X')$ is the generalisation of the concept of Dirac delta function in three dimensional space and $G(X; X')$ represents the effect at the point X due to a source function or delta function input applied at X' .

Multiplying equation (17) on both sides by $f(X')$ and integrating over the volume V with respect to X' , we obtain

$$L\left\{\int_{V_{X'}} G(X; X') f(X') dV_{X'}\right\} = \int_{V_{X'}} f(X') \delta(X - X') dV_{X'} = f(X)$$

Comparing it with (16), we have

$$u(X) = \int_{V_{X'}} G(X; X') f(X') dV_{X'} \quad \dots (19)$$

which is the solution of (16). This leads to the simple definition that if a function $u(x, y, z; x', y', z')$ is a fundamental solution of the equation, for example, $\nabla^2 u = 0$, then u is a solution of the non-homogeneous equation

$$\nabla^2 u = \delta(x, y, z; x', y', z')$$

The above concept can be easily extended to higher dimensions. Thus, the Green's function technique can be applied, in principle, to get the solution of any linear non-homogeneous partial differential equation. We have given formula (17) for the solution of a non-homogeneous partial differential equation. However, in practice, construction of the Green's function is not always very easy. In order to provide a motivation for constructing Green's function, we now proceed to present some singularity solutions (or fundamental solutions) to the well known operators.

Let us find the fundamental solution for a three-dimensional potential problem that is governed by the differential equation

$$\nabla^2 u = \delta(X) \quad \text{or} \quad \text{grad div } u = \delta(X). \quad \dots (20)$$

where u can be interpreted, for example, as the electrostatic potential. We propose to get a solution depending only on the source distance $r = |X|$. Then, for $r > 0$, $u(r)$ satisfies

*Physical interpretation of equation (17) in heat conduction or electrostatics. $G(X; X')$ can be regarded as the temperature (the electrostatic potential) at any point X in three dimensional space due to a unit source (or a unit charge) at X' .

$$\nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) = 0$$

Integrating it, we obtain $u = (A/r) + B$... (21)

Since the potential vanishes at infinity (i.e., $u \rightarrow 0$ as $r \rightarrow \infty$), (21) $\Rightarrow B = 0$.

Then, (21) reduces to $u = A/r$... (22)

Integrating (20) over a small sphere R_ϵ of radius ϵ whose surface is denoted by σ_ϵ , we get

$$\int_{R_\epsilon} (\text{div grad } u) dV = 1$$

or $\int_{\sigma_\epsilon} [\partial u / \partial r]_{r=\epsilon} dS = 1$, using the divergence theorem.

Substituting the value of u from (22) in the above relation, we get

$$\int_{\sigma_\epsilon} \left[-A/r^2 \right]_{r=\epsilon} dS = 1 \quad \text{or} \quad (-A/\epsilon^2) \times \int_{\sigma_\epsilon} dS = 1 \quad \text{or} \quad (-A/\epsilon^2) \times 4\pi \epsilon^2 = 1$$

Thus, $A = -1/4\pi$

Substituting this value of A in (22), the singularity solution or the fundamental solution of $\nabla^2 u = 0$ is given by $u = -(1/4\pi r)$... (23)

Next, let us find the fundamental solution for a two-dimensional potential problem that is governed by (20). As before, we propose to get a solution depending only on the source distance $r = |X|$. Then, for $r > 0$ and the present two dimensional case, (20) reduces to

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = 0 \quad \dots (24)$$

Integrating it, we get $u = A \log_e r + B$, ... (25)

where A and B are arbitrary constants.

If we write $u = 2q \log (1/r)$... (26)

with q as constant, then (25) is satisfied except possibly at the origin, where u is not defined. This solution has the property that if C is any circle with centre at the origin, the flux of u through that circle is $-4\pi q$. It therefore corresponds to a uniform line density q along the z -axis which appears as a point singularity in the two-dimensional theory.

Note 1 : If $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z')$ be two distinct points in three dimensional space, then the singularity solution or the fundamental solution of the Laplace equation $\nabla^2 u = 0$ is given by

$$u = 1/(4\pi |\mathbf{r} - \mathbf{r}'|) \quad \dots (26)$$

Note 2 : If $\mathbf{r} = (x, y)$ and $\mathbf{r}' = (x', y')$ be two distinct points in two dimensional space, then the singularity solution or the fundamental solution of the Laplace equation $\nabla^2 u = 0$ is given by

$$u = 2q \log (1/|\mathbf{r} - \mathbf{r}'|) \quad \dots (27)$$

12.9 THE THEORY OF GREEN'S FUNCTION FOR LAPLACE'S EQUATION

Let us reconsider the interior Dirichlet problem (refer Art. 12.2). To start with, we assume that the values of u and $\partial u / \partial n$ are known at every point of S of a finite region V and that

$$\nabla^2 u = 0 \quad \text{within } V \quad \dots (1)$$

Here, $\mathbf{n}$ is the unit vector normal to S drawn outwards from V and $\partial/\partial n$ denotes differentiation in that direction.

We now proceed to find the solution $u(\mathbf{r})$ of our problem at point P with position vector $\mathbf{r}$. Let C be a sphere with centre at P and radius ε . Let Σ be the region which is exterior to C and interior to S . Further, let the boundary of Σ be denoted by $\partial\Sigma$ and

$$u' = 1/|\mathbf{r} - \mathbf{r}'|, \quad \dots (2)$$

where $\mathbf{r}'$ is another point Q either in Σ or on the boundary $\partial\Sigma$.

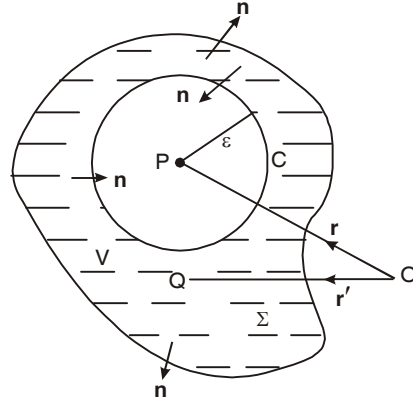


Figure 1

If u and u' are twice continuously differentiable functions in Σ and have first order derivatives on $\partial\Sigma$, then by Green's theorem in region Σ , we have

$$\iint_{\partial\Sigma} \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) dS' = \iiint_{\Sigma} (u \nabla^2 u' - u' \nabla^2 u) dV' \quad \dots (3)$$

where dS' and dV' denote elementary surface area on $\partial\Sigma$ and elementary volume of Σ respectively.

From (1) and (2), it follows that $\nabla^2 u = \nabla^2 u' = 0$ within $\partial\Sigma$. So, in the region Σ , (3) reduces to

$$\iint_{\partial\Sigma} \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) dS' = 0$$

or

$$\iint_C \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) dS' + \iint_S \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) dS' = 0,$$

where C denotes the surface of the circle with centre at P and radius ε . Substituting the value of u' given by (2) in the above relation, we obtain

$$\iint_C \left\{ u(\mathbf{r}') \frac{\partial}{\partial n} \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial u(\mathbf{r}')}{\partial n} \right\} dS' + \iint_S \left\{ u(\mathbf{r}') \frac{\partial}{\partial n} \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial u(\mathbf{r}')}{\partial n} \right\} dS' = 0, \quad \dots (4)$$

where the normals $\mathbf{n}$ are in the directions shown in the figure 1.

From the figure 1, we see that when Q is on C , we have

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\varepsilon} \quad \text{and} \quad \frac{\partial}{\partial n} \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\varepsilon^2} \quad \dots (5)$$

$$\text{Also,} \quad dS' = \text{the surface area on } C = \varepsilon^2 \sin \theta \, d\theta \, d\phi \quad \dots (6)$$

Further, on C ,

$$u(\mathbf{r}') = u(\mathbf{r}) + du$$

or

$$u(\mathbf{r}') = u(\mathbf{r}) + x(\partial u / \partial x) + y(\partial u / \partial y) + z(\partial u / \partial z)$$

or

$$u(\mathbf{r}') = u(\mathbf{r}) + \varepsilon \left(\sin \theta \cos \phi \frac{\partial u}{\partial x} + \sin \theta \sin \phi \frac{\partial u}{\partial y} + \cos \theta \frac{\partial u}{\partial z} \right)$$

Thus,

$$u(\mathbf{r}') = u(\mathbf{r}) + O(\varepsilon) \quad \text{on } C \quad \dots (7)$$

and hence

$$\frac{\partial u(\mathbf{r}')}{\partial n} = \frac{\partial u(\mathbf{r})}{\partial n} + O(\varepsilon) \quad \dots (8)$$

$$\begin{aligned} \text{Now, } \iint_C u(\mathbf{r}') \frac{\partial}{\partial n} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS' &= \iint_C \{u(\mathbf{r}) + O(\varepsilon)\} \frac{1}{\varepsilon^2} \times \varepsilon^2 \sin \theta d\theta d\phi, \text{ using (5), (6) and (7)} \\ &= u(\mathbf{r}) \iint_C \sin \theta d\theta d\phi + O(\varepsilon) = u(\mathbf{r}) \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin \theta d\theta d\phi + O(\varepsilon) = 4\pi u(\mathbf{r}) + O(\varepsilon) \quad \dots (9) \end{aligned}$$

$$\text{and } \iint_C \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial u(\mathbf{r}')}{\partial n} dS' = \iint_C \frac{1}{\varepsilon} \left\{ \frac{\partial u(\mathbf{r}')}{\partial n} + O(\varepsilon) \right\} \varepsilon^2 \sin \theta d\theta d\phi = O(\varepsilon) \quad \dots (10)$$

[using (5), (6) and (8)]

Substituting the results given by (9) and (10) in (4) and letting ε tend to zero, we obtain

$$4\pi u(\mathbf{r}) + \iint_S \left\{ u(\mathbf{r}') \frac{\partial}{\partial n} \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial u(\mathbf{r}')}{\partial n} \right\} dS' = 0$$

$$\text{Thus, } u(\mathbf{r}) = \frac{1}{4\pi} \iint_S \left\{ \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial u(\mathbf{r}')}{\partial n} - u(\mathbf{r}') \frac{\partial}{\partial n} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right\} dS' \quad \dots (11)$$

Therefore the value of u at an interior point of the region V can be determined in terms of the values of u and $\partial u / \partial n$ on the boundary S .

We now proceed to show that a similar result also holds in the case of the exterior Dirichlet problem. In this case we take the region Σ (as chosen is case of the interior Dirichlet problem) to be region bounded by S , a sphere with centre at P and radius ε and Σ' a sphere with centre at the origin and large radius R as shown in the figure 2. Taking the directions of the normals to be as indicated in figure 2 and proceeding as in the case of the interior Dirichlet problem, we shall arrive at

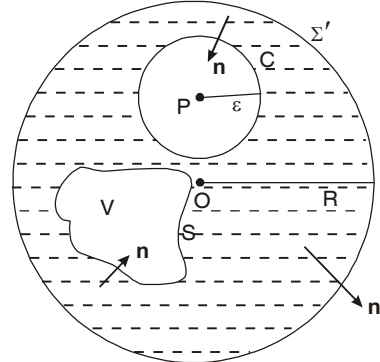


Figure 2

$$4\pi u(\mathbf{r}) + O(\varepsilon) + \iint_S \left\{ \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial u(\mathbf{r}')}{\partial n} - u(\mathbf{r}') \frac{\partial}{\partial n} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right\} dS' + \iint_{\Sigma'} \left(\frac{1}{R} \times \frac{\partial u}{\partial n} + \frac{u}{R^2} \right) dS' = 0 \quad \dots (12)$$

Letting $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$, we find that the solution (11) is valid in the case of the exterior Dirichlet problem provided that Ru and $R^2(\partial u / \partial n)$ remain finite as $R \rightarrow \infty$.

From equation (3), it appears that to get a solution of Dirichlet problem we need to know not only the value of the function u but also the value of $\partial u / \partial n$. But this is not so, as can be seen by the introduction of the concept of a Green's function defined as follows.

We define a Green's function $G(\mathbf{r}, \mathbf{r}')$, by the equation

$$G(\mathbf{r}, \mathbf{r}') = H(\mathbf{r}, \mathbf{r}') + 1/|\mathbf{r} - \mathbf{r}'|, \quad \dots (13)$$

where the function $H(\mathbf{r}, \mathbf{r}')$ satisfies the relations

$$(\partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2) H(\mathbf{r}, \mathbf{r}') = 0, \quad \dots (14)$$

i.e. the function $H(\mathbf{r}, \mathbf{r}')$ is harmonic in V

$$\text{and } H(\mathbf{r}, \mathbf{r}') + 1/|\mathbf{r} - \mathbf{r}'| = 0 \text{ on } S \quad \dots (15)$$

Thus, the Green's function for the Dirichlet problem involving the Laplace operator is a function $G(\mathbf{r}, \mathbf{r}')$ which satisfies the following properties :

$$(i) \quad \nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \text{ in } V \quad \dots (16)$$

$$(ii) \quad G(\mathbf{r}, \mathbf{r}') = 0 \text{ on } S \quad \dots (17)$$

$$(iii) \quad G(\mathbf{r}, \mathbf{r}') \text{ is symmetric, i.e., } G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r}) \quad \dots (18)$$

$$(iv) \quad G \text{ is continuous, but } \partial G / \partial n \text{ has a discontinuity at the point } \mathbf{r}, \text{ which is given by the equation}$$

$$\lim_{\epsilon \rightarrow 0} \iint_C \frac{\partial G}{\partial n} dS = 1 \quad \dots (19)$$

Following the method already used in the derivation of (11) and replacing u' by $G(\mathbf{r}, \mathbf{r}')$, we can prove that

$$u(\mathbf{r}) = \frac{1}{4\pi} \iint_S \left\{ G(\mathbf{r}, \mathbf{r}') \frac{\partial u(\mathbf{r}')}{\partial n} - u(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \right\} dS' \quad \dots (20)$$

From (13) and (15), we have $G(\mathbf{r}, \mathbf{r}') = 0$ on S . Hence, from (20), it follows that the solution of the Dirichlet problem is given by the relation

$$u(\mathbf{r}) = -\frac{1}{4\pi} \iint_S u(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} dS', \quad \dots (21)$$

showing that the solution of the Dirichlet problem can be reduced to the determination of the Green's function $G(\mathbf{r}, \mathbf{r}')$.

Physical interpretation of the Green's function. Let S be a grounded electrical conductor (boundary potential zero). Then if a unit charge is located at the source point $\mathbf{r}$, then $G(\mathbf{r}, \mathbf{r}')$ is the sum of the potential at the point $\mathbf{r}'$ due to the charge at the source point $\mathbf{r}$ in free space and the potential due to the charges induced on S . Thus

$$G(\mathbf{r}, \mathbf{r}') = H(\mathbf{r}, \mathbf{r}') + 1/|\mathbf{r} - \mathbf{r}'|$$

It follows that the property (i) given by equation (16) implies that $\nabla^2 G = 0$ everywhere except at the source point $\mathbf{r}$.

Proof of some important properties of Green's function

Theorem I. The Green's function $G(\mathbf{r}_1, \mathbf{r}_2)$ has the symmetric property, $G(\mathbf{r}_1, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_1)$ i.e. if P_1 and P_2 are two points within a finite region bounded by a surface S , then the value at P_2 of the Green's function for the point P_1 and the surface S is equal to the value at P_1 of the Green's function for the point P_2 and the surface S .

Proof. The proof of the theorem depends upon the following Lemma which we now state and prove.

Lemma. Consider a sphere with centre at the origin and radius ' a '. By using the divergence theorem to the sphere show that $\nabla^2 (1/r) = -4\pi\delta(r)$, where $\delta(r)$ is a Dirac delta function.

Proof. Applying the divergence theorem to $\nabla(1/r)$, we get

$$\iiint_V \nabla \cdot \nabla(1/r) dV = \iint_S \nabla(1/r) \cdot \hat{\mathbf{n}} dS, \quad \dots (23)$$

where $\hat{\mathbf{n}}$ is an outward drawn normal to the given sphere. If $u = u(r, \theta, \phi)$, then

$$\nabla u = \hat{\mathbf{e}}_r \frac{\partial u}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial u}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{\sin \theta}{r} \frac{\partial u}{\partial \phi}$$

$$\therefore \quad \iint_S \nabla(1/r) \cdot \hat{\mathbf{e}}_r dS = \iint_S \frac{\partial}{\partial r} \left(\frac{1}{r} \right) dS = - \iint_S \frac{1}{r^2} dS = -\frac{1}{a^2} \times 4\pi a^2 = -4\pi$$

Note that $\nabla^2(1/r)$ has the following three properties

- (i) It vanishes if $r \neq 0$
- (ii) It is undefined at the origin
- (iii) Its integral over any sphere with centre at the origin is -4π .

It follows that

$$\iiint_V \nabla^2(1/r) dV = -4\pi \delta(r) \quad \dots (23)$$

Hence the lemma.

Proof of the theorem. We define

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} + H(\mathbf{r}, \mathbf{r}'), \quad \dots (24)$$

where $H(\mathbf{r}, \mathbf{r}')$, is harmonic so that

$$\nabla^2 H(\mathbf{r}, \mathbf{r}') = 0 \quad \dots (25)$$

Then, by the above lemma, we have

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi \delta(\mathbf{r} - \mathbf{r}') \quad \dots (26)$$

From (24), (25) and (26), we obtain

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) + \nabla^2 H(\mathbf{r}, \mathbf{r}') = -4\pi \delta(\mathbf{r} - \mathbf{r}') \quad \dots (27)$$

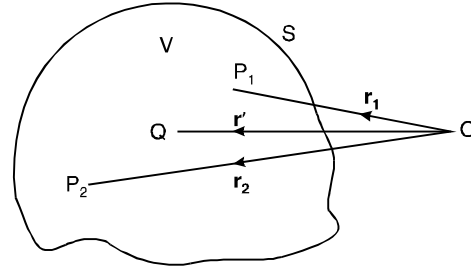


Figure 3

Suppose that P_1 and P_2 are two points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively, which lie in the interior of a finite region V bounded by a surface S . Let $\mathbf{r}'$ be the position vector of any variable point Q lying in V .

Let $u = \text{Green's function for } P_1 = G(\mathbf{r}_1, \mathbf{r}')$

and $u' = \text{Green's function for } P_2 = G(\mathbf{r}_2, \mathbf{r}')$

Then $G(\mathbf{r}_1, \mathbf{r}') = 0$ on S ... (28)

and $\nabla^2 G(\mathbf{r}_1, \mathbf{r}') = -4\pi \delta(\mathbf{r}_1 - \mathbf{r}')$, by (27) ... (29)

Also, $G(\mathbf{r}_2, \mathbf{r}') = 0$ on S ... (30)

and $\nabla^2 G(\mathbf{r}_2, \mathbf{r}') = -4\pi \delta(\mathbf{r}_2 - \mathbf{r}')$, by (27) ... (31)

If u and u' are twice continuously differentiable functions in V and have first order derivatives on S , then by Green's theorem in region V , we have

$$\iiint_V (u \nabla^2 u' - u' \nabla^2 u) dV = \iint_S \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) dS, \quad \dots (32)$$

where $\mathbf{n}$ is the unit vector normal to dS drawn outwards from V and $\partial/\partial n$ denotes differentiation in that direction.

Since $u = G(\mathbf{r}_1, \mathbf{r}')$ and $u' = G(\mathbf{r}_2, \mathbf{r}')$, (32) yields

$$\begin{aligned} & \iiint_V \{G(\mathbf{r}_1, \mathbf{r}') \nabla^2 G(\mathbf{r}_2, \mathbf{r}') - G(\mathbf{r}_2, \mathbf{r}') \nabla^2 G(\mathbf{r}_1, \mathbf{r}')\} dV \\ &= \iint_S \left\{ G(\mathbf{r}_1, \mathbf{r}') \frac{\partial G(\mathbf{r}_2, \mathbf{r}')}{\partial n} - G(\mathbf{r}_2, \mathbf{r}') \frac{\partial G(\mathbf{r}_1, \mathbf{r}')}{\partial n} \right\} dS \end{aligned}$$

or $-4\pi \iiint_V \{G(\mathbf{r}_1, \mathbf{r}') \delta(\mathbf{r}_2 - \mathbf{r}') - G(\mathbf{r}_2, \mathbf{r}') \delta(\mathbf{r}_1 - \mathbf{r}')\} dV = 0$, using (28), (29), (30), (31)

or
$$\iiint_V G(\mathbf{r}_1, \mathbf{r}') \delta(\mathbf{r}_2 - \mathbf{r}') dV = \iiint_V G(\mathbf{r}_2, \mathbf{r}') \delta(\mathbf{r}_1 - \mathbf{r}') dV$$

or $G(\mathbf{r}_1, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_1)$, using the shifting property of Dirac delta function (See Art. 10.7).

Hence the result.

Theorem II. If the Green's function G is continuous and $\partial G / \partial n$ has discontinuity at $\mathbf{r}$, in particular, then to show that

$$\lim_{\epsilon \rightarrow 0} \iint_{\partial C} \frac{\partial G}{\partial n} dS = 1$$

Proof. As shown in figure 1, let C be a sphere with radius ϵ and bounded by ∂C . From (16), it follows that G satisfies

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}') \quad \dots (33)$$

Integrating both sides of (33) over the sphere C , we obtain

$$\iiint_C \nabla^2 G dV = 1 \quad \dots (34)$$

Re-writing (34),

$$\lim_{\epsilon \rightarrow 0} \iiint_C \nabla^2 G dV = 1$$

or

$$\lim_{\epsilon \rightarrow 0} \iint_{\partial C} \frac{\partial G}{\partial n} dS = 1, \text{ using the divergence theorem}$$

12.10 Construction of the Green's function with help of the method of images.

The following examples will illustrate the method of images given in Art 12.4A on page 12.14.

Example 1. Use Green's function technique to solve the Dirichlet's problem for a semi-infinite space. (Kanpur 2011)

Solution. The Dirichlet's problem in the given semi-infinite space is governed by the following boundary value problem :

$$\nabla^2 u = 0, 0 \leq x < \infty, -\infty < y < \infty, -\infty < z < \infty \quad \dots (1)$$

$$u = f(y, z) \quad \text{on} \quad x = 0 \quad \dots (2)$$

$$\text{and} \quad u \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty, \quad \dots (3)$$

where

$$r = (x^2 + y^2 + z^2)^{1/2}$$

By definition, the Green's function for the given problem must satisfy the following conditions :

$$G(\mathbf{r}, \mathbf{r}') = H(\mathbf{r}, \mathbf{r}') + 1/|\mathbf{r} - \mathbf{r}'| \quad \dots (4)$$

$$\text{where} \quad (\partial^2 / \partial x'^2 + \partial^2 / \partial y'^2 + \partial^2 / \partial z'^2) H(\mathbf{r}, \mathbf{r}') = 0 \quad \dots (5)$$

$$\text{and} \quad G(\mathbf{r}, \mathbf{r}') = 0 \quad \text{on the plane} \quad x = 0 \quad \dots (6)$$

Suppose that Q with position vector $\mathbf{p}$, is the image in the plane $x = 0$ (i.e., yz -plane) of the point P with position vector $\mathbf{r}$. If we take

$$H(\mathbf{r}, \mathbf{r}') = -(1/|\mathbf{p} - \mathbf{r}'|), \quad \dots (7)$$

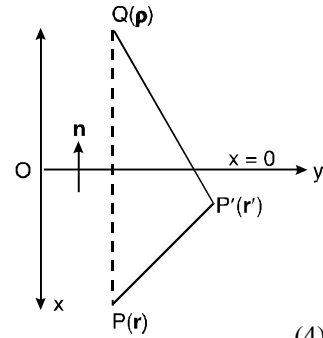
then it is obvious that equation (5) is satisfied.

From the figure-1, we find that $PP' = QP'$ whenever P' lies on $x = 0$. Thus $|\mathbf{p} - \mathbf{r}'| = |\mathbf{r} - \mathbf{r}'|$ on the plane $x = 0$. Then, from (4) and (7), we see that

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{|\mathbf{p} - \mathbf{r}'|} + \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{|\mathbf{r} - \mathbf{r}'|} = 0,$$

showing that equation (6) is also satisfied.

Thus, with help of the method of images, the required Green's function of the present



problem is given by
$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{p} - \mathbf{r}'|}, \quad \dots (8)$$

where if $\mathbf{r} = (x, y, z)$, then $\mathbf{p} = (-x, y, z)$. Take $\mathbf{r}' = (x', y', z')$, $\dots (9)$

The solution of the Dirichlet problem described by (1) – (3) is given by (refer equation (21) of Art. 12.9)

$$u(\mathbf{r}) = -\frac{1}{4\pi} \iint_S u(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} dS, \quad \dots (10)$$

where dS is the surface element of the given semi-infinite space.

From (8) and (9), we have

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{1/2}} - \frac{1}{\{(x+x')^2 + (y-y')^2 + (z-z')^2\}^{1/2}}$$

Since
$$\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} = -\frac{\partial}{\partial x'} \left[\frac{1}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{1/2}} - \frac{1}{\{(x+x')^2 + (y-y')^2 + (z-z')^2\}^{1/2}} \right],$$

it follows that on the plane $x' = 0$

$$\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} = -\frac{2x}{\{x^2 + (y-y')^2 + (z-z')^2\}^{3/2}} \quad \dots (11)$$

Substituting the value of $\partial G(\mathbf{r}, \mathbf{r}') / \partial n$ as given by (11) and noting that $u(\mathbf{r}') = f(y', z')$, (10) takes the form

$$u(x, y, z) = \frac{x}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(y', z') dy' dz'}{\{x^2 + (y-y')^2 + (z-z')^2\}^{1/2}}, \quad \dots (12)$$

giving the solution of the Dirichlet problem given by (1) - (3). If the nature of the function $f(y', z')$ is explicitly known, then (12) can be integrated to find the final solution.

Example 2. Obtain the solution of the interior Dirichlet problem for a sphere using the Green's function method and hence derive the Poisson integral formula.

Solution. The interior Dirichlet problem for a sphere of radius a is governed by the following boundary value problem :

$$\nabla^2 u = 0, \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi \quad \dots (1)$$

$$u(a, \theta, \phi) = f(\theta, \phi) \quad \text{on} \quad r = a \quad \dots (2)$$

By definition, the Green's function for the given problem must satisfy the following conditions :

$$G(\mathbf{r}, \mathbf{r}') = H(\mathbf{r}, \mathbf{r}') + 1/|\mathbf{r} - \mathbf{r}'| \quad \dots (3)$$

$$(\partial^2 / \partial x'^2 + \partial^2 / \partial y'^2 + \partial^2 / \partial z'^2) H(\mathbf{r}, \mathbf{r}') = 0 \quad \dots (4)$$

and
$$G(\mathbf{r}, \mathbf{r}') = 0, \text{ on the surface of the sphere } r = a \quad \dots (5)$$

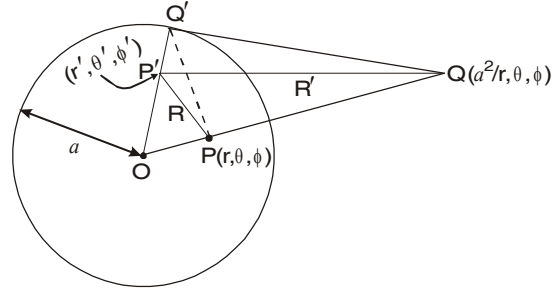
Let $P(r, \theta, \phi)$ be a point inside the given sphere, where we place a unit charge with position vector $\mathbf{r}$ and let its inverse point with respect to the sphere be Q with position vector $\mathbf{p}$. Then $\overrightarrow{OP} = \mathbf{r}$ and $\overrightarrow{OQ} = \mathbf{p}$. Let $\mathbf{r}'$ be the position vector of an arbitrary point P' inside the sphere such that $\overrightarrow{OP'} = \mathbf{r}'$. Then $r = |\mathbf{r}| = OP$, $r' = |\mathbf{r}'| = OP'$ and $\rho = |\mathbf{p}| = OQ$.

Since Q is the inverse point of P , we have

$$\begin{aligned} OP \times OQ &= a^2 \quad \text{so that} \\ OQ &= a^2 / OP = a^2 / r. \quad \dots (6) \end{aligned}$$

Hence spherical polar coordinates of P and Q may be taken as (r, θ, ϕ) and $(a^2 / r, \theta, \phi)$ respectively.

If Q' be a variable point on the surface of the sphere, then from similar triangles $OQ'P$ and $OQ'Q$, we have



$$PQ' / QQ' = r / a = a / \rho \quad \text{so that} \quad PQ' = (r / a) \times QQ',$$

which is valid for all points on the spherical surface. Hence, the harmonic function is given by

$$H(\mathbf{r}, \mathbf{r}') = -\frac{a}{r |\overrightarrow{Q'Q}|} = -\frac{a}{r |\overrightarrow{OQ} - \overrightarrow{OQ'}|} = -\frac{a}{r |\boldsymbol{\rho} - \mathbf{r}'|} \quad \dots (7)$$

$$\text{But} \quad \boldsymbol{\rho} = \overrightarrow{OQ} = OQ (\mathbf{r} / r) = (a^2 / r) (\mathbf{r} / r) = (a^2 / r^2) \mathbf{r}$$

$$\therefore (7) \text{ yields} \quad H(\mathbf{r}, \mathbf{r}') = -\frac{a}{r |(a^2 / r^2) \mathbf{r} - \mathbf{r}'|} \quad \dots (8)$$

Then it is obvious that equation (4) is satisfied. Again, from a well-known proposition of elementary geometry it is known that if P' lies on the surface of the sphere, then $PP' = (r / a) QP'$ and hence equation (5) is also satisfied. Therefore, the Green's function for the present problem is given by

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{(a / r)}{|(a^2 / r^2) \mathbf{r} - \mathbf{r}'|} \quad \dots (9)$$

$$\text{or} \quad G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\overrightarrow{PP'}|} - \frac{(a / r)}{|\overrightarrow{QP'}|} = \frac{1}{R} - \frac{(a / r)}{R'} \quad \dots (10)$$

where $|\overrightarrow{PP'}| = PP' = R$, say and $|\overrightarrow{QP'}| = QP' = R'$, say

By cosine law in solid geometry, we have

$$(PP')^2 = R^2 = r^2 + r'^2 - 2rr' \cos \ominus \quad \dots (11)$$

$$(QP')^2 = R'^2 = (a^2 / r)^2 + r'^2 - 2 \times (a^2 / r) \times r' \times \cos \ominus, \quad \dots (12)$$

$$\text{where} \quad \cos \ominus = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \quad \dots (13)$$

From equation (10), we have

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r'} = -\frac{1}{R^2} \frac{\partial R}{\partial r'} + \frac{(a / r)}{R'^2} \frac{\partial R'}{\partial r'} = -\frac{1}{R^3} \left(R \frac{\partial R}{\partial r'} - \frac{a}{r} \times \frac{R^3}{R'^3} \times R' \times \frac{\partial R'}{\partial r'} \right) \quad \dots (14)$$

But, we also have $PP' / QP' = R / R' = r / a$. Hence (14) yields

$$\frac{\partial G}{\partial n} = -\frac{1}{R^3} \left\{ R \frac{\partial R}{\partial r'} - \frac{a}{r} \times \left(\frac{r}{a} \right)^3 R' \frac{\partial R'}{\partial r'} \right\} \quad \dots (15)$$

Now, from equations (11) and (12), we have

$$2R \frac{\partial R}{\partial r'} = 2r' - 2r \cos \Theta \quad \text{and} \quad 2R' \frac{\partial R'}{\partial r'} = 2r' - 2 \times (a^2 / r) \cos \Theta$$

Substituting the above values in (14), we obtain

$$\left[\frac{\partial G}{\partial n} \right]_{r'=a} = -\frac{1}{R^3} \left[(a - r \cos \Theta) - \frac{r^2}{a^2} \left(a - \frac{a^2}{r} \cos \Theta \right) \right] = -\frac{1}{R^3} \left(a - \frac{r^2}{a} \right) = \frac{r^2 - a^2}{aR^3}$$

$$\text{Thus,} \quad \left[\frac{\partial G}{\partial n} \right]_{r'=a} = \frac{r^2 - a^2}{a(r^2 + a^2 - 2ar \cos \Theta)^{3/2}}, \text{ using (11)} \quad \dots (16)$$

The solution of the Dirichet problem described by (1) and (2) is given by (refer equation (21) of Art. 12.9)

$$u(r) = -\frac{1}{4\pi} \iint_S u(r') \frac{\partial G(r, r')}{\partial n} dS \quad \dots (17)$$

where dS is the surface element of sphere of radius a and centre at origin O .

Substituting the value of $\partial G(r, r') / \partial n$ as given by (16), noting that $u(r') = f(\theta', \phi')$, and $dS = a^2 \sin \theta' d\theta' d\phi' =$ surface element of sphere of radius a , (17) takes the form

$$u(r, \theta, \phi) = -\frac{1}{4\pi} \int_{\theta'=0}^{\pi} \int_{\phi'=0}^{2\pi} \frac{f(\theta', \phi') \times (-1) \times (a^2 - r^2) \times a^2 \sin \theta' d\theta' d\phi'}{a(r^2 + a^2 - 2ar \cos \Theta)^{3/2}}$$

$$\text{or} \quad u(r, \theta, \phi) = \frac{a(a^2 - r^2)}{4\pi} \int_{\phi'=0}^{2\pi} d\phi' \int_{\theta'=0}^{\pi} \frac{f(\theta', \phi') \sin \theta' d\theta'}{(r^2 + a^2 - 2ar \cos \Theta)^{3/2}}, \quad \dots (18)$$

where Θ is defined by equation (13)

Hence the solution of the interior Dirichlet problem described by (1) and (2) is given by (18). The integral on the right side of (18) is known as the *Poisson integral*. The result (18) is known as the *Poisson integral formula*. (Compare this with equation (20) of Art. 12.4)

Exercise : Show that the solution of the corresponding exterior Dirichlet problem is given by

$$u(r, \theta, \phi) = \frac{a(r^2 - a^2)}{4\pi} \int_0^{2\pi} d\phi' \int_0^{\pi} \frac{f(\theta', \phi') \sin \theta' d\theta'}{(r^2 + a^2 - 2ra \cos \Theta)^{3/2}}$$

Solution. Left as an exercise.

12.11 GREEN'S FUNCTION FOR THE TWO-DIMENSIONAL LAPLACE EQUATION.

The theory of the Green's function for the two-dimensional Laplace equation may be developed along the lines similar to those of Art. 12.9. We shall use Green's identity given below.

Green's identity of calculus : If $P(x, y)$ and $Q(x, y)$ are functions defined inside and on the boundary C of the closed area D , then by Green's theorem, we have

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dS = \int_C (Pdx + Qdy) \quad \dots (1)$$

Putting $P = -u(\partial u' / \partial y)$ and $Q = u(\partial u' / \partial x)$ in (1), and making use of the fact that

$$(\partial u' / \partial x) dy - (\partial u' / \partial y) dx = \partial u' / \partial n \quad \text{and} \quad \nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \quad \text{for two dimensions,}$$

$$\text{we get} \quad \iint_D u \nabla^2 u' dS + \int_D \left(\frac{\partial u}{\partial x} \frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial u'}{\partial y} \right) dS = \int_C u \frac{\partial u'}{\partial n} ds, \quad \dots (2)$$

where $\partial u' / \partial n$ denotes the derivative of u' in the direction of the outward normal to the boundary

curve C and ds denotes the elementary arc length of the curve C .

On interchanging u and u' in (2), we have

$$\iint_D u' \nabla^2 u \, dS + \iint_D \left(\frac{\partial u'}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial u'}{\partial y} \frac{\partial u}{\partial y} \right) dS = \int_C u' \frac{\partial u}{\partial n} ds \quad \dots (3)$$

Subtracting (3) from (2), we find

$$\iint_D (u \nabla^2 u' - u' \nabla^2 u) \, dS = \int_C \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) ds, \quad \dots (4)$$

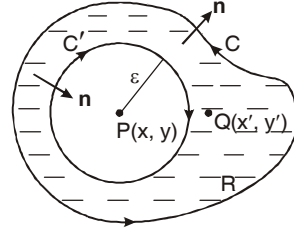
which is known as *Green's identity*.

Consider a point $P(x, y)$ inside the region D bounded by the smooth curve C . Assume that $u(x, y)$ is a harmonic function in D . Draw a circle C' with centre P and small radius ε .

$$\text{Let} \quad u' = \log \frac{1}{|\mathbf{r} - \mathbf{r}'|}, \quad \dots (6)$$

$$\text{where} \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} \quad \text{and} \quad \mathbf{r}' = x'\mathbf{i} + y'\mathbf{j}, \quad \dots (7)$$

Here (x', y') are coordinates of any point Q in D . From (6), we note that u' is a harmonic function in any region that does not contain the point $P(x, y)$. This function is known as the *fundamental solution of Laplace's equation in two dimensions*.



Let R be the region which is exterior to C' and interior to C . Further let the boundary of R be denoted by Γ . Then, we have

$$\nabla^2 u = 0 \quad \text{and} \quad \nabla^2 u' = 0 \quad \text{in } R. \quad \dots (8)$$

If u and u' are twice continuously differentiable functions in R and have first order derivatives on Γ , then by Green's identity (4), we have

$$\int_{\Gamma} \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) ds' = \iint_R (u \nabla^2 u' - u' \nabla^2 u) dS \quad \dots (9)$$

where s is measured in the directions shown in the figure and ds' is an elementary arc of Γ .

$$\text{From (9),} \quad \int_{\Gamma} \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) ds' = 0, \text{ using (8)}$$

$$\text{or} \quad \int_{C'} \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) ds' + \int_C \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) ds' = 0$$

Substituting the value of u' given by (6) in the above relation, we have

$$\begin{aligned} \int_{C'} \left\{ u(x', y') \frac{\partial}{\partial n} \log \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \log \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial u}{\partial n} \right\} ds' \\ + \int_C \left\{ u(x', y') \frac{\partial}{\partial n} \log \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \log \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial u}{\partial n} \right\} ds' = 0, \quad \dots (10) \end{aligned}$$

where the normals $\mathbf{n}$ are in the directions shown in the figure.

Proceeding as in the three dimensional case (refer Art. 12.9), we can prove that

$$\int_{C'} u \frac{\partial}{\partial n} \log \frac{1}{|\mathbf{r} - \mathbf{r}'|} ds' = 2\pi u(x, y) + O(\varepsilon) \quad \dots (11)$$

and

$$\left| \int_{C'} \log \frac{1}{|\underline{r} - \underline{r}'|} \frac{\partial u}{\partial n} ds' \right| \leq -2\pi M \varepsilon \log \varepsilon, \quad \dots (12)$$

where M is an upper bound of $\partial u / \partial r$.

Substituting the results given by (11) and (12) in (10) and letting ε tend to zero, we obtain

$$u(x, y) = \frac{1}{2\pi} \int_C \left\{ \log \frac{1}{|\underline{r} - \underline{r}'|} \frac{\partial u(x', y')}{\partial n} - u(x', y') \frac{\partial}{\partial n} \log \frac{1}{|\underline{r} - \underline{r}'|} \right\} ds' \quad \dots (13)$$

The equation (13) would seem at first right to indicate that to obtain a solution of Dirichlet problem we need to know not only the value of the function u but also the value of $\partial u / \partial n$. That this is not in fact so can be demonstrated by introduction of the concept of a Green's function in two dimensions defined as follows :

We define a Green's function $G(x, y; x', y')$ by the equation

$$G(x, y; x', y') = W(x, y; x', y') + \log \frac{1}{|\underline{r} - \underline{r}'|}, \quad \dots (14)$$

where the function $W(x, y; x', y')$ satisfies the relations

$$(\partial^2 / \partial x'^2 + \partial^2 / \partial y'^2) W(x, y; x', y') = 0 \quad \dots (15)$$

and

$$G(x, y; x', y') = 0 \quad \text{on } C. \quad \dots (16)$$

Applying Green's identity (4) for $u(x', y')$ and $W(x, y; x', y')$ and noting that u and W are harmonic functions (*i.e.*, $\nabla^2 u = 0$ and $\nabla^2 W = 0$), we have

$$0 = \int_C \left(u \frac{\partial W}{\partial n} - W \frac{\partial u}{\partial n} \right) ds'$$

or

$$\frac{1}{2\pi} \int_C \left\{ u(x', y') \frac{\partial W}{\partial n} - W \frac{\partial u(x', y')}{\partial n} \right\} ds' = 0 \quad \dots (17)$$

Subtracting (17) from (13), we have

$$u(x, y) = \frac{1}{2\pi} \int_C \left\{ \frac{\partial u(x', y')}{\partial n} \left(W + \log \frac{1}{|\underline{r} - \underline{r}'|} \right) - u(x', y') \frac{\partial}{\partial n} \left(W + \log \frac{1}{|\underline{r} - \underline{r}'|} \right) \right\} ds'$$

or

$$u(x, y) = -\frac{1}{2\pi} \int_C u(x', y') \frac{\partial G(x, y; x', y')}{\partial n} ds' \quad \dots (18)$$

$$\left[\because \text{on } C, \quad G = W + \log \frac{1}{|\underline{r} - \underline{r}'|} = 0, \text{ by (14) and (16)} \right]$$

Hence the solution of the Dirichlet problem

$$\nabla^2 u = 0 \quad \text{within } D, \quad u = f(x, y) \quad \text{on } C$$

is given by (18) in the form

$$u(x, y) = -\frac{1}{2\pi} \int_C f(x', y') \frac{\partial G(x, y; x', y')}{\partial n} ds', \quad \dots (19)$$

when $\underline{n}$ is the outward drawn normal to the boundary curve C .

12.12 CONSTRUCTION OF THE GREEN'S FUNCTION WITH HELP OF THE METHOD OF IMAGES (Refer Art. 12.4A)

The following examples will illustrate the method of images given in Art. 12.4A. We shall deal with two special cases of result (18) of Art. 12.11.

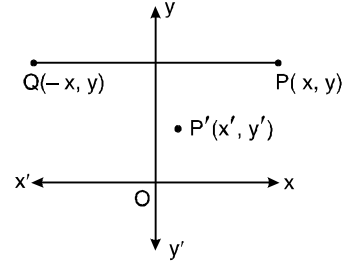
Example 1. (a) Use Green's function technique to solve Dirichlet's problem for a half plane.

Solution. The Dirichlet's problem in the given semi-infinite half plane is governed by the following boundary value problem :

$$\nabla^2 u = 0 \quad \text{for} \quad x \geq 0, \quad -\infty < y < \infty \quad \dots (1)$$

$$u = f(y) \quad \text{on} \quad x = 0 \quad \dots (2)$$

and $u \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty \quad \dots (3)$



By definition, the Green's function for the given problem must satisfy the following conditions :

$$G(x, y; x', y') = W(x, y; x', y') + \log \frac{1}{|\mathbf{r} - \mathbf{r}'|}, \quad \dots (4)$$

where $(\partial^2 / \partial x'^2 + \partial^2 / \partial y'^2) W(x, y; x', y') = 0 \quad \dots (5)$

and $G(x, y, x', y') = 0 \quad \text{on} \quad C \quad \dots (6)$

Suppose that $Q(-x, y)$ with position vector $\mathbf{p}$, is the image in the plane $x = 0$ (i.e., y -axis) of the point $P(x, y)$ with position vector $\mathbf{r}$. Let $P'(x', y')$ be any point in given region with position vector $\mathbf{r}'$. Then, from (4), we have

$$G = W + \log(1/PP'), \quad \text{as} \quad |\mathbf{r} - \mathbf{r}'| = PP' \quad \dots (7)$$

If we take $W = -\log \frac{1}{|\mathbf{p} - \mathbf{r}'|} = -\log \frac{1}{P'Q}, \quad \dots (8)$

$$[\because |\mathbf{p} - \mathbf{r}'| = |\vec{OQ} - \vec{OP'}| = |\vec{P'Q}| = P'Q]$$

then it is obvious that equation (5) is satisfied.

From the figure, we find that $PP' = QP'$ whenever P' lies on $x = 0$. Then, from (7) and (8), we observe that

$$G = -\log(1/QP') + \log(1/PP') = -\log(1/PP') + \log(1/PP') = 0,$$

showing that $G = 0$ on C and hence (6) is also satisfied.

Thus, with help of the method of images, the required Green's function for the present problem is given by

$$G(x, y; x', y') = -\log(1/QP') + \log(1/PP') = \log(QP'/PP')$$

or $G(x, y; x', y') = \frac{1}{2} \log \frac{(x+x')^2 + (y-y')^2}{(x-x')^2 + (y-y')^2} \quad \dots (9)$

Here, the outward drawn normal to the boundary is in the direction of the x -axis. Hence,

$$\frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial x'}\right)_{x'=0} = -\frac{1}{2} \left[\frac{2x}{(x+x')^2 + (y-y')^2} + \frac{2x}{(x-x')^2 + (y-y')^2} \right]_{x'=0}$$

Thus, $\partial G / \partial n = -(2x) / \{x^2 + (y-y')^2\} \quad \dots (10)$

Substituting the value of $\partial G / \partial n$ as given by (10) and noting that $u(x, y) = f(y)$ so that $u(x', y') = f(y')$, relation (18) of Art. 12.11 reduces to

$$u(x, y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} f(y') \left\{ \frac{-2x}{x^2 + (y-y')^2} \right\} dy' = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{f(y') dy'}{x^2 + (y-y')^2},$$

giving the solution of the Dirichlet problem given by (1) – (3) for the given half plane. If the nature of the function $f(y')$ is explicitly known, then the above equation can be integrated to find the final solution.

Example 1. (b) Solve $\nabla^2 u = 0$ in the upper half plane defined by $y \geq 0, -\infty < x < \infty$, using Green's function method, subject to the condition $u = f(x)$ on $y = 0$.

Hint. Proceeds as in example 1(a).

Ans. Green's function $= G(\mathbf{r}, \mathbf{r}') = \frac{1}{2} \log_e \frac{(x-x')^2 + (y+y')^2}{(x-x')^2 + (y-y')^2}$ and solution is given by

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x')}{(x-x')^2 + y^2} dx'$$

Example 2. Obtain the solution of the interior Dirichlet problem for a circle using the Green's method.

Solution. The interior Dirichlet problem for a circle of radius a this is governed by the following boundary value problem.

$$\nabla^2 u = 0, \quad r < a \quad \dots (1)$$

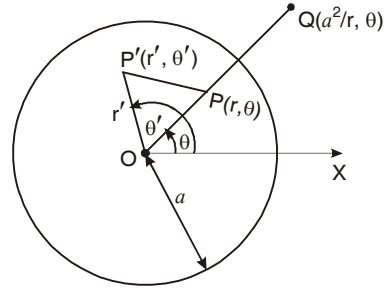
$$u = f(\theta) \quad \text{on} \quad r = a \quad \dots (2)$$

By definition, the Green's function for the given problem must satisfy the following conditions :

$$G(r, \theta; r', \theta') = W(r, \theta; r', \theta') + \log \frac{1}{|\mathbf{r} - \mathbf{r}'|} \quad \dots (3)$$

$$\nabla^2 W = 0 \quad \dots (4)$$

$$\text{and} \quad G(r, \theta; r', \theta') = 0, \text{ on the circular boundary } r = a \quad \dots (5)$$



Let $P(r, \theta)$ be a point inside the given circle, where we place a unit charge with position vector $\mathbf{r}$ and let its inverse point with respect to the circle be Q with position vector $\mathbf{\rho}$. Then $\overrightarrow{OP} = \mathbf{r}$ and $\overrightarrow{OQ} = \mathbf{\rho}$. Let $\mathbf{r}'$ be the position vector of an arbitrary point $P'(r', \theta')$ inside the circle such that $\overrightarrow{OP'} = \mathbf{r}'$. Then, let

$$r = |\mathbf{r}| = OP, \quad r' = |\mathbf{r}'| = OP' \quad \text{and} \quad \rho = |\mathbf{\rho}| = OQ.$$

Since Q is the inverse point of P , we have

$$OP \times OQ = a^2 \quad \text{so that} \quad OQ = a^2/OP = a^2/r \quad \dots (6)$$

Hence polar coordinates of P and Q may be taken as (r, θ) and $(a^2/r, \theta)$ respectively.

If we take
$$W = \log \left(\frac{r \times P'Q}{a} \right), \quad \dots (7)$$

then it can be verified that $\nabla^2 W = 0$ and so (4) is satisfied.

Since $|\mathbf{r} - \mathbf{r}'| = PP'$, from (3) and (7), it follows that the required Green's function may be taken as

$$G(r, \theta; r', \theta') = \log \left(\frac{r \times P'Q}{a} \right) + \log \frac{1}{PP'} = \log \left(\frac{r}{a} \times \frac{P'Q}{PP'} \right) \quad \dots (8)$$

On the circle $r = a$, we have

$$G = \log \frac{P'Q}{P'P} = \log \left\{ \frac{P'Q}{(r/a) \times P'Q} \right\} = \log \frac{a}{a} = \log 1 = 0,$$

showing that the condition (5) is also satisfied.

Using cosine formula of Trigonometry, we have

$$(PP')^2 = r^2 + r'^2 - 2rr' \cos(\theta' - \theta)$$

and

$$(P'Q)^2 = r'^2 + (a^2/r)^2 - 2r' \times (a^2/r) \times \cos(\theta' - \theta)$$

Inserting these results into equation (8), we have

$$G(r, \theta; r', \theta') = \log \left[\frac{r}{a} \times \frac{\{r'^2 + (a^2/r)^2 - 2r' \times (a^2/r) \times \cos(\theta' - \theta)\}^{1/2}}{\{r^2 + r'^2 - 2rr' \cos(\theta' - \theta)\}^{1/2}} \right]$$

or

$$G(r, \theta; r', \theta') = \frac{1}{2} \log \frac{a^2 + r^2 r'^2 / a^2 - 2rr' \cos(\theta' - \theta)}{r^2 + r'^2 - 2rr' \cos(\theta' - \theta)} \quad \dots (9)$$

$$\text{Now on } C, \quad \frac{\partial G}{\partial n} = \left(\frac{\partial G}{\partial r'} \right)_{r'=a} = \frac{-(a^2 - r^2)}{a \{a^2 - 2ar \cos(\theta' - \theta) + r^2\}} \quad \dots (10)$$

Substituting the value of $\partial G / \partial n$ as given by (10) and noting that $u(r, \theta) = f(\theta)$ so that $u(r', \theta') = f(\theta')$, result (18) of Art 12.11 reduces to

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\theta') d\theta'}{a^2 - 2ar \cos(\theta' - \theta) + r^2}, \quad \dots (11)$$

giving the solution of the Dirichlet problem given by (1) and (2) for a circle. If the nature of the function $f(\theta')$ is explicitly known, then the above equation (11) can be integrated to find the final solution.

Applications of integral equations to mixed boundary value problems

13.1 INTRODUCTION.

In the present chapter we propose to study some of the mixed boundary value problems which occur in physical sciences rather frequently. We use the term “mixed” boundary value problem to distinguish this type of problem from the “uniform” problems of Dirichlet and Neumann. Recall that a problem in potential theory is known as a Dirichlet problem if the potential function whose form inside a region R is to be determined is prescribed at each point of the entire surface S bounding the region R , and a Neumann problem if its normal derivative is prescribed at each point of the entire surface S bounding the region R . In potential theory a typical problem of mixed kind would be one in which the potential function is prescribed over a part of the boundary, and its normal derivative is prescribed over the remaining part.

Various mathematical techniques have been employed to solve mixed boundary value problems. In the present chapter, however, we shall discuss an integral-equation method applicable to most of the mixed boundary value problems.

13.2 TWO-PART BOUNDARY VALUE PROBLEMS

Consider a Fredholm integral equation of the first kind of the form.

$$\int_0^a K_0(t, \rho) g(t) dt = f(\rho), \quad 0 < \rho < a \quad \dots (1)$$

where the function $f(\rho)$ and the kernel $K_0(t, \rho)$ are known and $g(t)$ is to be determined.

The solution of various mixed boundary value problems in potential theory, elastostatics, steady heat conduction, the flow of perfect fluid, and various other problems of equilibrium states are known to depend upon the solution of (1). The boundaries occurring in such problems are those of solids such as circular discs, elliptic discs, spherical caps, and spheroidal caps.

WORKING RULE FOR SOLVING (1)

We know that solution of (1), in general, cannot be easily obtained because (1) is a Fredholm integral equation of the first kind. However, it is possible to reduce the solution of (1) to that of a pair of Volterra integral equations of the first kind with simple kernels. This reduction is obtained for every kernel $K_0(t, \rho)$ which for all $g(t)$ satisfies the relation

$$\int_0^a K_0(t, \rho) g(t) dt = h_1(\rho) \int_0^\rho K_2(\omega, \rho) \{h_2(\omega)\}^2 \int_\omega^a K_2(\omega, t) g(t) h_3(t) dt d\omega, \quad 0 < \rho < a \quad \dots (2)$$

where h_1 , h_2 , h_3 , and K_2 are known functions. We also suppose that the kernel K_2 is such that the Volterra integral equations

$$\int_0^\rho K_2(t, \rho) g(t) dt = f(\rho), \quad 0 < \rho < a \quad \dots (3)$$

and
$$\int_{\rho}^a K_2(\rho, t) g(t) dt = f(\rho), \quad 0 < \rho < a \quad \dots (4)$$

possess explicit unique solutions for g in terms of f , for all arbitrary differential functions f .

Now, (1) can be easily solved with help of two functions $S(\rho)$ and $C(\rho)$ which are defined as follows :

$$S(\rho) = h_2(\rho) \int_{\rho}^a K_2(\rho, t) g(t) h_3(t) dt, \quad 0 < \rho < a \quad \dots (5)$$

and
$$f(\rho) = h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) C(\omega) h_2(\omega) d\omega, \quad 0 < \rho < a \quad \dots (6)$$

Using (2) and (6), (1) may be re-written as

$$\begin{aligned} h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) h_2(\omega) \left\{ h_2(\omega) \int_{\omega}^a K_2(\omega, t) g(t) h_3(t) dt \right\} d\omega \\ = h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) C(\omega) h_2(\omega) d\omega, \quad 0 < \rho < a \quad \dots (7) \end{aligned}$$

Replacing ρ by ω in (5), we have

$$S(\omega) = h_2(\omega) \int_{\omega}^a K_2(\omega, t) g(t) h_3(t) dt \quad \dots (8)$$

Using (8), (7) reduces to

$$h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) h_2(\omega) S(\omega) d\omega = h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) C(\omega) h_2(\omega) d\omega, \quad 0 < \rho < a$$

Thus,
$$S(\rho) = C(\rho), \quad 0 < \rho < a \quad \dots (9)$$

Hence, (6) reduces to
$$f(\rho) = h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) S(\omega) h_2(\omega) d\omega, \quad 0 < \rho < a \quad \dots (10)$$

Noting the assumptions already made about the solution of the Volterra integral equations (3) and (4), we first solve (10) to find the function $S(\rho)$. Substitute this value of $S(\rho)$ in (5). Next step will be to invert (5) and obtain the required unknown function $g(t)$.

The following solved examples will illustrate the above entire working rule.

Example 1. Solve the integral equation

$$\int_0^a t \phi(t) \int_0^{\infty} J_1(u\rho) J_1(ut) du dt = \Omega \rho, \quad 0 < \rho < a,$$

where $\phi(t)$ is the unknown function and $J_1(x)$ is the Bessel function. (See Art. 10.12)

Solution. Comparing the given equation with (1), we have

$$g(t) = t \phi(t), \quad f(\rho) = \Omega \rho, \quad K_0(t, \rho) = \int_0^{\infty} J_1(u\rho) J_1(ut) du \quad \dots (11)$$

Now, the kernel $K_0(t, \rho)$ satisfies (2) because, for all $g(t)$, we have

$$\begin{aligned} \int_0^a K_0(t, \rho) g(t) dt &= \int_0^a g(t) \int_0^{\infty} J_1(u\rho) J_1(ut) du dt \\ &= \int_0^a g(t) \int_0^{\infty} \frac{2u}{\pi \rho t} \int_0^{\rho} \int_0^t \frac{J_{1/2}(u\omega) J_{1/2}(u\nu) (\omega\nu)^{3/2} d\nu d\omega du dt}{(\rho^2 - \omega^2)^{1/2} (t^2 - \nu^2)^{1/2}} \\ &= \frac{2}{\pi \rho} \int_0^a \frac{g(t)}{t} \int_0^{\rho} \int_0^t \frac{\delta(\omega - \nu) (\omega\nu) d\nu d\omega dt}{(\rho^2 - \omega^2)^{1/2} (t^2 - \nu^2)^{1/2}} \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{\pi \rho} \int_0^a \frac{g(t)}{t} \int_0^{\min(\rho, t)} \frac{\omega^2 d\omega dt}{(\rho^2 - \omega^2)(t^2 - \omega^2)^{1/2}} \\
&= \frac{2}{\pi \rho} \int_0^\rho \frac{\omega^2}{(\rho^2 - \omega^2)^{1/2}} \int_\omega^a \frac{g(t) dt d\omega}{t(t^2 - \omega^2)^{1/2}}, \quad 0 < \rho < a, \quad \dots (12)
\end{aligned}$$

While arriving at the final form (12), we have employed the well known first Sonine integral

$$J_n(u\rho) = \left(\frac{2u}{\pi}\right)^{1/2} \frac{1}{\rho^n} \int_0^\rho \frac{J_{n-(1/2)}(u\omega) \omega^{n+(1/2)}}{(\rho^2 - \omega^2)^{1/2}} d\omega \quad \dots (13)$$

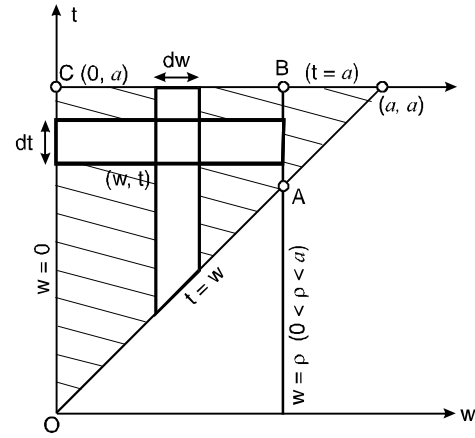
and the well known relation
$$\int_0^\infty u J_\mu(u\omega) J_\mu(u\nu) du = \frac{\delta(\omega - \nu)}{(\omega\nu)^{1/2}}, \quad \dots (14)$$

where δ is the well known Dirac delta function. We have also used the shifting property of Dirac delta function (refer Art. 10.7, chapter 10) and changed the order of integration as explained in the adjoining figure.

Comparing (2) and (12), the values of the functions h_1, h_2, h_3 and K_2 are given by

$$\left. \begin{aligned} h_1(\rho) &= 2/\pi\rho, \quad h_2(\rho) = \rho, \quad h_3(\rho) = 1/\rho \\ \text{and} \quad K_2(t, \rho) &= (\rho^2 - t^2)^{-1/2} \end{aligned} \right\} \quad \dots (15)$$

Moreover the simple form of the kernel K_2 will surely help in the process of the inversion of integral equations (3) and (4) (refer examples 3 and 4 in Art. 8.5, chapter 8). It follows that the present method is applicable to the given problem.



Now, if we take
$$S(\rho) = \rho \int_\rho^a \frac{\phi(t) dt}{(t^2 - \rho^2)^{1/2}} \quad \dots (16)$$

and
$$\Omega \rho = \frac{2}{\pi \rho} \int_0^\rho \frac{\omega S(\omega) d\omega}{(\rho^2 - \omega^2)^{1/2}}, \quad \dots (17)$$

then we can easily verify that the given integral equation (1) is identically satisfied.

Inversion of (17) yields
$$S(\rho) = \frac{\Omega}{\rho} \frac{d}{d\rho} \int_0^\rho \frac{t^3 dt}{(\rho^2 - t^2)^{1/2}} = 2 \Omega \rho \quad \dots (18)$$

Hence (16) reduces to
$$2 \Omega = \int_\rho^a \frac{\phi(t) dt}{(t^2 - \rho^2)^{1/2}} \quad \dots (19)$$

Inverting (19), we have

$$\phi(\rho) = -\frac{4\Omega}{\pi} \frac{d}{d\rho} \int_\rho^a \frac{u du}{(u^2 - \rho^2)^{1/2}} = -\frac{4\Omega \rho}{\pi(a^2 - \rho^2)^{1/2}} \quad \dots (20)$$

Remark. The reader should go through Art. 8.5 of Chapter 8 and then use the method to arrive at results (18) and (20).

Example 2. Solve the integral equation

$$f(\rho) = \int_0^a g(t) K_0(t, \rho) dt, \quad 0 < \rho < a \quad \dots (i)$$

where

$$K_0(t, \rho) = 2\pi \int_0^\infty J_n(u\rho) J_n(ut) du \quad \dots (ii)$$

Solution. Proceeding as in working rule of Art. 13.2, we have, for all $g(t)$

$$\begin{aligned} \int_0^a K_0(t, \rho) g(t) dt &= 2\pi \int_0^a g(t) \int_0^\infty J_n(u\rho) J_n(ut) du dt \\ &= 2\pi \int_0^a g(t) \int_0^\infty \frac{2u}{\pi(\rho t)^n} \int_0^\rho \int_0^t \frac{J_{n-1/2}(u\omega) J_{n-1/2}(uv) (\omega v)^{n+1/2}}{(\rho^2 - \omega^2)^{1/2} (t^2 - v^2)^{1/2}} dv d\omega du dt \\ &= \frac{4}{\rho^n} \int_0^a \frac{g(t)}{t^n} \int_0^\rho \int_0^t \frac{\delta(\omega - v) (\omega v)^n}{(\rho^2 - \omega^2)^{1/2} (t^2 - v^2)^{1/2}} dv d\omega dt \\ &= \frac{4}{\rho^n} \int_0^a \frac{g(t)}{t^n} \int_0^{\min(\rho, t)} \frac{\omega^{2n} d\omega dt}{(\rho^2 - \omega^2)^{1/2} (t^2 - \omega^2)^{1/2}} \\ &= \frac{4}{\rho^n} \int_0^\rho \frac{\omega^{2n}}{(\rho^2 - \omega^2)^{1/2}} \int_\omega^a \frac{g(t) dt d\omega}{t^n (t^2 - \omega^2)^{1/2}}, \quad 0 < \rho < a, \quad \dots (iii) \end{aligned}$$

where we have used the following standard results

$$J_n(u\rho) = \left(\frac{2u}{\pi}\right)^{1/2} \frac{1}{\rho^n} \int_0^\rho \frac{J_{n-(1/2)}(u\omega) \omega^{n+(1/2)}}{(\rho^2 - \omega^2)^{1/2}} d\omega \quad [\text{Sonine Integral}] \quad \dots (iv)$$

and

$$\int_0^\infty u J_\mu(u\omega) J_\mu(uv) du = \frac{\delta(\omega - v)}{(\omega v)^{1/2}}, \quad \dots (v)$$

where $\delta(\omega - v)$ is the well known Dirac delta function.

Since (i) is difficult to solve, we reduce its solution to that of a pair of Volterra integral equations of the first kind with simple kernels. The desired reduction is obtained for every kernel $K_0(t, \rho)$ which for all $g(t)$ satisfies the relation

$$\int_0^a K_0(t, \rho) g(t) dt = h_1(\rho) \int_0^\rho K_2(\omega, \rho) [h_2(\omega)]^2 \int_\omega^a K_2(\omega, t) g(t) h_3(t) dt d\omega, \quad 0 < \rho < a \quad \dots (vi)$$

where h_1, h_2, h_3 and K_2 are known functions. It is further assumed that the kernel K_2 is such that the Volterra integral equations

$$\int_0^a K_2(t, \rho) g(t) dt = f(\rho), \quad 0 < \rho < a \quad \dots (vii)$$

and

$$\int_\rho^a K_2(\rho, t) g(t) dt = f(\rho), \quad 0 < \rho < a \quad \dots (viii)$$

possess explicit unique solutions for g in terms of f , for all arbitrary differential functions f .

Comparing (iii) and (vi), the values of the functions h_1, h_2, h_3 and K_2 are given by

$$h_1(\rho) = 4/\rho^n, \quad h_2(\rho) = \rho^n, \quad h_3(\rho) = \rho^{-n}, \quad K_2(t, \rho) = (\rho^2 - t^2)^{-1/2} \quad \dots (ix)$$

Again the form of K_2 given by (ix) ensures the inversion of (vii) and (viii) and so the working rule of Art. 10.2 can be employed to solve the given problem. Accordingly, in order to solve (i), we define two functions $S(\rho)$ and $C(\rho)$ as follows :

$$S(\rho) = h_2(\rho) \int_{\rho}^a K_2(\rho, t) g(t) h_3(t) dt = \rho^n \int_{\rho}^a \frac{g(t) dt}{t^n (t^2 - \rho^2)^{1/2}}, \quad 0 < \rho < a \quad \dots (x)$$

$$f(\rho) = h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) S(\omega) h_2(\omega) d\omega = \frac{4}{\rho^n} \int_0^{\rho} \frac{\omega^n S(\omega) d\omega}{(\rho^2 - \omega^2)^{1/2}}, \quad 0 < \rho < a \quad \dots (xi)$$

Inverting (xi), we have (refer Chapter 8 for details)

$$S(\rho) = \frac{1}{2\pi\rho^n} \frac{d}{d\rho} \int_0^{\rho} \frac{t^{n+1} f(t) dt}{(\rho^2 - t^2)^{1/2}}, \quad \dots (xii)$$

Next, inverting (x), we find the value of the unknown function $g(t)$ in terms of $S(\omega)$:

$$g(t) = -\frac{2t^n}{\pi} \frac{d}{dt} \int_t^a \frac{\omega^{1-n} S(\omega) d\omega}{(\omega^2 - t^2)^{1/2}} \quad \dots (xiii)$$

We now substitute the value of $S(\omega)$ given by (xii) in (xiii) and get the desired value of $g(t)$.

Particular case : Let the disc be kept at a unit potential so that $f(\rho) = 1$ and $n = 0$. Then, (xii) and (xiii) reduce to

$$S(\rho) = \frac{1}{2\pi} \frac{d}{d\rho} \int_0^{\rho} \frac{t dt}{(\rho^2 - t^2)^{1/2}} = \frac{1}{2\pi} \quad \dots (xiv)$$

and
$$g(t) = -\frac{1}{\pi^2} \frac{d}{dt} \int_t^a \frac{\omega d\omega}{(\omega^2 - t^2)^{1/2}} = \frac{1}{\pi^2 (a^2 - t^2)^{1/2}}, \quad 0 < t < a \quad \dots (xv)$$

Remark. We now show that the capacity C of the disc can be obtained without finding the value of the unknown function g . Indeed, we know that

$$C = 2\pi \int_0^a g(t) dt \quad \dots (xvi)$$

For the particular case, putting $n = 0$ in (xiii) we have

$$g(t) = -\frac{2}{\pi} \frac{d}{dt} \int_t^a \frac{\omega S(\omega) d\omega}{(\omega^2 - t^2)^{1/2}} \quad \dots (xvii)$$

Substituting the value of $g(t)$ from (xvii) in (xvi), we have

$$C = 4 \int_0^a S(\omega) d\omega = 2a/\pi, \text{ using (xiv)}$$

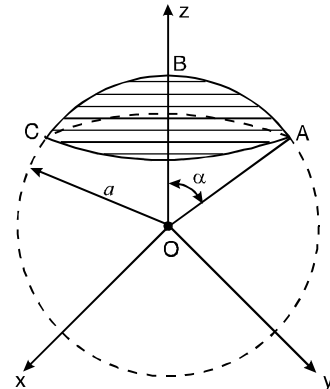
Example 3. Obtain electrostatic potential due to a spherical cap.

Solution. In terms of spherical polar coordinates

(r, θ, ϕ) , let the given spherical cap ABC be defined by $r = a$, $0 \leq \theta \leq \alpha$, $0 \leq \phi \leq 2\pi$ as shown in the adjoining figure. Consider the axially symmetric case so that the potential on the cap can be expressed by a function of θ only, say $f(\theta)$. Thus, we are to solve the boundary value problem given by

$$\nabla^2 V(r, \theta, \phi) = 0 \quad \text{in } D \quad \dots (i)$$

and
$$V(a, \theta, \phi) = f(\theta), \quad 0 \leq \theta \leq \alpha; \quad 0 \leq \phi < 2\pi, \quad \dots (ii)$$



where D is the region exterior to the cap. Then, it can be shown that the integral representation formula for (i) is given by

$$V(r, \theta, \phi) = a^2 \int_0^\alpha \int_0^{2\pi} \frac{\sigma(t)}{R} \sin t \, d\phi_1 \, dt, \quad \dots (iii)$$

where $\sigma(t)$ is the charge density at the point $Q(a, t, \phi_1)$ on the cap and

$$R = (r^2 + a^2 - 2ar \cos \gamma)^{1/2}, \quad \cos \gamma = \cos \theta \cos t + \sin \theta \sin t \cos(\phi - \phi_1)$$

Using the boundary condition (ii), (iii) reduces to the Fredholm integral equation of the first kind

$$f(\theta) = a^2 \int_0^\alpha \sin t \, \sigma(t) K_0(t, \theta) \, dt, \quad 0 < \theta < \alpha, \quad \dots (iv)$$

where

$$K_0(t, \theta) = \int_0^{2\pi} \frac{d\phi_1}{(2a^2 - 2a^2 \cos \gamma)^{1/2}} \quad \dots (v)$$

Now, expanding the integrand in (v) in terms of the spherical harmonics $Y_n^m(\theta, \phi)$ (refer Art. 10.17, chapter 10), we obtain

$$\frac{1}{(2a^2 - 2a^2 \cos \gamma)^{1/2}} = \frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^n \frac{(n - |m|)!}{(n + |m|)!} Y_n^m(\theta, \phi) \bar{Y}_n^m(t, \phi_1), \quad \dots (vi)$$

where the bar denotes the complex conjugate.

From (v) and (vi), we obtain

$$K_0(t, \theta) = \frac{2\pi}{a} \sum_{n=0}^{\infty} P_n(\cos \theta) P_n(\cos t), \quad \dots (vii)$$

where P_n is the Legendre polynomial.

Using the well known *Mehler-Dirichlet integral*

$$P_n(\cos \theta) = \frac{\sqrt{2}}{\pi} \int_0^\theta \frac{\cos \{(n + 1/2)\omega\} \, d\omega}{(\cos \omega - \cos \theta)^{1/2}}$$

and a standard result

$$\sum_{n=0}^{\infty} \cos \{(n + 1/2)\omega\} \cos \{(n + 1/2)v\} = (\pi/2) \delta(\omega - v), \quad 0 < \omega < \pi, \quad 0 < v < \pi$$

the $K_0(t, \theta)$ given by (vii) can be re-written in the integral form given by

$$K_0(t, \theta) = \frac{2}{a} \int_0^\theta \int_0^t \frac{\delta(\omega - v) \, dv \, d\omega}{(\cos \omega - \cos \theta)^{1/2} (\cos v - \cos t)^{1/2}} \quad \dots (viii)$$

According to working rule of Art. 13.2, the solution of

$$\int_0^\alpha K_0(t, \theta) g(t) \, dt = f(\theta), \quad 0 < \theta < \alpha \quad \dots (ix)$$

is obtained by reducing its solution to that of a pair of Volterra integral equations of the first kind with simple kernels. The proposed reduction is obtained for every kernel $K_0(t, \rho)$ that for all $g(t)$ satisfies the relation

$$\int_0^\alpha K_0(t, \theta) g(t) \, dt = h_1(\theta) \int_0^\theta K_2(\omega, \theta) [h_2(\omega)]^2 \int_\omega^\alpha K_2(\omega, t) g(t) h_3(t) \, dt \, d\omega, \quad 0 < \theta < \alpha \quad \dots (x)$$

where h_1, h_2, h_3 , and K_2 are known functions. It is further assumed that the kernel K_2 is such that the Volterra integral equations

$$\int_0^\theta K_2(t, \theta) g(t) \, dt = f(\theta), \quad 0 < \theta < \alpha \quad \dots (xi)$$

and
$$\int_0^\alpha K_2(\theta, t) g(t) dt = f(\theta), \quad 0 < \theta < \alpha \quad \dots (xii)$$

possess explicit unique solutions for g in terms of f , for all arbitrary functions f . Then, (1) is solved by introducing a function $S(\theta)$ given by

$$S(\theta) = h_2(\theta) \int_0^\alpha K_2(\theta, t) g(t) h_3(t) dt, \quad 0 < \theta < \alpha \quad \dots (xiii)$$

and
$$f(\theta) = h_1(\theta) \int_0^\theta K_2(\omega, \theta) S(\omega) h_2(\omega) d\omega, \quad 0 < \theta < \alpha \quad \dots (xiv)$$

Comparing (iv) and (ix), here
$$g(t) = a^2 \sin t \sigma(t) \quad \dots (xv)$$

For the given problem, the relation that corresponds to (x) is given by

$$\int_0^\alpha K_0(t, \theta) g(t) dt = \frac{2}{a} \int_0^\alpha g(t) \int_0^{\min(\theta, t)} \frac{d\omega dt}{(\cos \omega - \cos \theta)^{1/2} (\cos \omega - \cos t)^{1/2}}, \quad 0 < \theta < \alpha$$

or
$$\int_0^\alpha K_0(t, \theta) g(t) dt = \frac{2}{a} \int_0^\theta \frac{1}{(\cos \omega - \cos \theta)^{1/2}} \int_\omega^\alpha \frac{g(t) dt d\omega}{(\cos \omega - \cos t)^{1/2}}, \quad 0 < \theta < \alpha \quad \dots (xvi)$$

Comparing (x) and (xvi), we have

$$h_1(\theta) = 2/a, \quad h_2(\theta) = h_3(\theta) = 1, \quad K_2(t, \theta) = (\cos t - \cos \theta)^{-1/2}$$

Substituting the above values in (xiii) & (xiv) and using (xv), we obtain

$$S(\theta) = \int_0^\alpha \frac{a^2 \sigma(t) \sin t dt}{(\cos \theta - \cos t)^{1/2}} \quad \dots (xvii)$$

and
$$f(\theta) = \frac{2}{a} \int_0^\theta \frac{S(\omega) d\omega}{(\cos \omega - \cos \theta)^{1/2}} \quad \dots (xviii)$$

The Volterra integral equations (xvii) and (xviii) can easily be solved as done in solved examples 1 and 2 of Art. 8.5, chapter 8. Proceeding as indicated above, the solution of (xviii) yields

$$S(\omega) = \frac{a}{2\pi} \frac{d}{d\omega} \int_0^\omega \frac{f(\theta) \sin \theta d\theta}{(\cos \theta - \cos \omega)^{1/2}} \quad \dots (xix)$$

Likewise, the solution of (xvii) yields

$$\sigma(t) = -\frac{1}{\pi a^2} \frac{d}{dt} \int_t^\alpha \frac{S(\omega) \sin \omega d\omega}{(\cos t - \cos \omega)^{1/2}} \quad \dots (xx)$$

Substituting the value of $S(\omega)$ given by (xix) in (xx), we get the required value of the unknown function $\sigma(t)$ of the integral equation (iv).

Remark. The capacity C of the solid can be obtained without first finding the charge density $\sigma(t)$. We know that

$$C = 2\pi a^2 \int_0^\alpha \sin t \sigma(t) dt \quad \text{or} \quad C = 2\pi a^2 \int_0^\alpha \left(-\frac{1}{\pi a^2} \right) \frac{d}{dt} \int_t^\alpha \frac{S(\omega) \sin \omega d\omega}{(\cos t - \cos \omega)^{1/2}}, \quad \text{by (xx)}$$

Thus,
$$C = 2 \int_0^\alpha \frac{S(\omega) \sin \omega d\omega}{(1 - \cos \omega)^{1/2}} = 2\sqrt{2} \int_0^\alpha S(\omega) \cos(\omega/2) d\omega \quad \dots (xxi)$$

Particular case When the cap is kept at a unit potential, so that, $f(\theta) = 1$, then (xix) reduces

to
$$S(\omega) = \frac{a}{2\pi} \frac{d}{d\omega} \int_0^\omega \frac{\sin \theta d\theta}{(\cos \theta - \cos \omega)^{1/2}} = \frac{a}{\pi\sqrt{2}} \cos \frac{\omega}{2}$$

Substituting the above value of $S(\omega)$ in (xxi), we obtain

$$C = 2\sqrt{2} \int_0^\alpha \frac{a}{\pi\sqrt{2}} \cos^2 \frac{\omega}{2} d\omega = \frac{a}{\pi} \int_0^\alpha (1 + \cos \omega) d\omega = \frac{a}{\pi} (\alpha + \sin \alpha).$$

EXERCISE

1. Extend the analysis of solved example 2 in Art. 13.2 to the following two exercises

(i) The disc is bounded by a grounded cylindrical vessel of radius b such that $a/b \ll 1$. The disc and the cylinder have the common axis.

(ii) The disc is placed symmetrically between two grounded parallel plates $z = \pm b$ such that $a/b \ll 1$.

2. Instead of the whole disc, consider the case of an annular disc and extend the analysis of solved example 2 in Art. 13.2 accordingly. Do the same to the problems in exercises. 1(i) and 1(ii).

3. Solve exercises 1(i) and 1(ii) when the solid is a spherical cap instead of the circular disc.

13.3. THREE-PART BOUNDARY VALUE PROBLEMS

A three-part boundary value problem has an integral representation formula of the form

$$\int_b^a K_0(t, \rho) g(t) dt = f(\rho), \quad b < \rho < a \quad \dots (1)$$

where b and a are two given numbers such as the inner and outer radii of an annular disc or the bounding angles of an annular spherical cap. The function g is unknown while the functions f and the kernel K_0 are known.

Suppose that
$$f(\rho) = \sum_{r=-\infty}^{\infty} a_r \rho^r = f_1(\rho) + f_2(\rho) \quad \dots (2)$$

where
$$f_1(\rho) = \sum_{r=0}^{\infty} a_r \rho^r, \quad 0 \leq \rho < a \quad \dots (3)$$

and
$$f_2(\rho) = \sum_{r=-\infty}^{-1} a_r \rho^r, \quad b \leq \rho < \infty \quad \dots (4)$$

We also define two functions $g_1(\rho)$ and $g_2(\rho)$ such that

$$g_1(\rho) + g_2(\rho) = \begin{cases} 0, & 0 \leq \rho < b \\ g(\rho), & b \leq \rho \leq a \\ 0, & a < \rho < \infty \end{cases} \quad \dots (5)$$

In view of the relations (2), (3), (4) and (5), we see easily that (1) splits into the following integral equations :

$$\int_0^\infty K_0(t, \rho) g_1(t) dt = f_1(\rho), \quad 0 < \rho < a \quad \dots (6)$$

and
$$\int_0^\infty K_0(t, \rho) g_2(t) dt = f_2(\rho), \quad b < \rho < \infty \quad \dots (7)$$

We now proceed as in Art. 13.2. Suppose that the kernel $K_0(t, \rho)$ is such that for all $g(t)$ it satisfies the following relation for $0 < \rho < \infty$:

$$\int_0^\infty K_0(t, \rho) g(t) dt = \begin{cases} h_{11}(\rho) \int_0^\rho K_2(\omega, \rho) [h_{12}(\omega)]^2 \int_\omega^\infty K_2(\omega, t) g(t) h_{13}(t) dt d\omega \\ h_{21}(\rho) \int_\rho^\infty u_2(\rho, \omega) [h_{22}(\omega)]^2 \int_0^\omega K_2(t, \omega) g(t) h_{23}(t) dt d\omega \end{cases} \quad \dots (8)$$

where h_{ij} ($i = 1, 2; j = 1, 2, 3$) and the kernel K_2 are known functions.

We also assume that the kernel K_2 is such that the Volterra integral equations

$$\int_0^\rho K_2(t, \rho) g(t) dt = f(\rho), \quad 0 < \rho < \infty \quad \dots (9)$$

and
$$\int_\rho^\infty K_2(\rho, t) g(t) dt = f(\rho), \quad 0 < \rho < \infty \quad \dots (10)$$

possess unique solutions for g in terms of all arbitrary differentiable functions f .

Combining (6) with the first part of the relation (8), we have

$$h_{11}(\rho) \int_0^\rho K_2(\omega, \rho) \{h_{12}(\omega)\}^2 \int_\omega^\infty K_2(\omega, t) g_1(t) h_{13}(t) dt d\omega = f_1(\rho), \quad 0 < \rho < a \quad \dots (11)$$

Again, combining (7) with the second part of the relation (8), we have

$$h_{21}(\rho) \int_\rho^\infty K_2(\rho, \omega) \{h_{22}(\omega)\}^2 \int_0^\omega K_2(t, \omega) g_2(t) h_{23}(t) dt d\omega = f_2(\rho), \quad b < \rho < \infty \quad \dots (12)$$

In order to solve (1), we define six functions S_1, S_2, T_1, T_2, C_1 and C_2 as follows :

$$h_{12}(\rho) \int_\rho^\infty K_2(\rho, t) g_1(t) h_{13}(t) dt = \begin{cases} S_1(\rho), & 0 < \rho < a \\ -T_1(\rho), & a < \rho < \infty \end{cases} \quad \dots (13)$$

$$h_{22}(\rho) \int_0^\rho K_2(t, \rho) g_2(t) h_{23}(t) dt = \begin{cases} -T_2(\rho), & 0 < \rho < b \\ S_2(\rho), & b < \rho < \infty \end{cases} \quad \dots (14)$$

$$h_{11}(\rho) \int_0^\rho K_2(\omega, \rho) C_1(\omega) h_{12}(\omega) d\omega = f_1(\rho), \quad 0 < \rho < a, \quad \dots (15)$$

and
$$h_{21}(\rho) \int_\rho^\infty K_2(\rho, \omega) C_2(\omega) h_{22}(\omega) d\omega = f_2(\rho), \quad b < \rho < \infty. \quad \dots (16)$$

The above mentioned integral equation are similar to (9) and (10), whose solutions are assumed to be known.

With help of (11), (13) and (15), we obtain

$$h_{11}(\rho) \int_0^\rho K_2(\omega, \rho) h_{12}(\omega) S_1(\omega) d\omega = h_{11}(\rho) \int_0^\rho K_2(\omega, \rho) C_1(\omega) h_{12}(\omega) d\omega, \quad 0 < \rho < a \quad \dots (17)$$

or
$$S_1(\rho) = C_1(\rho), \quad 0 < \rho < a \quad \dots (18)$$

Again, with help of (12), (14) and (16), we obtain

$$S_2(\rho) = C_2(\rho), \quad b < \rho < \infty \quad \dots (19)$$

Using (18) and (19), (15) and (16) may be re-written as

$$h_{11}(\rho) \int_0^\rho K_2(\omega, \rho) S_1(\omega) h_{12}(\omega) d\omega = f_1(\rho), \quad 0 < \rho < a \quad \dots (20)$$

and
$$h_{21}(\rho) \int_\rho^\infty K_2(\rho, \omega) S_2(\omega) h_{22}(\omega) d\omega = f_2(\rho), \quad b < \rho < \infty \quad \dots (21)$$

The functions S_1 and S_2 can be obtained in terms of the known functions f_1 and f_2 from equations (20) and (21). Hence we are left with determination of only two functions T_1 and T_2 . To this end, we use relation (5) and (13) and obtain

$$h_{12}(\rho) \int_{\rho}^{\infty} K_2(\rho, t) g_2(t) h_{13}(t) dt = T_1(\rho), \quad a < \rho < \infty \quad \dots (22)$$

Similarly, from (5) and (14), we obtain

$$h_{22}(\rho) \int_0^{\rho} K_2(t, \rho) g_1(t) h_{23}(t) dt = T_2(\rho), \quad 0 < \rho < b \quad \dots (23)$$

Next step is to invert (14) for getting the value of $g_2(t)$ in terms of T_2 and S_2 and substitute this value of $g_2(t)$ in (22). Thus, we shall arrive at an integral equation containing two unknown functions T_1 and T_2 . Similarly, from the relations (13) and (23), we arrive at another integral equation containing two unknowns T_1 and T_2 . Both these equations are Fredholm integral equations and hence they can be easily solved by well known iterative method. (refer chapter 5)

Now, we illustrate the above analysis with help of the following solved example.

Example. Solve the integral equation

$$\int_b^a t \phi(t) \int_0^{\infty} J_1(u\rho) J_1(ut) du dt = \Omega \rho, \quad b < \rho < a \quad \dots (24)$$

[This equation embodies the solution of the torsion of an isotropic and homogeneous elastic half-space due to a uniformly rotating annular disc with inner radius b and outer radius a]

Solution. Comparing (24) with (1) and (5), we have

$$\left. \begin{aligned} g(t) &= t \phi(t), & g_1(t) &= t \phi_1(t); & g_2(t) &= t \phi_2(t) \\ f(\rho) &= \Omega \rho, & f_1(\rho) &= \Omega \rho, & f_2(\rho) &= 0, \end{aligned} \right\} \quad \begin{aligned} &0 \leq \rho < a; \\ &b < \rho < \infty \end{aligned} \quad \dots (25)$$

and

$$K_0(t, \rho) = \int_0^{\infty} J_1(u\rho) J_1(ut) du, \quad \dots (26)$$

where

$$\phi_1(\rho) + \phi_2(\rho) = \begin{cases} 0, & 0 \leq \rho < b \\ \phi(\rho), & b \leq \rho \leq a \\ 0, & a < \rho < \infty \end{cases} \quad \dots (27)$$

Also, the kernel $K_0(t, \rho)$ must satisfy the condition (8). Therefore, for all $g(t)$, we have

$$\begin{aligned} \int_0^{\infty} K_0(t, \rho) g(t) dt &= \int_0^{\infty} g(t) \int_0^{\infty} J_1(u\rho) J_1(ut) du dt, \text{ by (26)} \\ &= \int_0^{\infty} g(t) \int_0^{\infty} \frac{2u}{\pi \rho t} \int_0^{\rho} \int_0^t \frac{J_{1/2}(u\omega) J_{1/2}(uv) (\omega v)^{1/2} dv d\omega du dt}{(\rho^2 - \omega^2)^{1/2} (t^2 - v^2)^{1/2}} \\ &= \frac{2}{\pi \rho} \int_0^{\infty} \frac{g(t)}{t} \int_0^{\rho} \int_0^t \frac{\delta(\omega - v) (\omega v) dv d\omega dt}{(\rho^2 - \omega^2)^{1/2} (t^2 - v^2)^{1/2}} \\ &= \frac{2}{\pi \rho} \int_0^{\infty} \frac{g(t)}{t} \int_0^{\min(\rho, t)} \frac{\omega^2 d\omega dt}{(\rho^2 - \omega^2)^{1/2} (t^2 - \omega^2)^{1/2}} \\ &= \frac{2}{\pi \rho} \int_0^{\rho} \frac{\omega^2}{(\rho^2 - \omega^2)^{1/2}} \int_{\omega}^{\infty} \frac{g(t) dt d\omega}{t (t^2 - \omega^2)^{1/2}}, \quad 0 < \rho < \infty \end{aligned} \quad \dots (28)$$

and

$$\begin{aligned}
\int_0^\infty K_0(t, \rho) g(t) dt &= \int_0^\infty g(t) \int_0^\infty J_1(u\rho) J_1(ut) du dt \\
&= \int_0^\infty g(t) \int_0^\infty \frac{2u\rho t}{\pi} \int_\rho^\infty \int_t^\infty \frac{J_{3/2}(u\omega) J_{3/2}(uv)}{(\omega v)^{1/2} (\omega^2 - \rho^2)^{1/2} (v^2 - t^2)^{1/2}} dv d\omega du dt \\
&= \frac{2\rho}{\pi} \int_0^\infty t g(t) \int_\rho^\infty \int_t^\infty \frac{\delta(\omega - v) dv d\omega dt}{\omega v (\omega^2 - \rho^2)^{1/2} (v^2 - t^2)^{1/2}} \\
&= \frac{2\rho}{\pi} \int_0^\infty t g(t) \int_{\max(\rho, t)}^\infty \frac{d\omega dt}{\omega^2 (\omega^2 - \rho^2)^{1/2} (\omega^2 - t^2)^{1/2}} \\
&= \frac{2\rho}{\pi} \int_\rho^\infty \frac{1}{\omega^2 (\omega^2 - \rho^2)^{1/2}} \int_0^\infty \frac{t g(t) dt d\omega}{(\omega^2 - t^2)^{1/2}}, \quad 0 < \rho < \infty \quad \dots (29)
\end{aligned}$$

where we have used the following formulas

$$J_n(u\rho) = \left(\frac{2u}{\pi}\right)^{1/2} \frac{1}{\rho^n} \int_0^\rho \frac{J_{n-(1/2)}(u\omega) \omega^{n+(1/2)}}{(\rho^2 - \omega^2)^{1/2}}, \quad [\text{First sonine Integral}]$$

$$\int_0^\infty u J_\mu(u\omega) J_\mu(uv) du = \delta(\omega - v) / (\omega v)^{1/2}$$

and

$$J_n(u\rho) = \left(\frac{2u}{\pi}\right)^{1/2} \rho^n \int_\rho^\infty \frac{J_{n+(1/2)}(u\omega) \omega^{-\{n-(1/2)\}}}{(\omega^2 - \rho^2)^{1/2}} d\omega$$

In addition to the application of these three formulas, we have changed the order of integration in the steps leading to formulas (28) and (29) as explained in following figures (i) and (ii)

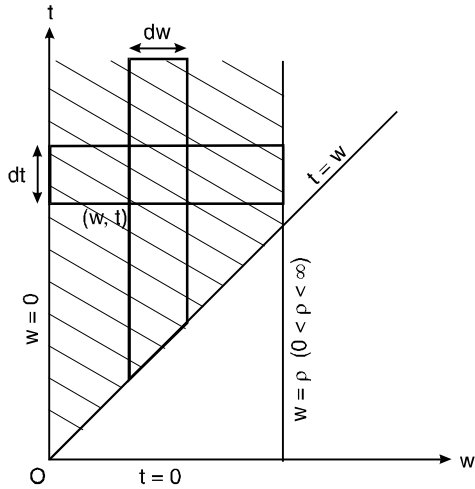


Fig. (i)

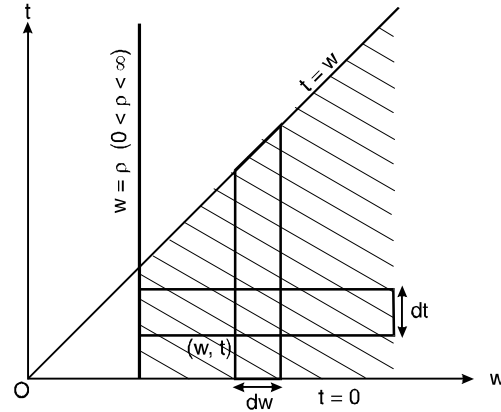


Fig. (ii)

Comparing (28) and (29) with (8), we have

$$\left. \begin{aligned}
h_{11}(\rho) &= 2/\pi\rho, & h_{12}(\rho) &= \rho, & h_{13}(\rho) &= 1/\rho, & h_{21}(\rho) &= 2\rho/\pi, \\
h_{22}(\rho) &= 1/\rho, & h_{23}(\rho) &= \rho, & \text{and} & & K_2(t, \rho) &= (\rho^2 - t^2)^{-1/2}
\end{aligned} \right\} \quad \dots (30)$$

Since the kernel K_2 is such that the Volterra integral equations (9) and (10) can be easily solved, it follows that the method prescribed in article 13.3 is applicable for solving the given problem.

The system of integral equations that corresponds to the system (13), (14), (20), (21), (22) and (23) for the present problem are given by

$$\rho \int_{\rho}^{\infty} \frac{\phi_1(t) dt}{(t^2 - \rho^2)^{1/2}} = \begin{cases} S_1(\rho), & 0 < \rho < a \\ -T_1(\rho), & a < \rho < \infty \end{cases} \quad \dots (31)$$

$$\frac{1}{\rho} \int_0^{\rho} \frac{t^2 \phi_2(t) dt}{(\rho^2 - t^2)^{1/2}} = \begin{cases} -T_2(\rho), & 0 < \rho < b \\ S_2(\rho), & b < \rho < \infty \end{cases} \quad \dots (32)$$

$$\frac{2}{\pi \rho} \int_0^{\rho} \frac{\omega S_1(\omega) d\omega}{(\rho^2 - \omega^2)^{1/2}} = \Omega \rho, \quad 0 < \rho < a \quad \dots (33)$$

$$\frac{2\rho}{\pi} \int_{\rho}^{\infty} \frac{S_2(\omega) d\omega}{\omega(\omega^2 - \rho^2)^{1/2}} = 0, \quad b < \rho < \infty \quad \dots (34)$$

$$\rho \int_{\rho}^{\infty} \frac{\phi_2(t) dt}{(t^2 - \rho^2)^{1/2}} = T_1(\rho), \quad a < \rho < \infty \quad \dots (35)$$

$$\frac{1}{\rho} \int_0^{\rho} \frac{t^2 \phi_1(t) dt}{(\rho^2 - t^2)^{1/2}} = T_2(\rho), \quad 0 < \rho < b \quad \dots (36)$$

Using methods employed in examples 3 and 4 of Art. 8.5 (chapter 8) and example 16 of Art. 9.5 (chapter 9), the above integral equations (31), (32), (33) and (34) can be inverted to yield

$$\phi_1(\rho) = -\frac{2}{\pi} \frac{d}{d\rho} \left[\int_{\rho}^a \frac{S_1(u) du}{(u^2 - \rho^2)^{1/2}} - \int_a^{\infty} \frac{T_1(u) du}{(u^2 - \rho^2)^{1/2}} \right] \quad \dots (37)$$

$$\phi_2(\rho) = \frac{2}{\pi \rho^2} \frac{d}{d\rho} \left[-\int_0^b \frac{u^2 T_2(u) du}{(\rho^2 - u^2)^{1/2}} + \int_b^{\rho} \frac{u^2 S_2(u) du}{(\rho^2 - u^2)^{1/2}} \right] \quad \dots (38)$$

$$S_1(\rho) = \frac{\Omega}{\rho} \frac{d}{d\rho} \int_0^{\rho} \frac{t^3 dt}{(\rho^2 - t^2)^{1/2}} = 2 \Omega \rho, \quad \dots (39)$$

$$S_2(\rho) = 0 \quad \dots (40)$$

Substitute, the values of $S_1(u)$ and $S_2(u)$ as given by (39) and (40) in (37) and (38) respectively. Then substituting the resulting values of $\phi_2(t)$ and $\phi_1(t)$ so obtained in (35) and (36) respectively, we finally obtain

$$\begin{aligned} T_1(\rho) &= \frac{2\rho}{\pi} \int_0^b u^2 T_2(u) \int_{\rho}^{\infty} \frac{dt du}{t (t^2 - \rho^2)^{1/2} (t^2 - u^2)^{3/2}} \\ &= \frac{1}{\rho (\pi)^{1/2} \Gamma(5/2)} \int_0^b \frac{u^2 T_2(u) {}_2F_1(1/2, 1; 5/2; u^2/\rho^2) du}{(\rho^2 - u^2)} \quad a < \rho < \infty \quad \dots (41) \end{aligned}$$

$$\begin{aligned} \text{and} \quad T_2(\rho) &= \frac{4\Omega}{\pi \rho} \int_0^{\rho} \frac{t^3 dt}{(\rho^2 - t^2)^{1/2} (a^2 - t^2)^{1/2}} + \frac{2}{\pi \rho} \int_0^{\rho} \frac{t^3}{(\rho^2 - t^2)^{1/2}} \int_a^{\infty} \frac{T_1(u) du dt}{(u^2 - t^2)^{3/2}} \\ &= \frac{8 \Omega \rho^2 a^3}{3 \pi (\rho^2 - a^2)^2} {}_2F_1\left(1, \frac{3}{2}; \frac{5}{2}; \frac{4a^2 \rho^2}{(\rho^2 + a^2)^2}\right) \\ &\quad + \frac{\rho^2}{(\pi)^{1/2} \Gamma(5/2)} \int_a^{\infty} \frac{T_1(u) {}_2F_1(1/2, 1; 5/2; \rho^2/u^2)}{u (u^2 - \rho^2)} du, \quad 0 < \rho < b \quad \dots (42) \end{aligned}$$

In the above relations (41) and (42), ${}_2F_1$ stands for the well known hypergeometric function. In the process of getting these relations, we have also used the following well known relations containing the hypergeometric functions

$$\begin{aligned}\int_{\rho}^{\infty} \frac{dt}{t(t^2 - \rho^2)^{1/2}(t^2 - u^2)^{3/2}} &= \frac{\pi^{1/2}}{2\rho^2 \Gamma(5/2)(\rho^2 - u^2)} {}_2F_1\left(\frac{1}{2}, 1; \frac{5}{2}; \frac{u^2}{\rho^2}\right), \quad u < \rho \\ \int_0^{\rho} \frac{t^3 dt}{(\rho^2 - t^2)^{1/2}(u^2 - t^2)^{3/2}} &= \frac{\pi^{1/2} \rho^3}{2 \Gamma(5/2) u (u^2 - \rho^2)} {}_2F_1\left(1/2, 1; 5/2; \rho^2/u^2\right), \quad \rho < u \\ \int_0^{\rho} \frac{t^3 dt}{(\rho^2 - t^2)^{1/2}(a^2 - t^2)^{1/2}} &= \frac{\pi^{1/2} (a\rho)^3}{2 \Gamma(5/2) (\rho^2 + a^2)^2} {}_2F_1\left(1, \frac{3}{2}; \frac{5}{2}; \frac{4a^2 \rho^2}{(\rho^2 + a^2)^2}\right), \quad \rho < a\end{aligned}$$

Here (41) and (42) are two simultaneous Fredholm Fredholm integral equations of the second kind. We now proceed to solve them approximately by iteration with help of introduction of the parameter $\lambda = b/a$, such that $\lambda \ll 1$. In fact, it is known that the hypergeometric function ${}_2F_1$ involved in (41) and (42) can be reduced to an elementary function as follows.

$${}_2F_1\left(\frac{1}{2}, 1; \frac{5}{2}; \frac{x^2}{y^2}\right) = \frac{3y}{4x^3} \left[2xy - (y^2 - x^2) \log \frac{y+x}{y-x} \right], \quad x < y \quad \dots (43)$$

Hence (41) and (42) can be transformed into simple forms given by

$$T_1(a\rho) = \frac{1}{\pi} \int_0^1 T_2(bu) \left[\frac{2\lambda\rho}{\rho^2 - \lambda^2 u^2} - \frac{1}{u} \log \left(\frac{\rho + \lambda u}{\rho - \lambda u} \right) \right] du, \quad 1 < \rho < \infty \quad \dots (44)$$

$$\begin{aligned}T_2(b\rho) &= \frac{8\Omega\lambda^2\rho^2 a^2}{3\pi(1+\lambda^2\rho^2)^2} {}_2F_1\left(1, \frac{3}{2}; \frac{5}{2}; \frac{4\lambda^2\rho^2}{(1+\lambda^2\rho^2)^2}\right) \\ &\quad + \frac{1}{\lambda\rho\pi} \int_1^{\infty} T_1(au) \left[\frac{2\lambda u\rho}{u^2 - \lambda^2\rho^2} - \log \frac{u + \lambda\rho}{u - \lambda\rho} \right] du, \quad 0 < \rho < 1 \dots (45)\end{aligned}$$

We first solve (45) by noting that

$${}_2F_1\left(1, \frac{3}{2}; \frac{5}{2}; x^2\right) = 1 + 3 \sum_{n=1}^{\infty} \frac{x^{2n}}{2n+3} \quad \dots (46)$$

Using (46), the first iteration for T_2 is given by

$$T_2(b\rho) = \frac{8\Omega a}{3\pi} \left\{ \lambda^2 \rho^2 + \frac{2\rho^4 \lambda^4}{5} + O(\lambda^6) \right\} \quad \dots (47)$$

Substituting this value of T_2 in (44), we can solve the resulting equation approximately to yield

$$T_1(a\rho) = \frac{32\Omega a \lambda^5}{45\pi^2} \left[\frac{1}{\rho^3} \left(1 + \frac{2\lambda^2}{7} \right) + \frac{6\lambda^2}{7\rho^5} + O(\lambda^4) \right], \quad \dots (48)$$

While arriving at the above approximations (47) and (48), we have included only those terms which are required to find the torque experienced by the annulus up to $O(\lambda^9)$.

Now, substituting the values of S_1 , S_2 , T_1 and T_2 as given by (39), (40), (47) and (48) in (37) and (38), we have

$$\phi_1(\rho) = \frac{4\Omega}{\pi} \left(\frac{(\rho/a)}{\{1 - (\rho^2/a^2)\}^{1/2}} + \frac{16\lambda^5}{45\pi^2} \left[\left(1 + \frac{2\lambda^2}{7} \right) \frac{a^3}{2\rho^3} \left\{ -\frac{3a}{\rho} \sin^{-1} \frac{\rho}{a} + \left(1 - \frac{\rho^2}{a^2} \right)^{1/2} \right\} + 2 \left(1 - \frac{\rho^2}{a^2} \right)^{-1/2} \right] \right)$$

$$+ \frac{3\lambda^2 a^5}{28\rho^5} \left\{ -\frac{15a}{\rho} \sin^{-1} \frac{\rho}{a} - 2 \left(1 - \frac{\rho^2}{a^2} \right)^{3/2} + 9 \left(1 - \frac{\rho^2}{a^2} \right)^{1/2} + 8 \left(1 - \frac{\rho^2}{a^2} \right)^{-1/2} \right\} + O(\lambda^9) \dots (49)$$

and
$$\phi_2(\rho) = -\frac{4\Omega}{3\pi^2} \left[2\lambda \left\{ \frac{3\rho}{b} \sin^{-1} \frac{b}{\rho} - \left(1 - \frac{b^2}{\rho^2} \right)^{1/2} - 2 \left(1 - \frac{b^2}{\rho^2} \right)^{-1/2} \right\} \right. \\ \left. + \frac{\lambda^3 \rho^2}{5b^2} \left\{ \frac{15\rho}{b} \sin^{-1} \frac{b}{\rho} + 2 \left(1 - \frac{b^2}{\rho^2} \right)^{3/2} - 9 \left(1 - \frac{b^2}{\rho^2} \right)^{1/2} - 8 \left(1 - \frac{b^2}{\rho^2} \right)^{-1/2} \right\} + O(\lambda^5) \right] \dots (50)$$

Substituting the values of $\phi_1(\rho)$ and $\phi_2(\rho)$ as given by (49) and (50) in the relation

$$\phi(\rho) = \phi_1(\rho) + \phi_2(\rho),$$

we get the required solution of the integral equation (24).

13.4. GENERALIZED TWO-PART BOUNDARY VALUE PROBLEMS

Consider an integral equation of a more general type such as

$$\int_0^a g(t) K_1(t, \rho) dt = f(\rho), \quad 0 < \rho < a \quad \dots (1)$$

where the kernels $K_1(t, \rho)$ can be perturbed on the kernel $K_0(t, \rho)$ of Art. 13.2. Then (1) can also be solved by employing the present method. Hence, in order to solve (1), we split $K_1(t, \rho)$ as

$$K_1(t, \rho) = K_0(t, \rho) + G(t, \rho), \quad \dots (2)$$

where we assume that the kernel $G(t, \rho)$ is in some sense smaller than $K_0(t, \rho)$. From (1) and (2), we obtain

$$\int_0^a K_0(t, \rho) g(t) dt = f(\rho) - \int_0^a G(t, \rho) g(t) dt, \quad 0 < \rho < a \quad \dots (3)$$

The integral equation (3) is a Fredholm integral equation of the first kind and is therefore, in general, difficult to solve. However, it is possible to reduce the solution of (3) to that of a pair of Volterra integral equations of the first kind with simple kernels. This reduction is obtained for every kernel $K_0(\rho, t)$ which for all $g(t)$ satisfies the relation

$$\int_0^a K_0(t, \rho) g(t) dt = h_1(\rho) \int_0^\rho K_2(\omega, \rho) \{h_2(\omega)\}^2 \int_\omega^a K_2(\omega, t) g(t) h_3(t) dt d\omega, \quad 0 < \rho < a \quad \dots (4)$$

where h_1 , h_2 , h_3 and K_2 are known functions. It is further assumed that the K_2 is such that the Volterra integral equations

$$\int_0^\rho K_2(t, \rho) g(t) dt = f(\rho), \quad 0 < \rho < a \quad \dots (5)$$

and
$$\int_\rho^a K_2(\rho, t) g(t) dt = f(\rho), \quad 0 < \rho < a \quad \dots (6)$$

possess explicit unique solutions for g in terms of f , for all arbitrary differentiable functions f .

From (3) and (4), we have

$$h_1(\rho) \int_0^\rho K_2(\omega, \rho) \{h_2(\omega)\}^2 \int_\omega^a K_2(\omega, t) g(t) h_3(t) dt d\omega = f(\rho) - \int_0^a G(t, \rho) g(t) dt, \quad 0 < \rho < a \quad \dots (7)$$

Now, (1) can be easily solved with help of three functions $S(\rho)$, $C(\rho)$ and $L(v, \omega)$ which are defined as follows :

$$S(\rho) = h_2(\rho) \int_{\rho}^a K_2(\rho, t) g(t) h_3(t) dt, \quad 0 < \rho < a \quad \dots (8)$$

$$f(\rho) = h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) C(\omega) h_2(\omega) d\omega, \quad 0 < \rho < a \quad \dots (9)$$

$$G(t, \rho) = h_1(\rho) h_3(t) \int_0^{\rho} \int_0^t K_2(\omega, \rho) K_2(v, t) h_2(\omega) h_2(v) L(v, \omega) dv d\omega \quad \dots (10)$$

We now proceed to put the right hand side of (7) in the form of the left hand side with help of (8), (9) and (10) as we did in Art. 13.2. Then, the integral on the right side of (7) takes the form

$$\begin{aligned} \int_0^a G(t, \rho) g(t) dt &= \int_0^a g(t) h_1(\rho) h_3(t) \int_0^{\rho} \int_0^t K_2(\omega, \rho) K_2(v, t) h_2(\omega) h_2(v) L(v, \omega) dv d\omega dt \\ &= h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) h_2(\omega) \int_0^a L(v, \omega) h_2(v) \int_v^a K_2(v, t) g(t) h_3(t) dt dv d\omega, \quad 0 < \rho < a \quad \dots (11) \end{aligned}$$

where it is assumed that various orders of integration can be interchanged.

Substituting (8), (9) and (11) in (7), we get

$$\begin{aligned} h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) h_2(\omega) S(\omega) d\omega &= h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) C(\omega) h_2(\omega) d\omega \\ &\quad - h_1(\rho) \int_0^{\rho} K_2(\omega, \rho) h_2(\omega) \int_0^a L(v, \omega) S(v) dv d\omega, \quad 0 < \rho < a \quad \dots (12) \end{aligned}$$

$$(12) \Rightarrow S(\rho) = C(\rho) - \int_0^a L(v, \rho) S(v) dv, \quad 0 < \rho < a \quad \dots (13)$$

which is a Fredholm integral equation of the second kind. Solve (13) for $S(\rho)$ and substitute the value of $S(\rho)$ so obtained in (8) and then invert the resulting equation to get the required function $g(t)$ as solution of the given integral equation (1).

We now proceed to illustrate the above analysis with the following examples :

Example 1. Solve the integral equation

$$\int_0^a t \phi(t) \int_0^{\infty} (1/\gamma) \{u J_1(u\rho) J_1(ut)\} du dt = \Omega \rho, \quad 0 < \rho < a \quad \dots (14)$$

where
$$\gamma = \begin{cases} -i(k^2 - u^2)^{1/2}, & k \geq u \\ (u^2 - k^2)^{1/2}, & u \geq k \end{cases} \quad \dots (15)$$

[The above equation can be used to solve the problem of the torsional oscillations of an isotropic and homogeneous elastic half-space due to a rigid circular disc of radius a which is performing simple harmonic oscillations]

Solution. Comparing (14) with (1), we have

$$g(t) = t \phi(t), \quad f(\rho) = \Omega \rho, \quad K_1(t, \rho) = \int_0^{\infty} (1/\gamma) \{u J_1(u\rho) J_1(ut)\} du \quad \dots (16)$$

We now split $K_1(t, \rho)$ as in (2) with

$$K_0(t, \rho) = \int_0^{\infty} J_1(u\rho) J_1(ut) du, \quad \dots (17)$$

and
$$G(t, \rho) = \int_0^\infty (u/\gamma - 1) J_1(u\rho) J_1(ut) du$$

$$= \frac{2}{\pi \rho t} \int_0^\rho \int_0^t \frac{(\omega v)^{3/2}}{(\rho^2 - \omega^2)^{1/2} (t^2 - \omega^2)^{1/2}} \left\{ \int_0^\infty u(u/\gamma - 1) J_{1/2}(uv) J_{1/2}(u\omega) du \right\} dv d\omega \quad \dots (18)$$

Using the procedure of Art. 13.4 and making use of the known results of Art. 13.2, we obtain, as usual, the following results :

$$h_1(\rho) = 2/\pi\rho, \quad h_2(\rho) = \rho, \quad h_3(\rho) = 1/\rho, \quad K_2(t, \rho) = (\rho^2 - t^2)^{-1/2} \quad \dots (19)$$

$$L(v, \omega) = (v\omega)^{1/2} \int_0^\infty u(u/\gamma - 1) J_{1/2}(uv) J_{1/2}(u\omega) du, \quad \dots (20)$$

$$S(\rho) = \rho \int_\rho^a \frac{\phi(t) dt}{(t^2 - \rho^2)^{1/2}} \quad \dots (21)$$

$$\Omega\rho = \frac{2}{\pi\rho} \int_0^\rho \frac{\omega C(\omega) d\omega}{(\rho^2 - \omega^2)^{1/2}} \quad \dots (22)$$

and
$$S(\rho) = C(\rho) - \int_0^a L(v, \rho) S(v) dv, \quad 0 < \rho < a \quad \dots (23)$$

Now, inverting (22), we obtain (as in Art. 13.2)

$$C(\rho) = \frac{\Omega}{\rho} \frac{d}{d\rho} \int_0^\rho \frac{t^3 dt}{(\rho^2 - t^2)^{1/2}} = 2\Omega\rho. \quad \dots (24)$$

Substituting the above value of $C(\rho)$ in (23), we have

$$S(\rho) = 2\Omega\rho - \int_0^a L(v, \rho) S(v) dv, \quad 0 < \rho < a \quad \dots (25)$$

Converting the infinite integral (20) into a finite integral (refer Appendix at the end of this chapter), we have

$$L(v, \omega) = \begin{cases} i(v\omega)^{1/2} \int_0^k \left\{ u^2/(k^2 - u^2) \right\}^{1/2} H_{1/2}^{(1)}(uv) J_{1/2}(u\omega) du, & v \geq \omega \\ i(v\omega)^{1/2} \int_0^k \left\{ u^2/(k^2 - u^2) \right\}^{1/2} J_{1/2}(uv) H_{1/2}^{(1)}(u\omega) du, & \omega \geq v \end{cases} \quad \dots (26)$$

where $H_{1/2}^{(1)}$ is a Hankel function of the first kind. This form of the kernel is useful for small values of k .

We now proceed to solve (25) approximately for small values of ak , which is dimensionless parameter. To this end, we re-write (25) as

$$S(a\rho) = 2\Omega a\rho - a \int_0^1 L(av, a\rho) S(av) dv, \quad 0 < \rho < 1. \quad \dots (27)$$

Using the series expansions for the Hankel and Bessel functions, (26) yields the following approximate value of the kernel $aL(av, a\rho)$.

$$aL(av, a\rho) \begin{cases} (a^2\rho)/2 + (4i\alpha^3\rho v)/3\pi - \alpha^4(3\rho v^2 + \rho^3)/16 - 8i\alpha^5(\rho^3 v + \rho v^3)/45\pi \\ + \alpha^6(5\rho v^4 + 10\rho^3 v^2 + \rho^5)/384 + 4i\alpha^7(3\rho v^5 + 10\rho^3 v^3 + 3\rho^5 v)/1575\pi \\ + O(\alpha^8), \quad v \geq \rho \\ (a^2 v)/2 + (4i\alpha^3 \rho v)/3\pi - \alpha^4(3\rho^2 v + v^3)/16 - 8i\alpha^5(\rho v^3 + \rho^3 v)/45\pi \\ + \alpha^6(5\rho^4 v + 10\rho^2 v^3 + v^5)/384 + 4i\alpha^7(3\rho v^5 + 10\rho^3 v^3 + 3\rho^5 v)/1575\pi \\ + O(\alpha^8), \quad \rho \geq v \end{cases} \quad \dots (28)$$

where $\alpha = ak$. By using the usual iteration method to (27), we arrive at an approximate value for $S(a\rho)$ given by

$$S(a\rho) = 2\Omega a \{C_1(\alpha)\rho + C_3(\alpha)\rho^3 + C_5(\alpha)\rho^5 + C_7(\alpha)\rho^7 + O(\alpha^8)\} \quad \dots (29)$$

where
$$C_1(\alpha) = 1 - \frac{\alpha^2}{4} - \frac{4i\alpha^3}{9\pi} + \frac{19\alpha^4}{192} + \frac{53i\alpha^5}{225\pi} - \left(\frac{16}{81\pi^2} + \frac{143}{3840}\right)\alpha^6 - \frac{8051i\alpha^7}{58800\pi},$$

$$C_3(\alpha) = \alpha^2/12 + \alpha^4/96 + i\alpha^5/45\pi - 11\alpha^6/2304 - 47i\alpha^7/4200\pi$$

$$C_5(\alpha) = -\alpha^4/960 - \alpha^6/3840 - i\alpha^7/1680\pi, \quad C_7(\alpha) = \alpha^6/80640$$

Now, inverting (21), we obtain

$$\phi(\rho) = -\frac{2}{\pi} \frac{d}{d\rho} \int_{\rho}^a \frac{S(u) du}{(u^2 - \rho^2)^{1/2}} = -\frac{2}{\pi} \frac{d}{d\rho} \int_{\rho/a}^1 \frac{S(au) du}{\{u^2 - (\rho^2/a^2)\}^{1/2}} \quad \dots (30)$$

Using (29), (30) reduces to

$$\begin{aligned} \phi(\rho) = \frac{(4\Omega\rho)/a}{\pi \{1 - (\rho^2/a^2)\}^{1/2}} & \left[1 - \alpha^2 \left\{ \frac{1}{6} + \frac{1}{6} \left(1 - \frac{\rho^2}{a^2} \right) \right\} - \frac{4i\alpha^3}{9\pi} + \alpha^4 \left\{ \frac{13}{120} - \frac{1}{60} \left(1 - \frac{\rho^2}{a^2} \right) - \frac{1}{360} \left(1 - \frac{\rho^2}{a^2} \right)^2 \right\} \right. \\ & + \frac{i\alpha^5}{\pi} \left\{ \frac{58}{225} - \frac{2}{45} \left(1 - \frac{\rho^2}{a^2} \right) \right\} - \alpha^6 \left\{ \frac{16}{81\pi^2} + \frac{17}{1680} - \frac{53}{5040} \left(1 - \frac{\rho^2}{a^2} \right) + \frac{1}{1680} \left(1 - \frac{\rho^2}{a^2} \right)^2 \right. \\ & \left. \left. + \frac{1}{25200} \left(1 - \frac{\rho^2}{a^2} \right)^3 \right\} - \frac{i\alpha^7}{\pi} \left\{ \frac{1093}{7350} - \frac{13}{525} \left(1 - \frac{\rho^2}{a^2} \right) + \frac{1}{630} \left(1 - \frac{\rho^2}{a^2} \right)^2 \right\} + O(\alpha^8) \right] \quad \dots (31) \end{aligned}$$

Remark When $\alpha \rightarrow 0$, (31) reduces to the result (20) of Art. 13.2.

Example 2. Solve the integral equation

$$1 = \int_0^a t g(t) K_1(t, \rho) dt, \quad 0 < \rho < a, \quad \text{where} \quad K_1(t, \rho) = \int_0^\infty (1/\gamma) \{u J_0(u\rho) J_0(ut)\} du$$

Hint : To solve the given integral equation, we split the kernel K_1 as

$$K_1(t, \rho) = K_0(t, \rho) + G(t, \rho)$$

where
$$K_0(t, \rho) = \int_0^\infty J_0(u\rho) J_0(ut) du, \quad G(t, \rho) = \int_0^\infty (u/\gamma - 1) J_0(u\rho) J_0(ut) du.$$

Now proceed as in example 1 for getting the solution.

10.5. GENERALIZED THREE-PART BOUNDARY VALUE PROBLEMS

Consider an integral equation of a more general type such as

$$\int_b^a K_1(t, \rho) g(t) dt = f(\rho), \quad b < \rho < a \quad \dots (1)$$

where the kernels $K_1(t, \rho)$ can be perturbed on the kernel $K_0(t, \rho)$ of Art. 13.3. In order to solve (1), we split the kernel $K_1(t, \rho)$ as

$$K_1(t, \rho) = K_0(t, \rho) + G(t, \rho) \quad \dots (2)$$

where we assume that the kernel $G(t, \rho)$ is in some sense smaller than $K_0(t, \rho)$.

Suppose that
$$f(\rho) = \sum_{r=-\infty}^{\infty} a_r \rho^r = f_1(\rho) + f_2(\rho), \quad \dots (3)$$

where
$$f_1(\rho) = \sum_{r=0}^{\infty} a_r \rho^r, \quad 0 \leq \rho \leq a \quad \dots (4)$$

and
$$f_2(\rho) = \sum_{r=-\infty}^{-1} a_r \rho^r, \quad b \leq \rho \leq \infty \quad \dots (5)$$

We also define two functions $g_1(\rho)$ and $g_2(\rho)$ such that

$$g_1(\rho) + g_2(\rho) = \begin{cases} 0, & 0 \leq \rho \leq b \\ g(\rho), & b \leq \rho \leq a \\ 0, & a < \rho < \infty \end{cases} \quad \dots (6)$$

Using (2), (3), (4), (5) and (6), the integral equation (1) becomes equivalent to the following pair of integral equations

$$\int_0^{\infty} K_0(t, \rho) g_1(t) dt = f_1(\rho) - \int_0^{\infty} G(t, \rho) g_1(t) dt, \quad 0 < \rho < a \quad \dots (7)$$

$$\int_0^{\infty} K_0(t, \rho) g_2(t) dt = f_2(\rho) - \int_0^{\infty} G(t, \rho) g_2(t) dt, \quad b < \rho < \infty \quad \dots (8)$$

We assume that the kernel $K_0(t, \rho)$ is such that for all $g(t)$ it satisfies the relation

$$\int_0^{\infty} K_0(t, \rho) g(t) dt = \begin{cases} h_{11}(\rho) \int_0^{\rho} K_2(\omega, \rho) \{h_{12}(\omega)\}^2 \int_{\omega}^{\infty} K_2(\omega, t) g(t) h_{13}(t) dt d\omega, & 0 < \rho < \infty \\ h_{21}(\rho) \int_{\rho}^{\infty} K_2(\rho, \omega) \{h_{22}(\omega)\}^2 \int_0^{\omega} K_2(t, \omega) g(t) h_{23}(t) dt d\omega, & 0 < \rho < \infty \end{cases} \quad \dots (9)$$

where h_{ij} ($i = 1, 2, j = 1, 2, 3$), and the kernel K_2 are known functions. Moreover, we suppose that the kernel K_2 is such that the Volterra integral equations

$$\int_0^{\rho} K_2(t, \rho) g(t) dt = f(\rho), \quad 0 < \rho < \infty \quad \dots (10)$$

and
$$\int_{\rho}^{\infty} K_2(\rho, t) g(t) dt = f(\rho), \quad 0 < \rho < \infty \quad \dots (11)$$

possess unique solutions for g in terms of all arbitrary differentiable functions f .

Now we extend the analysis of Art. 10.4 and define two new kernels $L_1(v, \omega)$ and $L_2(v, \omega)$ such that

$$G(t, \rho) = \begin{cases} h_{11}(\rho) h_{13}(t) \int_0^{\rho} \int_0^t K_2(\omega, \rho) K_2(v, t) h_{12}(\omega) h_{12}(v) L_1(v, \omega) dv d\omega, \\ h_{21}(\rho) h_{23}(t) \int_{\rho}^{\infty} \int_t^{\infty} K_2(\rho, \omega) K_2(t, v) h_{22}(\omega) h_{22}(v) L_2(v, \omega) dv d\omega, \end{cases} \quad \dots (12)$$

Hence, the integrals on the right side of the relations (7) and (8) can be reduced in the following forms :

$$\int_0^{\infty} G(t, \rho) g_1(t) dt = \int_0^{\infty} g_1(t) h_{11}(\rho) h_{13}(t) \int_0^{\rho} \int_0^t K_2(\omega, \rho) K_2(v, t) h_{12}(\omega) h_{12}(v) L_1(v, \omega) dv d\omega dt$$

$$= h_{11}(\rho) \int_0^\rho K_2(\omega, \rho) h_{12}(\omega) \int_0^\infty L_1(v, \omega) h_{12}(v) \int_v^\infty K_2(v, t) g_1(t) h_{13}(t) dt dv d\omega, \quad 0 < \rho < a \quad \dots (13)$$

$$\text{and } \int_0^\infty G(t, \rho) g_2(t) dt = \int_0^\infty g_2(t) h_{21}(\rho) h_{23}(t) \int_\rho^\infty \int_t^\infty K_2(\rho, \omega) K_2(t, v) h_{22}(\omega) h_{22}(v) L_2(v, \omega) dv d\omega dt$$

$$= h_{21}(\rho) \int_\rho^\infty K_2(\rho, \omega) h_{22}(\omega) \int_0^\infty L_2(v, \omega) h_{22}(v) \int_0^v K_2(t, v) g_2(t) h_{23}(t) dt dv d\omega, \quad b < \rho < \infty \quad \dots (14)$$

where it is assumed that various orders of integration may be interchanged.

Using (7), (13) and the first part of (9), we obtain

$$h_{11}(\rho) \int_0^\rho K_2(\omega, \rho) \{h_{12}(\omega)\}^2 \int_\omega^\infty K_2(\omega, t) g_1(t) h_{13}(t) dt d\omega = f_1(\rho) - h_{11}(\rho) \int_0^\rho K_2(\omega, \rho) h_{12}(\omega)$$

$$\times \int_0^\infty L_1(v, \omega) h_{12}(v) \int_v^\infty K_2(v, t) g_1(t) h_{13}(t) dt dv d\omega, \quad 0 < \rho < a \quad \dots (15)$$

Similarly, (8), (14) and the second part of (9) yields

$$h_{21}(\rho) \int_\rho^\infty K_2(\rho, \omega) \{h_{22}(\omega)\}^2 \int_0^\omega K_2(t, \omega) g_2(t) h_{23}(t) dt d\omega = f_2(\rho) - h_{21}(\rho) \int_\rho^\infty K_2(\rho, \omega) h_{22}(\omega)$$

$$\times \int_0^\infty L_2(v, \omega) h_{22}(v) \int_0^v K_2(t, v) g_2(t) h_{23}(t) dt dv d\omega, \quad b < \rho < \infty \quad \dots (16)$$

In order to solve (1), we define six functions S_1, S_2, T_1, T_2, C_1 and C_2 as follows :

$$h_{12}(\rho) \int_\rho^\infty K_2(\rho, t) g_1(t) h_{13}(t) dt = \begin{cases} S_1(\rho), & 0 < \rho < a \\ -T_1(\rho), & a < \rho < \infty \end{cases} \quad \dots (17)$$

$$h_{22}(\rho) \int_0^\rho K_2(t, \rho) g_2(t) h_{23}(t) dt = \begin{cases} -T_2(\rho), & 0 < \rho < b \\ S_2(\rho), & b < \rho < \infty \end{cases} \quad \dots (18)$$

$$h_{11}(\rho) \int_0^\rho K_2(\omega, \rho) C_1(\omega) h_{12}(\omega) d\omega = f_1(\rho), \quad 0 < \rho < a \quad \dots (19)$$

$$h_{21}(\rho) \int_\rho^\infty K_2(\rho, \omega) C_2(\omega) h_{22}(\omega) d\omega = f_2(\rho), \quad b < \rho < \infty \quad \dots (20)$$

Now, we use (17), (18), (19) and (20) in (15) and (16) and proceed as in Art. 10.3. Then, as before, the functions C_1 and C_2 can be obtained in terms of f_1 and f_2 . Again, it can be shown that the four unknown functions S_1, S_2, T_1 and T_2 satisfy the following two simultaneous Fredholm integral equations of the second kind :

$$S_1(\rho) + \int_0^a L_1(v, \rho) S_1(v) dv = C_1(\rho) + \int_a^\infty L_1(v, \rho) T_1(v) dv, \quad 0 < \rho < a \quad \dots (21)$$

$$\text{and } S_2(\rho) + \int_b^\infty L_2(v, \rho) S_2(v) dv = C_2(\rho) + \int_0^b L_2(v, \rho) T_2(v) dv, \quad b < \rho < \infty \quad \dots (22)$$

From relations (6) and (17), we have

$$h_{12}(\rho) \int_\rho^\infty K_2(\rho, t) g_2(t) h_{13}(t) dt = T_1(\rho), \quad a < \rho < \infty \quad \dots (23)$$

Similarly, from relations (6) and (18), we have

$$h_{22}(\rho) \int_0^\rho K_2(t, \rho) g_1(t) h_{23}(t) dt = T_2(\rho), \quad 0 < \rho < b \quad \dots (24)$$

When the values of g_1 and g_2 are substituted in terms of S_1 , T_1 , S_2 and T_2 in (23) and (24), we arrive at two Fredholm integral equations in addition to two Fredholm integral equation (21) and (22) which have been already obtained. Thereby, we have obtained a system of four Fredholm integral equations for getting four functions S_1 , S_2 , T_1 and T_2 . The system is solved by iteration as in the previous articles of this chapter.

We now illustrate the above analysis with the following example.

Example. Solve the integral equation

$$\int_b^a t \phi(t) \int_0^\infty (1/\gamma) \{u J_1(u\rho) J_1(ut)\} du dt = \Omega \rho, \quad b < \rho < a \quad \dots (25)$$

where

$$\gamma = \begin{cases} -i(k^2 - u^2)^{1/2}, & k \geq u \\ (u^2 - k^2)^{1/2}, & u \geq k \end{cases} \quad \dots (26)$$

[Equation (25) arises in the problem of torsional oscillations of an elastic half-space due to a rigid annular disc]

Solution. Comparing (25) with (1), we have

$$g(t) = t \phi(t), \quad f(\rho) = \Omega \rho, \quad K_1(t, \rho) = \int_0^\infty (1/\gamma) \{u J_1(u\rho) J_1(ut)\} du \quad \dots (27)$$

We now split $K_1(t, \rho)$ as in (2) with

$$K_0(t, \rho) = \int_0^\infty J_1(u\rho) J_1(ut) du, \quad \dots (28)$$

$$\text{and} \quad G(t, \rho) = \int_0^\infty (u/\gamma - 1) J_1(u\rho) J_1(ut) du$$

$$= \frac{2}{\pi \rho t} \int_0^\rho \int_0^t \frac{(\omega v)^{3/2}}{(\rho^2 - \omega^2)^{1/2} (t^2 - \omega^2)^{1/2}} \left\{ \int_0^\infty u (u/\gamma - 1) J_{1/2}(uv) J_{1/2}(u\omega) du \right\} dv d\omega \quad \dots (29)$$

For the present problem the relations (28) and (29) of Art. 13.3 are valid and it can be proved that $G(t, \rho)$ is given by

$$\begin{aligned} G(t, \rho) &= \int_0^\infty (u/\gamma - 1) J_1(u\rho) J_1(ut) du \\ &= \begin{cases} \frac{2}{\pi \rho t} \int_0^\rho \int_0^t \frac{(\omega v)^{3/2}}{(\rho^2 - \omega^2)^{1/2} (t^2 - \omega^2)^{1/2}} \left\{ \int_0^\infty u (u/\gamma - 1) J_{1/2}(uv) J_{1/2}(u\omega) du \right\} dv d\omega \\ \frac{2 \rho t}{\pi} \int_\rho^\infty \int_t^\infty \frac{(\omega v)^{-1/2}}{(\omega^2 - \rho^2)^{1/2} (v^2 - t^2)^{1/2}} \left\{ \int_0^\infty u (u/\gamma - 1) J_{3/2}(uv) J_{3/2}(u\omega) du \right\} dv d\omega \end{cases} \quad \dots (30) \end{aligned}$$

which corresponds to equation (12). Hence, the functions h_{ij} ($i = 1, 2, j = 1, 2, 3$) and the kernel K_2 are given by

$$\begin{aligned} h_{11}(\rho) &= 2/\pi\rho, \quad h_{12}(\rho) = \rho, \quad h_{13}(\rho) = 1/\rho, \quad h_{21}(\rho) = 2\rho/\pi, \\ h_{22}(\rho) &= 1/\rho, \quad h_{23}(\rho) = \rho \quad \text{and} \quad K_2(t, \rho) = (\rho^2 - t^2)^{-1/2} \end{aligned} \quad \dots (31)$$

Again the kernels $L_1(v, \omega)$ and $L_2(v, \omega)$ are given by

$$L_1(v, \omega) = (\omega v)^{1/2} \int_0^\infty u (u/\gamma - 1) J_{1/2}(uv) J_{1/2}(u\omega) du \quad \dots (32)$$

and

$$L_2(v, \omega) = (\omega v)^{1/2} \int_0^\infty [u (u/\gamma) - 1] J_{3/2}(uv) J_{3/2}(u\omega) du \quad \dots (33)$$

Observe that $L_1(v, \omega)$ is the same as $L(v, \omega)$ as given by (20) in Art. 13.4

Proceeding as before four Fredholm integral equations of the second kind for the unknown functions S_1 , S_2 , T_1 and T_2 are given by

$$T_1(\rho) = l_2(\rho) + \frac{1}{\rho(\pi)^{1/2}\Gamma(5/2)} \int_0^b \frac{u^2 T_2(u) {}_2F_1(1/2, 1; 5/2; u^2/\rho^2) du}{\rho^2 - u^2}, \quad a < \rho < \infty \quad \dots (34)$$

$$T_2(\rho) = l_1(\rho) + \frac{\rho^2}{(\pi)^{1/2}\Gamma(5/2)} \int_a^\infty \frac{T_1(u) {}_2F_1(1/2, 1; 5/2; \rho^2/u^2) du}{u(u^2 - \rho^2)}, \quad 0 < \rho < b \quad \dots (35)$$

$$S_1(\rho) + \int_0^a L_1(v, \rho) S_1(v) dv = 2\Omega\rho + \int_a^\infty L_1(v, \rho) T_1(v) dv, \quad 0 < \rho < a \quad \dots (36)$$

$$S_2(\rho) + \int_b^\infty L_2(v, \rho) S_2(v) dv = \int_0^b L_2(v, \rho) T_2(v) dv, \quad b < \rho < \infty \quad \dots (37)$$

where
$$l_1(\rho) = -\frac{2}{\pi\rho} \int_0^\rho \frac{t^2}{(\rho^2 - t^2)^{1/2}} \frac{d}{dt} \int_t^a \frac{S_1(u) du dt}{(u^2 - t^2)^{1/2}}, \quad 0 < \rho < b \quad \dots (38)$$

$$l_2(\rho) = \frac{2\rho}{\pi} \int_\rho^\infty \frac{1}{t^2(t^2 - \rho^2)^{1/2}} \frac{d}{dt} \int_b^t \frac{u^2 S_2(u) du dt}{(t^2 - u^2)^{1/2}}, \quad a < \rho < \infty \quad \dots (39)$$

We now proceed to solve equation (34), (35), (36) and (37) approximately by iteration when the parameters ka and b/a are small.

In the above equations (34) and (35) ${}_2F_1$ stands for the hypergeometric function. Indeed, the hypergeometric functions ${}_2F_1$ occurring under the integral signs in these equations is reducible to an elementary function :

$${}_2F_1\left(\frac{1}{2}, 1; \frac{5}{2}, \frac{x^2}{y^2}\right) = \frac{3y}{4x^3} \left\{ 2xy - (y^2 - x^2) \log \frac{y+x}{y-x} \right\}, \quad x < y \quad \dots (40)$$

Using (40), the equation (34) can be re-written in the form

$$T_1(a\rho) = l_2(a\rho) + \frac{1}{\pi} \int_0^1 T_2(bu) \left\{ \frac{2\lambda\rho}{\rho^2 - \lambda^2 u^2} - \frac{1}{u} \log \left(\frac{\rho + \lambda u}{\rho - \lambda u} \right) \right\} du, \quad 1 < \rho < \infty \quad \dots (41)$$

where $\lambda = b/a$. Likewise, (35) may be re-written as

$$T_2(b\rho) = l_1(b\rho) + \frac{1}{\lambda\rho\pi} \int_1^\infty T_1(au) \left\{ \frac{2\lambda u\rho}{u^2 - \lambda^2 \rho^2} - \log \left(\frac{u + \lambda\rho}{u - \lambda\rho} \right) \right\} du, \quad 0 < \rho < 1 \quad \dots (42)$$

Note that the parameters involved in the present problem are

$$\alpha = ak, \quad \beta = bk, \quad \lambda = b/a = \beta/\alpha. \quad \dots (43)$$

and the discussion, which now follows, is based on the assumption that

$$\alpha = O(\lambda) \quad \text{so that} \quad \beta = \alpha\lambda = O(\alpha^2). \quad \dots (44)$$

Now, re-writing (36), we have

$$S_1(a\rho) + a \int_0^1 L_1(av, a\rho) S_1(av) dv = 2\Omega a\rho + a \int_1^\infty L_1(av, a\rho) T_1(av) dv, \quad 0 < \rho < 1 \quad \dots (45)$$

In order to solve (45), we assume that

$$S_1(a\rho) = X_1(a\rho) + W_1(a\rho), \quad \dots (46)$$

where
$$X_1(a\rho) = 2\Omega a\rho - a \int_0^1 L_1(av, a\rho) X_1(av) dv, 0 < \rho < 1 \quad \dots (47)$$

and
$$W_1(a\rho) = a \int_1^\infty L_1(av, a\rho) T_1(av) dv - a \int_0^1 L_1(av, a\rho) W_1(av) dv, 0 < \rho < 1 \quad \dots (48)$$

As already observed L and L_1 are identical and hence the integral equation (47) is same as equation (27) of Art. 13.4. Hence, $X_1(a\rho)$ is given by the expression on the R.H.S. of equation (29) of Art. 13.4.

Proceeding likewise, the integral equation (37) can re-written as

$$S_2(b\rho) + b \int_1^\infty L_2(bv, b\rho) S_2(bv) dv = b \int_0^1 L_2(bv, b\rho) T_2(bv) dv, 1 < \rho < \infty \quad \dots (49)$$

whose kernel can be reduced to the following form (refer Appendix at the end of this chapter)

$$L_2(v, \rho) = \begin{cases} i(\rho v)^{1/2} \int_0^k \left\{ u^2 / (k^2 - u^2)^{1/2} \right\} J_{3/2}(u\rho) H_{3/2}^{(1)}(uv) du, & v \geq \rho \\ i(\rho v)^{1/2} \int_0^k \left\{ u^2 / (k^2 - u^2)^{1/2} \right\} J_{3/2}(uv) H_{3/2}^{(1)}(u\rho) du, & \rho \geq v \end{cases} \quad \dots (50)$$

Using the expansions for Bessel and Hankel functions, we arrive at the approximate formula

$$b L_2(bv, b\rho) = \begin{cases} \alpha^2 \lambda^2 \left\{ (\rho^2 / 6v) + O(\alpha^2) \right\}, & v \geq \rho \\ \alpha^2 \lambda^2 \left\{ (v^2 / 6\rho) + O(\alpha^2) \right\}, & \rho \geq v \end{cases} \quad \dots (51)$$

The functions occurring in the system of equations (34)- (51) must be obtained in the order $X_1, l_1, T_2, S_2, l_2, T_1, W_1, S_1$. We have already obtained X_1 . We now proceed to compute the other functions of the sequence of functions already mentioned. The required results, obtained by one iteration, are given by

$$l_1(b\rho) = \frac{8\Omega a\rho^2 \lambda^2}{3\pi} \left\{ \left(1 - \frac{\alpha^2}{3} - \frac{4i\alpha^3}{9\pi} \right) + \frac{2\rho^2 \lambda^2}{5} + O(\alpha^4) \right\}, 0 < \rho < 1 \quad \dots (52)$$

$$T_2(b\rho) = l_1(b\rho) + O(\alpha^7), 0 < \rho < 1 \quad \dots (53)$$

$$S_2(b\rho) = \frac{4\Omega a\alpha^2 \lambda^4}{45\pi} \left\{ \frac{1}{\rho} + O(\alpha^4) \right\}, 1 < \rho < \infty \quad \dots (54)$$

$$l_2(a\rho) = \frac{8\Omega a\alpha^2 \lambda^5}{45\pi^2} \left\{ \frac{1}{\rho} + O(\alpha^2) \right\}, 1 < \rho < \infty \quad \dots (55)$$

$$T_1(a\rho) = \frac{32\Omega a\lambda^5}{45\pi^2} \left\{ \frac{\alpha^2}{4\rho} + \frac{1}{\rho^3} \left(1 - \frac{\alpha^3}{3} + \frac{2\lambda^2}{7} \right) + \frac{6\lambda^2}{7\rho^5} + O(\alpha^3) \right\}, 1 < \rho < \infty \quad \dots (56)$$

$$W_1(a\rho) = \frac{8\Omega a\alpha^2 \lambda^5}{45\pi^2} \{ \rho + O(\alpha) \}, 0 < \rho < 1 \quad \dots (57)$$

$$S_1(a\rho) = X_1(a\rho) + (8\Omega a\rho \alpha^2 \lambda^5) / (45\pi^2) + O(\alpha^8), 0 < \rho < 1 \quad \dots (58)$$

In the above approximation, we have included only those terms that are required to compute the value of the torque experienced by the annular disc to $O(\alpha^8)$

The values of g_1 and g_2 are obtained by inverting (17) and (18) with T_2, S_2, T_1 and S_1 as given by (53), (54), (56) and (58) respectively. Then, using (6), the value of g can be evaluated. Next, using (27), we get

$$\phi(\rho) = \rho^{-1} g(\rho).$$

Remark. The method explained in the previous articles must be modified (as indicated below) while dealing with problems connected with spherical caps and annular spherical caps (or spherical rings). In this connection we have the following results :

Replace a by α , b by β , ρ by θ , ∞ by π and equation (2) of Art. 13.3 by

$$f(\theta) = \sum_{r=-\infty}^{\infty} a_r \{\tan(\theta/2)\}^r = f_1(\theta) + f_2(\theta) \quad \dots (i)$$

where

$$f_1(\theta) = \sum_{r=0}^{\infty} a_r \{\tan(\theta/2)\}^r, \quad 0 \leq \theta \leq \alpha \quad \dots (ii)$$

and

$$f_2(\theta) = \sum_{r=-\infty}^{-1} a_r \{\tan(\theta/2)\}^r, \quad \beta < \theta < \pi, \quad \dots (iii)$$

where θ is the polar angle and α, β are the bounding angles of the annular cap.

13.6 APPENDIX

With help of the complex integration method, show that

$$\int_0^{\infty} u \left(\frac{u}{\gamma} - 1 \right) J_{\mu}(uv) J_{\mu}(u\rho) du = \begin{cases} i \int_0^k \frac{u^2}{(k^2 - u^2)^{1/2}} H_{\mu}^{(1)}(uv) J_{\mu}(u\rho) du, & v \geq \rho \\ i \int_0^k \frac{u^2}{(k^2 - u^2)^{1/2}} J_{\mu}(uv) H_{\mu}^{(1)}(u\rho) du, & \rho \geq v \end{cases} \quad \dots (1)$$

$$\text{where } \gamma = \begin{cases} -i(k^2 - u^2)^{1/2}, & k \geq u \\ (u^2 - k^2)^{1/2}, & u \geq k \end{cases} \quad \text{and} \quad v, \rho > 0. \quad \dots (2)$$

Proof. Let $v \geq \rho$ and let the complex plane be $s = \sigma + i\tau$.

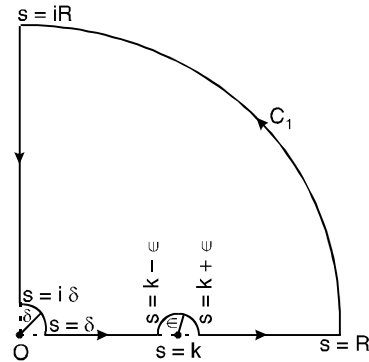
Integrating $\{s^2/(s^2 - k^2)^{1/2} - s\} H_{\mu}^{(1)}(sv) J_{\mu}(s\rho)$ around the circle C_1 in the upper right quadrant passing over the branch point $s = k$, as shown in the adjoining figure, we obtain

$$\oint_{C_1} \left\{ \frac{s^2}{(s^2 - k^2)^{1/2}} - s \right\} H_{\mu}^{(1)}(sv) J_{\mu}(s\rho) ds = 0,$$

because there are no singularities within this contour. Let $\delta, \epsilon \rightarrow 0$ and $R \rightarrow \infty$. Then the contributions from the corresponding arcs tend to zero. Hence, we obtain

$$\begin{aligned} \int_0^k \left\{ \frac{\sigma^2}{i(k^2 - \sigma^2)^{1/2}} - \sigma \right\} H_{\mu}^{(1)}(\sigma v) J_{\mu}(\sigma\rho) d\sigma + \int_k^{\infty} \left\{ \frac{\sigma^2}{(\sigma^2 - k^2)^{1/2}} - \sigma \right\} H_{\mu}^{(1)}(\sigma v) J_{\mu}(\sigma\rho) d\sigma \\ + i \int_{\infty}^0 \left\{ \frac{-\tau^2}{i(\tau^2 + k^2)^{1/2}} - i\tau \right\} H_{\mu}^{(1)}(i\tau v) J_{\mu}(i\tau\rho) d\tau = 0 \quad \dots (3) \end{aligned}$$

Similarly, integrating $\{s^2/(s^2 - k^2)^{1/2} - s\} H_{\mu}^{(2)}(sv) J_{\mu}(s\rho)$ around a contour C_2 in the lower right hand quadrant and passing under the branch point $s = k$, we obtain



$$\int_0^k \left\{ \frac{\sigma^2}{-i(k^2 - \sigma^2)^{1/2}} - \sigma \right\} H_{\mu}^{(2)}(\sigma v) J_{\mu}(\sigma \rho) d\sigma + \int_k^{\infty} \left\{ \frac{\sigma^2}{(\sigma^2 - k^2)^{1/2}} - \sigma \right\} H_{\mu}^{(2)}(\sigma v) J_{\mu}(\sigma \rho) d\sigma$$

$$- i \int_{\infty}^0 \left\{ \frac{\tau^2}{i(\tau^2 + \sigma^2)} + i\tau \right\} H_{\mu}^{(2)}(-i\tau v) J_{\mu}(-i\tau \rho) d\tau = 0 \quad \dots (4)$$

We know that

$$H_{\mu}^{(1)}(i\tau v) J_{\mu}(i\tau \rho) = -H_{\mu}^{(2)}(-i\tau v) J_{\mu}(-i\tau \rho) \quad \dots (5)$$

Using (5) and adding (3) and (4), we obtain, for $v \geq \rho$

$$\int_k^{\infty} \left\{ \frac{\sigma^2}{(\sigma^2 - k^2)^{1/2}} - \sigma \right\} J_{\mu}(\sigma v) J_{\mu}(\sigma \rho) d\sigma$$

$$= \int_0^k \sigma J_{\mu}(\sigma v) J_{\mu}(\sigma \rho) d\sigma - \int_0^k \frac{\sigma^2}{(k^2 - \sigma^2)^{1/2}} Y_{\mu}(\sigma v) J_{\mu}(\sigma \rho) d\sigma, \quad \dots (6)$$

where Y_{μ} is the Bessel functions of the second kind. Using (3) and (6), we have

$$\text{L.H.S. of (1)} = \int_0^{\infty} u \left(\frac{u}{\gamma} - 1 \right) J_{\mu}(uv) J_{\mu}(u\rho) du$$

$$= \int_0^k \left\{ \frac{i u^2}{(k^2 - u^2)^{1/2}} - u \right\} J_{\mu}(uv) J_{\mu}(u\rho) du + \int_k^{\infty} \left\{ \frac{u^2}{(u^2 - k^2)^{1/2}} - u \right\} J_{\mu}(uv) J_{\mu}(u\rho) du$$

$$= i \int_0^k \frac{u^2}{(k^2 - u^2)^{1/2}} \{ J_{\mu}(uv) + i Y_{\mu}(uv) \} J_{\mu}(u\rho) du = i \int_0^k \frac{u^2}{(u^2 - k^2)^{1/2}} H_{\mu}^{(1)}(uv) J_{\mu}(u\rho) du, \quad v \geq \rho,$$

which proves the first part of formula (1).

Second part of formula (1) can be proved similarly.

Integral Equation Perturbation Techniques

14.1 INTRODUCTION

In chapter 13, we have discussed method to solve the Fredholm integral equations of the first kind by converting them to Volterra integral equations and to Fredholm integral equations of the second kind. Since we had to deal with problems involving one variable of integration, the required formulation was rather simple. Hence in order to solve the integral equations relating to boundaries such as a cylinder or a sphere, we require new techniques. In the present chapter, we propose to discuss three-dimensional problems and present approximate techniques for solving them. The analysis for the corresponding plane problems is simpler once the technique is well understood.

14.2 WORKING RULE FOR SOLVING AN INTEGRAL EQUATION BY PERTURBATION TECHNIQUES.

We propose to present approximate techniques for solving the Fredholm integral equations of the first kind

$$f(P) = \int_S K(P, Q) g(Q) dS, \quad P \in S \quad \dots (1)$$

with $P = x$ and $Q = \xi$.

In chapter 13, we have already observed that certain perturbation parameters are naturally involved in physical problems. Let ε be such a parameter occurring in (1). Then, we expand all the three functions K, f and g in power series in ε as shown below

$$K = K_0 + \varepsilon K_1 + \varepsilon^2 K_2 + \dots \quad \dots (2)$$

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots \quad \dots (3)$$

$$g = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \dots \quad \dots (4)$$

Substituting the values of K, f and g as given by (2), (3) and (4) in (1) and equating equal powers of ε , we arrive at the following set of integral equations

$$\int_S K_0 g_0 dS = f_0,$$

$$\int_S K_0 g_1 dS = f_1 - \int_S K_1 g_0 dS, \quad \dots (6)$$

$$\int_S K_0 g_2 dS = f_2 - \int_S K_1 g_1 dS - \int_S K_2 g_0 dS \quad \dots (7)$$

and so on.

The following three conditions must be satisfied in order that the present technique be applicable in solving the given integral equation (1).

- (i) $K_0(P, Q)$ is the dominant part of $K(P, Q)$ as in chapter 13.
- (ii) The integral equation (5) can be solved.
- (iii) The functions $g_0, g_1, \dots$ are such that the integrals occurring on the right side of equations (6), (7) etc. can be easily evaluated.

It is worth noting that, in the present technique, we have only to solve (5) because the other integral equations (6), (7) etc. in the sequence have the same kernel.

Sometimes, an approximation to order (ε) can be very easily obtained. Let the function K_1 occurring in the expansion (2) be a constant C (say). Then (1) can be re-written as

$$f(P) + \varepsilon f' = \int_S K_0(P, Q) g(Q) dS + O(\varepsilon^2), \quad \dots (8)$$

where

$$f' = -C \int_S g(Q) dS, \quad \dots (9)$$

is an unknown constant. Then, to order ε , (8) is similar to (5), whose solution is assumed known. Hence (8) can also be solved. The quantity f' can then be evaluated from different considerations.

The occurrence of a constant C can be demonstrated by the kernel

$$K(P, Q) = \frac{\exp\{i\varepsilon |\mathbf{x} - \boldsymbol{\xi}|\}}{|\mathbf{x} - \boldsymbol{\xi}|} = \frac{\exp(i\varepsilon r)}{r} = \frac{1}{r} + i\varepsilon + O(\varepsilon^2),$$

where $r = |\mathbf{x} - \boldsymbol{\xi}|$ and $\exp(a) = e^a$. Clearly, here $K_0(P, Q) = 1/r$ and $C = i$.

A special case of the above analysis. Let $K(P, Q) = K_0(P, Q) + C$, where C is a constant. Assume that solution $G(P)$ of the integral equation

$$\int_S K_0(P, Q) G(Q) dS = 1, \quad P \in S \quad \dots (10)$$

is known. Then, suppose we are required to solve the integral equation

$$\int_S \{C + K_0(P, Q)\} g(Q) dS = 1, \quad \dots (11)$$

which is obtained by setting $K(P, Q) = K_0(P, Q) + C$ and $f(P) = 1$ in (1). Re-writing (11), we get

$$\int_S K_0(P, Q) g(Q) dS = 1 - C \int_S g(Q) dS \quad \dots (12)$$

Clearly, the value of the integral on the right side of (12) is a constant, although as yet unknown. It follows that the R.H.S of (12) is a constant D (say). Then, (12) reduces to

$$\int_S K_0(P, Q) g(Q) dS = D, \quad \dots (13)$$

where

$$D = 1 - C \int_S g(Q) dS \quad \dots (14)$$

$$\text{Re-writing (13), we have} \quad \int_S K_0(P, Q) \{g(Q)/D\} dS = 1 \quad \dots (15)$$

Comparing (15) with (10), where the value of $G(Q)$ is known, we get

$$g(Q)/D = G(Q) \quad \text{or} \quad g(Q) = D G(Q) \quad \dots (16)$$

Integrating both sides of (16) over the surface S , we obtain

$$\int_S g(Q) dS = D \int_S G(Q) dS \quad \text{or} \quad \int_S g(Q) dS = \left\{1 - C \int_S g(Q) dS\right\} \int_S G(Q) dS, \quad \text{using (14)}$$

or

$$\left\{1 + C \int_S G(Q) dS\right\} \int_S g(Q) dS = \int_S G(Q) dS$$

or
$$\int_S g(Q) dS = \left[\int_S G(Q) dS \right] / \left[1 + C \int_S G(Q) dS \right] \quad \dots (17)$$

Substituting the value of D as given by (14) in (16), we have

$$g(Q) = \left\{ 1 - C \int_S g(Q) dS \right\} G(Q) \quad \dots (18)$$

Substituting the value of $\int_S g(Q) dS$ as given by (17) in (18), we get

$$g(Q) = \left\{ 1 - \frac{C \int_S G(Q) dS}{1 + C \int_S G(Q) dS} \right\} G(Q) \quad \text{or} \quad g(Q) = \frac{G(Q)}{1 + C \int_S G(Q) dS}$$

so that
$$g(P) = G(P) / \left\{ 1 + C \int_S G(Q) dS \right\} \quad \dots (19)$$

which gives the desired solution $g(P)$ of the integral equation (11).

While dealing with the above special case, we took $f(P) = 1$ and $K(P, Q) = K_0(P, Q) + C$ in (1), where C is a constant. We now extend the above analysis by taking a separable kernel with finite terms of the form

$$K(P, Q) = K_0(P, Q) + \sum_{i=1}^n \phi_i(Q) \psi_i(P) \quad \dots (22)$$

where the $\phi_i(Q)$ are known. Then, (1) takes the form

$$\int_S \left\{ K_0(P, Q) + \sum_{i=1}^n \phi_i(Q) \psi_i(P) \right\} g(Q) dS = f(P). \quad \dots (23)$$

Now, suppose that we know the solutions of integral equations

$$\int_S K_0(P, Q) G_0(Q) dS = f(P), \quad P \in S \quad \dots (24)$$

and
$$\int_S K_0(P, Q) G_i(Q) dS = \psi_i(P), \quad P \in S, \quad i = 1, 2, \dots, n \quad \dots (25)$$

then we propose to show that (23) can also be solved. To this end we re-write (23) as

$$\int_S K_0(P, Q) g(Q) dS = f(P) - \sum_{i=1}^n C_i \psi_i(P), \quad P \in S, \quad \dots (26)$$

where
$$C_i = \int_S \phi_i(Q) g(Q) dS. \quad \dots (27)$$

are constants, although as yet unknown. The rest of the steps are exactly similar to those employed in solving (11)

Remark. The techniques of the Art. 14.2 can be applied to solve problems in various disciplines of mathematical physics and engineering. In what follows, we shall let $G(x; \xi)$ be the generic notation for the Green's function as in chapters 11 and 12. In view of the expansion (2) for the kernel K , we shall write $G_0(x; \xi)$ for $E(x; \xi)$.

14.3. APPLICATIONS OF PERTUACTION TECHNIQUES TO ELETROSTATICS

Suppose that there are conductors with surfaces S_1 and S_2 ; S_1 is completely contained in S_2 and is kept at a unit potential, while the potential on S_2 is zero. Let ' a ' denote a characteristic length of S_1 and b denote the minimum distance between a point of S_1 and a point of S_2 . Then the perturbation parameter can be taken as $\varepsilon = a/b$ which is assumed to be much smaller than unity.

In Art. 12.4, we presented an integral representation formula for the electrostatic potential in the region D between S_2 and S_1 :

$$f(P) = \int_{S_1} G(P, Q) \sigma(Q) dS, \quad \dots (1)$$

in terms of the Green's function $G(P, Q)$ and the charge density σ . Applying the boundary condition on S_1 , (1) reduces to

$$1 = \int_{S_1} G(P, Q) \sigma(Q) dS, \quad P \in S_1 \quad \dots (2)$$

As explained in working rule of Art. 14.2, we write

$$G(P, Q) = G_0(P, Q) + G_1(P, Q) \quad \dots (3)$$

where $G_0(P, Q)$ is the free-space Green's function and $G_1(P, Q)$ is the perturbation term. Using (3), (2) reduces to

$$1 = \int_{S_1} G_0(P, Q) \sigma(Q) dS + \int_{S_1} G_1(P, Q) \sigma(Q) dS, \quad P \in S_1, \quad \dots (4)$$

Now, in the absence of the conductor S_2 , there will be only the first integral on the R.H.S. of (4). It follows, that the second integral on the R.H.S. of (4) must represent the effect of the conductor S_2 on the potential of S_1 . Using the hypothesis of Art. 14.2, we assume that we can solve (4) in the absence of the second integral in it.

It can be easily seen that it is always possible to introduce a constant A and write

$$G_1(P, Q) = A + G_2(P, Q), \quad \dots (5)$$

where $G_2(P, Q) = O(A\epsilon)$. For example, one possible value of A is the value $G_1(P, Q)$ for an arbitrary pair of points P and Q on S_1 . Using (5), (4) reduces

$$1 = \int_{S_1} G_0(P, Q) \sigma(Q) dS + A \int_{S_1} \sigma(Q) dS + \int_{S_1} G_2(P, Q) \sigma(Q) dS \quad \dots (6)$$

Let us introduce a new charge density σ' , defined by

$$\sigma'(P) = \sigma(P) / \left\{ 1 - A \int_{S_1} \sigma(Q) dS \right\}. \quad \dots (7)$$

or
$$\left\{ 1 - A \int_{S_1} \sigma(Q) dS \right\} \sigma'(P) = \sigma(P)$$

Integration of both sides of the above equation over the surface S_1 yields

$$\left\{ 1 - A \int_{S_1} \sigma dS \right\} \int_{S_1} \sigma' dS = \int_{S_1} \sigma dS \quad \text{or} \quad \int_{S_1} \sigma' dS = \left\{ 1 + A \int_{S_1} \sigma' dS \right\} \int_{S_1} \sigma dS$$

$$\therefore \int_{S_1} \sigma dS = \left[\int_{S_1} \sigma' dS \right] / \left\{ 1 + A \int_{S_1} \sigma' dS \right\} \quad \dots (8)$$

Again, re-writing (6) in terms of the density σ' , we have

$$1 = \int_{S_1} G_0(P, Q) \sigma'(Q) dS + \int_{S_1} G_2(P, Q) \sigma'(Q) dS. \quad \dots (9)$$

From the above discussion, it follows that the second integral on right side of (9) is $O(\epsilon^2)$ times the first one. Accordingly, if we neglect the terms of this order, then σ' is the electrostatic charge density on S_1 when it is raised to a unit potential in free space. Hence we conclude that (8) gives

the capacity C of the condenser formed by S_1 and S_2 in terms of the free-space capacity C_0 of S_1 ,

$$\text{that is,} \quad C / C_0 = (1 + AC_0)^{-1} + O(\varepsilon^2) \quad \dots (10)$$

$$\text{or} \quad C / C_0 = 1 - AC_0 + O(\varepsilon^2). \quad \dots (11)$$

Suppose we interpret A as the value of $G_1(P, Q)$ for any pair of points P, Q on S_1 . Then, the capacity C given by (11) is exactly the same that would have been obtained had we used the perturbation procedure (2) – (6). The formula (10) is very useful for finding the electrostatic capacity because in many problems it is possible to show that, by a suitable choice of A , the formula (10) is valid for much higher order in ε .

We now illustrate the above analysis by help of the following example.

Example. Consider a sphere of radius a placed with its centre on the axis of an infinite cylinder of radius b . We have already studied an integral equation formulation of a general axially symmetric problem of this type in solved example 2 of Art. 12.5, chapter 12. In terms of cylindrical polar co-ordinates (ρ, ϕ, z) , we have proved that

$$G_1(\rho, \phi, z; \rho_1, \phi_1, z_1) = \sum_{r=0}^{\infty} (2 - \delta_{0r}) G_1^{(r)}(\rho, z; \rho_1, z_1) \cos r(\phi - \phi_1) \quad \dots (12)$$

$$\text{where} \quad G_1^{(r)} = -\frac{2}{\pi} \int_0^{\infty} \frac{I_r(u\rho) I_r(u\rho_1) K_r(ub)}{I_r(ub)} \cos u(z - z_1) du, \quad \dots (13)$$

where (ρ, ϕ, z) and (ρ_1, ϕ_1, z_1) , respectively are cylindrical polar coordinates of points P and Q respectively. Due to axial symmetry, we require only the term $G^{(0)}$ in (12).

We now proceed to find the constant A in the relation (6). To this end, we take

$$z = a \cos \theta \quad \text{and} \quad \rho = a \sin \theta, \quad \dots (14)$$

where θ is the angle between Oz and OP , O being the origin. Then (13) reduces to

$$G_1^{(0)} = -\frac{2}{\pi} \operatorname{Re} \int_0^{\infty} \frac{I_0(ua \sin \theta) I_0(ua \sin \theta_1) K_0(ub)}{I_0(ub)} e^{iua(\cos \theta - \cos \theta_1)} du \quad \dots (15)$$

where the “Re” means that we take the real part of the expression. In addition, we use the formula

$$I_0(ua \sin \theta) e^{iua \cos \theta} = \sum_{n=0}^{\infty} \frac{(iua \sin \theta)^n}{n!} P_n(\cos \theta), \quad \dots (16)$$

where P_n are the Legendre polynomials. Using (16), (15) reduces to

$$G_1^{(0)}(P, Q) = -\frac{2}{\pi} \operatorname{Re} \int_0^{\infty} \sum_{m,n=0}^{\infty} (-1)^m (i)^{m+n} \frac{(ua \sin \theta)^n}{n!} \frac{(ua \sin \theta_1)^m}{m!} \frac{K_0(ub)}{I_0(ub)} P_n(\cos \theta) P_m(\cos \theta_1) du \quad \dots (17)$$

$$\begin{aligned} &= -\frac{2}{\pi b} \operatorname{Re} \int_0^{\infty} (-1)^m (i)^{m+n} \frac{(\varepsilon \sin \theta)^n}{n!} P_n(\cos \theta) \frac{(\varepsilon \sin \theta_1)^m}{m!} P_m(\cos \theta_1) v^{m+n} \frac{K_0(v)}{I_0(v)} dv \\ &= -\frac{1}{b} \operatorname{Re} \left[\sum_{n=0}^{\infty} \varepsilon^n A_n P_n(\cos \theta) \right] \left[\sum_{n=0}^{\infty} \varepsilon^n A_n P_n(\cos \theta_1) \right], \quad \dots (18) \end{aligned}$$

where $v = ub$ and $\varepsilon = a/b$ (the dimensionless parameter of the problem). Also, the constants A_n are given by the formula

$$A_m A_n = \frac{2}{\pi} \int_0^{\infty} (-1)^m (i)^{m+n} v^{m+n} \frac{K_0(v)}{I_0(v)} dv \quad \dots (19)$$

From (19) we conclude that when $(m + n)$ is odd, $A_m A_n$ is an imaginary quantity.

Using (18) and (19), the desired constant A is given by

$$A = -\frac{A_0^2}{b} = -\frac{2}{\pi b} \int_0^\infty \frac{K_0(v)}{I_0(v)} dv \quad \dots (20)$$

Substituting the value of A as given by (20) in (10), we obtain

$$C / C_0 \approx [1 - (2 / \pi b) C_0 I(0)]^{-1}, \quad \dots (21)$$

where

$$I(2m) = (2m + 1) \int_0^\infty \frac{v^{2m} K_0(v)}{I_0(v)} dv, \quad \dots (22)$$

for which numerical tables are available.

EXERCISE

1. Instead of spherical coordinates (14) of Art. 14.3, take the oblate-spherical coordinates

$$z = a \varepsilon \xi, \quad \rho = ae \left\{ (1 - \xi^2) (1 + \xi^2) \right\}^{1/2}$$

and solve the electrostatic potential problem for the case of an oblate spheroid placed symmetrically inside a grounded infinite cylinder of radius b . Show that the capacity of this condenser is

$$C = \frac{ae}{\sin^{-1} e} \left[1 - \frac{2 \varepsilon e}{\pi \sin^{-1} e} \left\{ I(0) + \frac{(e \varepsilon)^2 I(2)}{81} + \frac{2(e \varepsilon)^4 I(4)}{225} \right\} \right]^{-1} + O(\varepsilon^6),$$

where the quantities $I(2m)$ are defined by (22) of Art. 14.3

$$\text{Hint. Use the expansion} \quad I_0(u\rho)e^{iuz} = \sum B_n(u) P_n(\xi) P_n(i\xi),$$

where the P_n are Legendre polynomials.

2. Using a method similar to exercise 1, solve the electrostatic problem for a prolate spheroid.
3. Solve exercises 1 and 2 when the spheroids are placed between two grounded parallel plates.

14.4 APPLICATIONS OF PERTURBATION TECHNIQUES TO LOW-REYNOLDS NUMBER HYDRODYNAMICS

The flow of an incompressible viscous fluid is governed by the following two types of linearized equations :

1. Stokes equations
2. Oseen equations

In this article we propose to use the perturbation techniques to study these two kinds of flows.

14.4A STEADY STOKES FLOW.

We have already studied steady stokes flow in an unbounded motion is solved example 1, of Art. 12.7. Please read carefully that example before proceeding further. Recall that, for a free space, the boundary value problem is

$$\nabla^2 \mathbf{q} = \nabla p, \quad \nabla \cdot \mathbf{q} = 0 \quad \dots (1)$$

$$\mathbf{q} = \mathbf{e}_1, \text{ on } S_1; \quad q_1 \rightarrow 0 \text{ at } \infty; \quad \dots (2)$$

where this system has been made dimensionless with help of the uniform speed U of the given solid B and with its characteristic length a . Here $\mathbf{q}, p$ and $\mathbf{e}_1$ denote the velocity vector, pressure and the unit vector along the x_1 -axis respectively.

We have proved that the integral-equation formula for the boundary value problem (1)- (2) in terms of Green's tensor T_1 and Green's vector p_1 is given by

$$e_1 = - \int_{S_1} f \cdot T_1 dS, \quad P \in S_1, \quad \dots (3)$$

where

$$f = (\partial q / \partial n) - p n \quad \dots (4)$$

$$T_1 = (1/8\pi) (I \nabla^2 |x - \xi| - \text{grad grad } |x - \xi|) \quad \dots (5)$$

$$p_1 = - (1/8\pi) \text{grad } \nabla^2 |x - \xi| \quad \dots (6)$$

and

$$I_1 = \delta_{ij} = \text{the Kronecher delta} \quad \dots (7)$$

The corresponding formula for the resistance F_∞ (The subscript ∞ signifies that an infinite mass of fluid is under consideration) on the given body B can now be obtained by noting that the stress tensor has the value

$$I p + \{\nabla q + (\nabla q)'\}$$

where $(\nabla q)'$ stands for the transpose of ∇q . Using (4), we get

$$F_\infty = \int_{S_1} f_\infty dS$$

We now find relation between F_∞ and the so called resistance tensor Φ_∞ , which is defined to be such that the force exerted on a body with uniform velocity u is $\Phi_\infty \cdot u$. Thus, we have

$$F_\infty = F_\infty e = - \Phi \cdot u, \quad \dots (9)$$

where e is the unit vector in the direction of u and $F_\infty = |F_\infty|$.

From our study of fluid dynamics, the solutions of various boundary value problems for steady Stokes flow in an unbounded medium are already known. Hence the solution of the integral equation (3) can be found for these problems. It follows that the tensor T_1 corresponds to the kernel K_0 of Art. 14.2.

In what follows, we propose to show how the correction term may be obtained for more complicated cases with help of technique outlined in Art. 14.2.

14.4.B. BOUNDARY EFFECTS OF STOKES FLOW

As shown in example 2 of Art. 12.7, the presence of the boundary S_2 requires the introduction of a new tensor T and a corresponding vector p . These quantities satisfy the equations

$$\nabla^2 T - \nabla p = I \delta(x - \xi), \quad \nabla \cdot T = 0, \quad T = 0 \quad \text{on } S_2 \quad \dots (10)$$

When S_2 tends to infinity T and p reduce to T_1 and p_1 as given above. According to the present scheme we write $T = T_1 + T_2$ and $p = p_1 + p_2$, where T_2 and p_2 satisfy the homogeneous part of the system (10). It follows that the integral equation which is equivalent to the present problem is given by

$$e_1 = - \int_{S_1} f \cdot T ds = - \int_{S_1} f \cdot (T_1 + T_2) dS \quad \dots (11)$$

Let P , Q and the origin be taken on S_1 . Let ϵ be the parameter that gives the ratio of a , the standard geometric length of the given solid B , to the minimum distance between a point of S_1 and a point of S_2 . Using Taylor's theorem, we have

$$\mathbf{T}_2 = \mathbf{T}_2^0 + \mathbf{r} \cdot (\nabla \mathbf{T}_2)_{x=\xi=0} + \xi \cdot [\nabla^0 \mathbf{T}_2]_{x=\xi=0} + O(\varepsilon^3), \quad \dots (12)$$

where $\mathbf{T}_2^0 = \mathbf{T}_2(0,0)$ and the subscript zero on the ∇ implies differentiation with respect to the components of ξ . Taking only the first-order terms of the relation (12) in (11), we obtain

$$\mathbf{e}_1 + \mathbf{F} \cdot \mathbf{T}_2^0 = - \int_{S_1} \mathbf{f} \cdot \mathbf{T}_1 dS, \quad \dots (13)$$

where

$$\mathbf{F} = \int_{S_1} \mathbf{f} dS$$

Here, $\mathbf{F}$ is the resistance experienced by the given body B in the bounded medium.

Since the integral equation (13) has the same kernel as that of (3), it follows (13) can be regarded as giving the velocity field in an unbounded fluid when the given body B moves with uniform velocity $\mathbf{e}_1 + \mathbf{F} \cdot \mathbf{T}_2^0$. Using the concept of the resistance tensor Φ_∞ , as defined above, we obtain the force formula $\mathbf{F}$ given by

$$\mathbf{F} = -(\mathbf{e}_1 + \mathbf{F} \cdot \mathbf{T}_2^0) \cdot \Phi_\infty \quad \dots (14)$$

Solving (14), we obtain

$$\mathbf{F} = -\mathbf{e}_1 \cdot [\Phi_\infty^{-1} + \mathbf{T}_2^0]^{-1} \quad \dots (15)$$

Since replacement of $\mathbf{F}$ by $\mathbf{F}_\infty$ produces error of order ε^2 and hence, to order ε , (14) reduces to

$$\mathbf{F} = -(\mathbf{e}_1 + \mathbf{F}_\infty \cdot \mathbf{T}_2^0) \cdot \Phi_\infty \quad \dots (16)$$

Let us define the *principal axes of the resistance* of the given body B in such a manner that when B moves parallel to one of them in an infinite mass of fluid, the force is in the direction of motion. They are the unit eigenvectors of the resistance tensor Φ_∞ . We denote them by $\mathbf{i}_1, \mathbf{i}_2$ and $\mathbf{i}_3$ such that

$$\Phi_\infty = \Phi_{\infty 1} \mathbf{i}_1 \mathbf{i}_1 + \Phi_{\infty 2} \mathbf{i}_2 \mathbf{i}_2 + \Phi_{\infty 3} \mathbf{i}_3 \mathbf{i}_3 \quad \dots (17)$$

We now decompose $\mathbf{T}_2^0$ into components with these eigenvectors as the basis. Moreover, we set $\mathbf{e}_1 = \mathbf{i}_1$. Substituting these expressions in (14), we derive

$$\mathbf{F} / \mathbf{F}_\infty = 1 / (1 - \lambda \mathbf{F}_\infty) \quad \dots (18)$$

where λ is independent of the form of S_1 .

14.4C LONGITUDINAL OSCILLATIONS OF SOLIDS IN STOKES FLOW

We now propose to use the present analysis to compute an approximate value of the velocity field generated and the resistance experienced by a solid of any shape which is executing slow longitudinal vibrations in an unbounded viscous fluid. Suppose that the given body B oscillates about some mean position with velocity $U e^{i\omega t} \mathbf{e}_1$ and $\mathbf{q}$ and p have the same time dependence. Then, we know that the dimensionless Stokes equations for the steady state vibrations are given by

$$-\nabla p + \nabla^2 \mathbf{q} - iM^2 \mathbf{q} = 0, \quad \text{div } \mathbf{q} = 0, \quad \dots (19)$$

where $M^2 = a^2 \omega / \nu$ is the *rotational Reynolds number* and ν is the coefficient of kinematic viscosity.

The integral representation formulas are the same as for the steady Stokes flow (refer Art. 14.4 A), while $\mathbf{T}$ and $\mathbf{p}$ now satisfy the equations

$$-\nabla \mathbf{p} + \nabla^2 \mathbf{T} - i M^2 \mathbf{T} = \mathbf{I} \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \text{div } \mathbf{T} = 0, \quad \dots (20)$$

and $\mathbf{T} \rightarrow \mathbf{0}$ as $\mathbf{x} \rightarrow \infty$. Equations (20) are satisfied when $\mathbf{T}$ and $\mathbf{p}$ are given by

$$\mathbf{T} = \mathbf{I} \nabla^2 \phi - \text{grad grad } \phi, \quad \mathbf{p} = -\text{grad}(\nabla^2 - i M^2) \phi \quad \dots (21)$$

$$(\nabla^2 - i M^2) \nabla^2 \phi = \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \dots (22)$$

$$\phi = \frac{1 - \exp[-\{(1+i)M/\sqrt{2}\} |\mathbf{x} - \boldsymbol{\xi}|]}{4\pi i M^2 |\mathbf{x} - \boldsymbol{\xi}|} \quad \dots (23)$$

$$\text{Then,} \quad \mathbf{T} = \mathbf{T}_1 - \{(1+i)/6\pi\sqrt{2}\} M \mathbf{I} + O(M^2), \quad \dots (24)$$

where $\mathbf{T}_1$ is given by (5).

We now substitute the boundary value $\mathbf{q} = \mathbf{e}_1$ in the integral representation formula for the system (19) and note that, in view of (20) and Green's theorem, we finally obtain

$$\int_{S_1} \left(\frac{\partial \mathbf{T}}{\partial n} - p \mathbf{n} \right) dS = -i M^2 \int_{R_1} \mathbf{T} dV, \quad \dots (25)$$

where R_1 is the interior of S_1 . Then, we get the following Fredholm integral equation of the first kind for getting $\mathbf{f}$

$$\mathbf{e}_1 - \{(1+i)/6\pi\sqrt{2}\} M \mathbf{F} = - \int_{S_1} \mathbf{T}_1 \cdot \mathbf{f} dS + O(M^2), \quad P \in S_1 \quad \dots (26)$$

Proceeding as before, we have the formula

$$\mathbf{F} = \Phi \cdot \left[\mathbf{e}_1 - \{(1+i)/6\pi\sqrt{2}\} M (\Phi \cdot \mathbf{e}_1) \right] + O(M^2) \quad \dots (27)$$

When the given body B moves parallel to one of its axes of resistance (which is chosen as the x_1 -axis in our coordinate system), then (27) reduces to

$$\mathbf{F} = -F_\infty \left[1 + \{(1+i)/6\pi\sqrt{2}\} M F_\infty \right] \mathbf{e}_1 \quad \dots (28)$$

In particular, for a sphere, $F_\infty = 6\pi\mu aU$ in physical units, where a is the radius of the sphere and μ is the shear viscosity of the fluid. Then, in physical units, formula (28) yields

$$\mathbf{F} = -6\pi aU \{1 + (M/\sqrt{2})(1+i)\} \mathbf{e}_1 + O(M^2) \quad \dots (29)$$

14.4D STEADY ROTARY STOKES FLOW.

Since the pressure is taken to be constant for the rotation of axially symmetric bodies, the steady Stokes equations take the simple form:

$$\nabla^2 \mathbf{q} = \mathbf{0}, \quad \text{div } \mathbf{q} = 0, \quad p = \text{const.} \quad \dots (30)$$

We choose the z -axis of cylindrical polar coordinates (ρ, ϕ, z) to be the axis of symmetry of the given body. Let the streamlines be circles lying in planes perpendicular to Oz . Hence $\mathbf{q}$ has a nonzero component $v(\rho, z)$ in the ϕ directions only and is independent of ϕ . The equation of continuity is thus satisfied automatically, while the equation of motion (30) reduces to

$$\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial v}{\partial \rho} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{\rho^2} = 0, \quad \dots (31)$$

which has been made dimensionless with Ωa as the typical velocity. Here, Ω is the uniform angular velocity of the body and 'a' is its characteristic length. The boundary conditions are

$$v = \rho \quad \text{on } S_1; \quad v = 0 \quad \text{on } S_2, \quad \dots (32)$$

where S_1 is the surface of the rotating body and S_2 is the bounding surface.

From (31), we can easily verify that the function

$$w(\rho, \phi, z) = v(\rho, z) \cos \phi \quad \dots (33)$$

is harmonic. Hence it can be represented in terms of a *source density* $\sigma(Q) \cos \phi_1$ spread over S_1 , where $\xi = (\rho_1, \phi_1, z_1)$ are the coordinates of Q and $\sigma(Q)$ is independent of ϕ_1 . Therefore, the integral representation formula (32) of Art. 12.4 of chapter 12 can be employed for the harmonic function. Thus, we obtain

$$w(\rho, \phi, z) = \int_{S_1} G(P, Q) \sigma(Q) \cos \phi_1 dS, \quad \dots (34)$$

where P is the arbitrary point in the region between S_1 and S_2 . In view of (32), we have $v = \rho$ on S_1 and hence we obtain

$$\rho = \int_C \rho_1 G^{(1)}(P, Q) \sigma(Q) ds, \quad P \in S_1 \quad \dots (35)$$

where $(1/\pi) G^{(1)}(P, Q)$ is the coefficient of $\cos(\phi - \phi_1)$ in the Fourier expansion of $G(P, Q)$ and ds stands for the element of the arc length measured along the curve C . Here C is the bounding curve of S_1 lying in the meridian plane.

We have already dealt with the decomposition

$$G(P, Q) = (1/|x - \xi|) + G_1(P, Q) \quad \dots (36)$$

where $G_1(P, Q)$ is finite in the limit as $Q \rightarrow P$. Likewise, we can decompose the Fourier component $G^{(1)}$ into the sum

$$G^{(1)} = G_0^{(1)} + G_1^{(1)}, \quad \dots (37)$$

where $G_0^{(1)}$ arises from the Fourier expansion of $1/|x - \xi|$ and $G_1^{(1)}$ arises from the expansion of G_1 . Hence, (35) may be re-written as

$$\rho = \int_C \rho_1 G_0^{(1)} \sigma ds + \int_C \rho_1 G_1^{(1)} \sigma ds \quad \dots (38)$$

Let b denote the minimum distance between a point of S_1 and a point of S_2 . Then we take a small perturbation parameter $\varepsilon = a/b$. Clearly, the second integral on R.H.S. of (38) is at least of order ε of the first integral. For geometric configuration for which

$$G_1^{(1)} = \rho \rho_1 (A + G_2), \quad \dots (39)$$

where A is a constant and G_2 is of order $A \varepsilon$, (38) reduces to

$$\rho = \int_C \rho_1 G_0^{(1)} \sigma ds + A \rho \int_C \rho_1^2 \sigma ds + \rho \int_C \rho_1^2 G_2 \sigma ds, \quad \dots (40)$$

or

$$\rho = \int_C \rho_1 G_0^{(1)} \sigma' ds + \rho \int_C \rho_1^2 G_2 \sigma' ds, \quad \dots (41)$$

where

$$\sigma' = \sigma / \left(1 - A \int_C \rho_1^2 \sigma ds \right) \quad \dots (42)$$

It follows that now we have a situation already dealt in Art. 14.3, that is, σ' represents, with an error which is at most of order ε^2 , an appropriate source density for the body rotating in an infinite mass of fluid.

The tangential stress component τ on the surface S_1 in the direction of ϕ increasing is given by

$$\tau = \mu \rho \frac{\partial}{\partial n} \left(\frac{v}{\rho} \right), \quad \dots (43)$$

where $\partial/\partial n$ denotes differentiation along the normal drawn outward to S_1 . Also, from the analysis of chapter 12, it is known that the source density $\sigma(Q)$ on S_1 is related to v by

$$4\pi\sigma(Q) = -\rho \frac{\partial}{\partial n} \left(\frac{v}{\rho} \right) \quad \dots (44)$$

$$\text{Thus,} \quad \tau = -4\pi\mu\sigma. \quad \dots (45)$$

Using this value of the stress component, the value of the frictional torque N can be obtained.

$$\text{It is given by} \quad N = -8\pi^2\mu \int_C \rho^2 \sigma ds \quad \dots (46)$$

To find the relation between N and the torque N_∞ is an unbounded fluid integrate both sides of the relation (42) around the meridian section C of the axially symmetric body. The relation is

$$\text{given by} \quad N = N_\infty \left\{ 1 + (A/8\pi^2\mu\Omega) N_\infty \right\}^{-1}, \quad \dots (47)$$

with an error of order ε^2 . By a suitable choice of A , the formula (47) is valid in many cases to a much higher order in ε .

Remark. Equation (47) can be illustrated with many useful configurations. For example, the case of a sphere which is symmetrically placed in an infinite cylindrical shell can be discussed as in the analysis of Art. 14.3. It can be shown that (47) yields

$$N = N_\infty \left\{ 1 + (N_\infty / 8\pi\mu\Omega a^3) H_1 \right\}, \quad \dots (48)$$

where H_k is given by the integral

$$H_k = \frac{2}{\pi} \frac{(-1)^k}{(2k)!} \int_0^\infty x^{2k} \frac{K_1(x)}{I_1(x)} dx \quad \dots (49)$$

14.4E ROTARY OSCILLATIONS IN STOKES FLOW

Setting $p = \text{constant}$ in (19), the equations governing the steady-state rotary oscillations (with circular frequency ω) of axially symmetric solids in an incompressible viscous fluid are

$$(\nabla^2 - iM^2)\mathbf{q} = 0, \quad \nabla \cdot \mathbf{q} = 0 \quad \dots (50)$$

Let the z -axis of cylindrical polar coordinates (ρ, ϕ, z) be the axis of symmetry of the given solid bodies. Assuming that the streamlines are circles lying in planes perpendicular to Oz , then $\mathbf{q}$ has a non-zero component $v(\rho, z)$ in the ϕ direction only and is independent of ϕ . Then, the equation of continuity is thus satisfied automatically, and the differential equations (50) reduces to

$$\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial v}{\partial \rho} - \frac{v}{\rho^2} + \frac{\partial^2 v}{\partial z^2} - \beta^2 v = 0, \quad \dots (51)$$

where
$$\beta^2 = i M^2 \quad \dots (52)$$

We now discuss the present analysis for $\beta \ll 1$. The boundary values are given by

$$v = \rho \quad \text{on } S_1, \quad v = 0 \quad \text{on } S_2, \quad \dots (53)$$

where S_1 is the surface of the oscillating body and S_2 is the bounding surface.

Let
$$w = v \cos \phi. \quad \dots (54)$$

Then (51) and (53) reduce to the boundary value problem given by

$$(\nabla^2 - \beta^2)w = 0, \quad \dots (55)$$

$$w = \rho \cos \phi \quad \text{on } S_1 \quad \dots (56a)$$

$$w = 0 \quad \text{on } S_2. \quad (56b)$$

The Green's function $G(\mathbf{x}; \boldsymbol{\xi})$ appropriate to the boundary value problem (55) - (56) is given by

$$(\nabla^2 - \beta^2) G(\mathbf{x}; \boldsymbol{\xi}) = -4\pi \delta(\mathbf{x} - \boldsymbol{\xi}), \quad [G]_{S_2} = 0. \quad \dots (57)$$

Thus,
$$G(\mathbf{x}; \boldsymbol{\xi}) = \frac{\exp(-\beta |\mathbf{x} - \boldsymbol{\xi}|)}{|\mathbf{x} - \boldsymbol{\xi}|} + G_1(\mathbf{x}; \boldsymbol{\xi}), \quad \dots (58)$$

where $G_1(\mathbf{x}; \boldsymbol{\xi})$ is finite in the limit as $\boldsymbol{\xi} \rightarrow \mathbf{x}$.

Following the analysis of Art. 12.6, the integral representation formula for $w(\mathbf{x})$ is given by

$$w(\mathbf{x}) = \int_{S_1} \sigma(\rho_1, z_1) \cos \phi_1 G(\mathbf{x}; \boldsymbol{\xi}) dS, \quad \boldsymbol{\xi} \in S_1, \quad \mathbf{x} \in R, \quad \dots (59)$$

where R is the region between S_1 and S_2 , and $\sigma(\rho_1, z_1)$ is given by the formula (44). Using the boundary condition (56a), the desired Fredholm integral equation is given by

$$\rho \cos \phi = \int_{S_1} \sigma(\rho_1, z_1) \cos \phi_1 G(\mathbf{x}; \boldsymbol{\xi}) dS, \quad \text{with } \mathbf{x} \text{ and } \boldsymbol{\xi} \text{ on } S_1. \quad \dots (60)$$

Let $G_1^{(1)}(\rho, z; \rho_1, z_1)$ be the coefficient of $\cos(\phi - \phi_1)$ in the Fourier expansion of $G_1(\mathbf{x}; \boldsymbol{\xi})$. Then the integration over ϕ_1 , reduces the above integral to

$$\begin{aligned} \rho \cos \phi = \int_{S_1} \sigma(\rho_1, z_1) \cos \phi_1 \frac{\exp(-\beta |\mathbf{x} - \boldsymbol{\xi}|)}{|\mathbf{x} - \boldsymbol{\xi}|} dS \\ + \pi \cos \phi \int_C \sigma(\rho_1, z_1) G_1^{(1)}(\rho, z; \rho_1, z_1) \rho_1 ds \end{aligned} \quad \dots (61)$$

in the notation of equation (35)

We now expand σ as the perturbation series

$$\sigma = \sum_n \beta^n \sigma_n \quad \dots (62)$$

in equation (61). By direct expansion of the Green's function, we can prove that $G_1^{(1)} = O(\varepsilon^3)$, where ε is the ratio of the characteristic length of the vibrating body to the distance of its centre from the nearest point of S_2 . Let $q = \beta / \varepsilon = O(1)$.

Equating like powers on both sides of (61) and rejecting terms that trivially vanish, we obtain

$$\rho \cos \phi = \int_{S_1} \frac{\sigma_0(\rho_1, z_1)}{|\mathbf{x} - \boldsymbol{\xi}|} \cos \phi_1 dS. \quad \dots (63a)$$

$$0 = \int_{S_1} \frac{\sigma_1(\rho_1, z_1)}{|\mathbf{x} - \boldsymbol{\xi}|} \cos \phi_1 dS \quad \dots (63b)$$

$$0 = \int_{S_1} \frac{\sigma_2(\rho_1, z_1)}{|\mathbf{x} - \boldsymbol{\xi}|} \cos \phi_1 dS + \frac{1}{2} \int_{S_1} \sigma_0(\rho_1, z_1) |\mathbf{x} - \boldsymbol{\xi}| \cos \phi_1 dS \quad \dots (63c)$$

$$0 = \int_{S_1} \frac{\sigma_3(\rho_1, z_1)}{|\mathbf{x} - \boldsymbol{\xi}|} \cos \phi dS + \frac{1}{2} \int \sigma_1(\rho_1, z_1) |\mathbf{x} - \boldsymbol{\xi}| \cos \phi dS \\ - \frac{1}{6} \int_{S_1} \sigma_0(\rho_1, z_1) |\mathbf{x} - \boldsymbol{\xi}|^2 \cos \phi dS + \pi \cos \phi \int_C \sigma_0(\rho_1, z_1) H(\rho, z; \rho_1, z_1) \rho_1 ds \quad \dots (63d)$$

and so on, where

$$G_1^{(1)}(\rho, z; \rho_1, z_1) = \beta^2 H(\rho, z; \rho_1, z_1) + O(\beta^4) \quad \dots (64)$$

Relations (63a) – (63d) show that the source densities $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ etc can be obtained by solving potential problems in free space of the form already dealt in chapter 12.

The velocity field and the frictional torque can also be easily obtained. In fact, using the formula (43), the torque N is given by

$$N = \mu \int_S \rho^2 \frac{\partial}{\partial n} \left(\frac{v}{\rho} \right) dS = 2\pi\mu \int_C \rho^3 \frac{\partial}{\partial n} \left(\frac{v}{\rho} \right) ds \quad \dots (65)$$

Using the relation (44), (62) and (65), we get

$$N = -8\pi^2\mu \int_C \rho^2 (\sigma_0 + \beta\sigma_1 + \beta^2\sigma_2 + \beta^3\sigma_3) ds + O(\beta^4) \quad \dots (66)$$

Remark. Since the potential problems of the type given in equations (63a) – (63d) can be solved for various configurations such as a sphere, a spheroid, a lens, and a thin circular disc, we can solve our problems for all these geometric shapes. We now illustrate the above analysis by the following example :

Example. Suppose a thin circular disc is vibrating about its axis in a viscous fluid which is contained in an infinite circular cylinder. Assume that the axes of the disc and the cylinder coincide.

Solution. By following the steps of solved example 2, page 12.16 of Art. 12.5, the Green's function for an infinite cylinder $-\infty < z < \infty, 0 \leq \rho \leq b$ is given by

$$G(\mathbf{x}; \boldsymbol{\xi}) = \frac{\exp \{-\beta |\mathbf{x} - \boldsymbol{\xi}|\}}{|\mathbf{x} - \boldsymbol{\xi}|} - \frac{2}{\pi} \sum_{n=0}^{\infty} (2 - \delta_{0n}) \cos n(\phi - \phi_1) \\ \times \int_{\beta}^{\infty} \frac{K_n(ub)}{I_n(ub)} I_n(up) I_n(u\rho_1) \cos \{(u^2 - \beta^2)(z - z_1)\} \frac{u du}{(u^2 - \beta^2)^{1/2}} \quad \dots (67)$$

Using (67), $G_1^{(1)}$ for the disc $\rho \leq 1, 0 \leq \phi \leq 2\pi, z = 0$ is given by

$$G_1^{(1)}(\rho, \rho_1) = -\frac{\varepsilon^3 \rho \rho_1}{2\pi} \int_q^{\infty} \frac{K_1(y) y^3 dy}{I_1(y) (y^2 - q^2)^{1/2}} + O(\varepsilon^5) \quad \dots (68)$$

or
$$H(\rho, \rho_1) = -(1/2\pi q^3) \rho \rho_1 A(q), \quad \dots (69)$$

where $A(q)$ denotes the infinite integral in (68).

Using the methods of Art. 12.4 and 12.5, the integral equations (63a) to (63b) can be solved to yield

and
$$\left. \begin{aligned} \sigma_0 &= \frac{2\rho}{\pi^2 (1-\rho^2)^{1/2}}, & \sigma_1 &= 0, & \sigma_2 &= \frac{\rho(2-\rho^2)}{3\pi^2 (1-\rho^2)^{1/2}} \\ \sigma_3 &= \frac{4}{3\pi^2} \left\{ -\frac{2}{3} + \frac{A(q)}{\pi q^3} \right\} \frac{\rho}{(1-\rho^2)^{1/2}}, & 0 \leq \rho \leq 1 \end{aligned} \right\} \quad \dots (70)$$

Substituting the above values of $\sigma_0, \sigma_1, \sigma_2$ and σ_3 in (66), the required value of N (in physical units) is given by

$$N = -\frac{32}{3} \mu \Omega a^3 \left\{ 1 + \frac{\beta^2}{5} - \frac{4\beta^3}{9\pi} + \frac{4\varepsilon^3 A(q)}{3\pi^2} \right\} e^{i\omega t} + O(\beta^4, \varepsilon^5) \quad \dots (71)$$

14.4F OSCEEN FLOW-TRANSLATIONAL MOTION

The slow motion past a solid is governed by the dimensionless Oseen equations (refer solved example 3 of Art. 12.7)

$$\mathbb{R}(\partial \mathbf{T} / \partial x_1) = -\nabla p + \nabla^2 \mathbf{q}, \quad \text{div } \mathbf{q} = 0 \quad \dots (72)$$

$$\mathbf{q} = \mathbf{e}_1 \quad \text{on } S_1; \quad \mathbf{q} = \mathbf{0} \quad \text{on } S_2, \quad \dots (73)$$

where S_1 is the surface of the solid body and S_2 is the bounding surface. Then, using the analysis of solved 3, page 12.21 of Art. 12.7, the Fredholm integral equation of the first kind which is equivalent to the boundary value problem (72) - (73) is given

$$\mathbf{e}_1 = -\int_S \mathbf{T} \cdot \mathbf{f} dS, \quad \dots (74)$$

where the Green's tensor $\mathbf{T}$ and the Green's vector $\mathbf{p}$ are now defined as

$$\mathbf{T} = (1/8\pi) \{ \mathbf{I} \nabla^2 \phi - \text{grad grad } \phi \} \quad \dots (75)$$

$$\mathbf{p} = -(1/8\pi) \text{grad} (\nabla^2 \phi - \mathbb{R}(\partial \phi / \partial x_1)) \quad \dots (76)$$

$$\phi = \frac{1}{|\sigma|} \int_0^{|\sigma|s} \frac{1-e^{-t}}{t} dt \quad \dots (77)$$

$$s = |\mathbf{x} - \boldsymbol{\xi}| + (\mathbb{R} / |\mathbb{R}|)(x_1 - \xi_1) \quad \dots (78)$$

$$\mathbf{f} = (\partial \mathbf{q} / \partial n) - p \mathbf{n} \quad \dots (79)$$

Now,
$$\frac{1-e^{-t}}{t} = \frac{1}{t} \left\{ 1 - \left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right) \right\} = 1 - \frac{t}{2!} + \frac{t^2}{3!} \dots \quad \dots (80)$$

Using (80) and expanding ϕ in (77) in terms of the Reynolds number, the relation (75) reduces to

$$\mathbf{T} = \mathbf{T}_1 + O(\mathbb{R}), \quad \dots (81)$$

where T_1 is given by

$$T_1 = (1/8\pi) (I \nabla^2 | \mathbf{x} - \boldsymbol{\xi} | - \text{grad grad} | \mathbf{x} - \boldsymbol{\xi} |) \quad \dots (82)$$

The rest of the analysis is similar to the one given in the previous article.

14.4G OSEEN FLOW-ROTARY MOTION

With the help of the present technique, we now proceed to find the solutions of the Oseen equations for the steady rotations of axially symmetric solids.

For the rotation of axially symmetric bodies, the pressure is taken to be constant and the Oseen equations reduce to the following simple form :

$$\mathbb{R} (\partial \mathbf{q} / \partial x_1) - \nabla^2 \mathbf{q} = \mathbf{0}, \quad \text{div } \mathbf{q} = 0 \quad \dots (83)$$

Let the z -axis of cylindrical polar coordinates (ρ, ϕ, z) with $(z = x_1)$ be the axis of symmetry of the given bodies. Assuming that the streamlines are circles lying in planes perpendicular to Oz , then $\mathbf{q}$ has a nonzero component $V(\rho, z)$ in the ϕ direction only and is independent of ϕ . The equation of continuity is thus satisfied automatically, while the boundary value problem becomes

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} - \frac{V}{\rho^2} + \frac{\partial^2 V}{\partial z^2} - 2c \frac{\partial V}{\partial z} = 0 \quad \dots (84)$$

where $c = Ua / 2\nu = \mathbb{R} / 2$. Equation (84) has been made dimensionless with U as the typical velocity and 'a' as the characteristic length of the body.

The boundary conditions on V are given by

$$V = \rho \quad \text{on } S_1; \quad V = 0 \quad \text{on } S_2, \quad \dots (85)$$

where S_1 is the surface of the rotating solid and S_2 is the bounding surface.

$$\text{Let} \quad V = e^{cz} v(\rho, z) \quad \dots (86)$$

Then, the substitution (86) transforms the boundary value problem (84) - (85) in the form

$$\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial v}{\partial \rho} - \frac{v}{\rho^2} + \frac{\partial^2 v}{\partial z^2} - c^2 v = 0 \quad \dots (87)$$

$$v = \rho e^{cz} \quad \text{on } S_1; \quad v = 0 \quad \text{on } S_2 \quad \dots (88)$$

[Note that (87) is the same as (51) of Art 14.4 E with β replaced by c . whereas the boundary conditions (88) are different from those of (53) of Art. 14.4 E]

By repeating the algebraic steps (54) upto (61) of Art. 14.4 E, we finally obtain the Fredholm integral equation

$$\begin{aligned} \rho e^{cz} \cos \phi = & \int_{S_1} \frac{\sigma(\rho_1, z_1) \cos \phi_1 \exp(-c | \mathbf{x} - \boldsymbol{\xi} |)}{| \mathbf{x} - \boldsymbol{\xi} |} dS \\ & + \pi \cos \phi \int_C \sigma(\rho_1, z_1) G_1^{(1)}(\rho, z; \rho_1, z_1) \rho_1 ds \end{aligned} \quad \dots (89)$$

from which $\sigma(\rho, z)$ can be obtained, where $\sigma(\rho, z)$ is defined by

$$4 \pi \sigma(Q) = -\rho \frac{\partial}{\partial n} \left(\frac{v}{\rho} \right) \quad \dots (90)$$

In order to find solution of (89), we take the expansion

$$\sigma = \sum_n c^n \sigma_n \quad \dots (91)$$

and also expand $\rho e^{cz} \cos \phi$ in power series of c . On comparing the equal powers of c in (89), we finally get the following integral equations of potential theory

$$\rho \cos \phi = \int_{S_1} \frac{\sigma_0(\rho_1, z_1)}{|\mathbf{x} - \boldsymbol{\xi}|} \cos \phi_1 dS \quad \dots (92a)$$

$$\rho z \cos \phi = \int_{S_1} \frac{\sigma_1(\rho_1, z_1)}{|\mathbf{x} - \boldsymbol{\xi}|} \cos \phi_1 dS \quad \dots (92b)$$

$$\frac{\rho z^2 \cos \phi}{2} = \int_{S_1} \frac{\sigma_2(\rho_1, z_1)}{|\mathbf{x} - \boldsymbol{\xi}|} \cos \phi_1 dS + \frac{1}{2} \int_{S_1} \sigma_0(\rho_1, z_1) |\mathbf{x} - \boldsymbol{\xi}| \cos \phi_1 dS \quad \dots (92c)$$

$$\begin{aligned} \frac{\rho z^3 \cos \phi}{6} = & \int_{S_1} \frac{\sigma_3(\rho_1, z_1)}{|\mathbf{x} - \boldsymbol{\xi}|} \cos \phi_1 dS + \frac{1}{2} \int_{S_1} \sigma_1(\rho_1, z_1) |\mathbf{x} - \boldsymbol{\xi}| \cos \phi_1 dS \\ & - \frac{1}{6} \int_{S_1} \sigma_0(\rho_1, z_1) |\mathbf{x} - \boldsymbol{\xi}|^2 \cos \phi_1 dS + \pi \cos \phi \int_C \sigma_0(\rho_1, z_1) H(\rho, z; \rho_1, z_1) \rho_1 ds \quad \dots (92d) \end{aligned}$$

where $H(\rho, z; \rho_1, z_1)$ is defined by the relation.

$$G_1^{(1)}(\rho, z; \rho_1, z_1) = c^2 H(\rho, z; \rho_1, z_1) + O(c^4) \quad \dots (93)$$

Consider the particular case of a thin circular disk $z = 0, \rho \leq 1$. Then the system of equations (92a) – (92d) is the same as the system (63a) – (63d) of Art. 14.4 E. It follows that the solution for steady rotation problem for the disk in Oseen flow is the same as the corresponding solution for the steady state vibrations in Stokes flow (refer Art 14.4 E). For example, the value of the torque N for the problem under consideration can be deduced from the formula (71) of Art 14.4 E. It is gives by

$$N = -\frac{32\mu\Omega a^3}{3} \left\{ 1 + \frac{c^2}{5} - \frac{4c^3}{9\pi} + \frac{4\epsilon^3 A(q)}{3\pi^2} \right\} + O(c^4, \epsilon^5), \quad \dots (94)$$

where Ω is the uniform angular velocity of the solid.

For solids of other geometric shapes, we have to solve the integral equations (92a) – (92d) with non-zero left side. We illustrate this fact by the following example.

Example. Suppose a sphere of radius a be rotating. For spherical geometry, we shall use the spherical polar coordinates. For this problem, the value of the Green's function $G(\mathbf{x}; \boldsymbol{\xi})$ is the same as (67) of Art. 14.4 E with β replaced by c . The corresponding values of $G_1^{(1)}$ and $H(\theta, \theta_1)$ are given by

$$G_1^{(1)}(\theta, \theta_1) = -\frac{\epsilon^2 \sin \theta \sin \theta_1}{2\pi} \int_q^\infty \frac{K_1(y) y^3 dy}{I_1(y) (y^2 - q^2)^{1/2}} + O(\epsilon^5) \quad \dots (95)$$

and

$$H(\theta, \theta) = -\frac{\sin \theta \sin \theta_1 A(q)}{2\pi q^3} \quad \dots (96)$$

Using the method of chapter 12, the source densities $\sigma_0, \sigma_1, \sigma_2$, and σ_3 can be obtained from (92a)-(92d). The final results are

$$\sigma_0 = (3/4\pi)P_1^1(\cos\theta), \quad \sigma_1 = (5/12\pi)P_2^1(\cos\theta) \quad \dots (97)$$

$$\sigma_2 = (1/4\pi)\{(3/2)P_1^1(\cos\theta) + (7/15)P_3^1(\cos\theta)\} \quad \dots (98)$$

$$\sigma_3 = -(3/4\pi)P_1^1(\cos\theta)\{(1/3) + (1/2\pi)q^3 A(q)\}, \text{ where } 0 < \theta < \pi. \quad \dots (99)$$

For the present problem torque N is given by

$$N = -8\pi^2\mu \int_C \rho^3 (\sigma_0 + c\sigma_1 + c^2\sigma_2 + c^3\sigma_3) ds + O(c^4) \quad \dots (100)$$

It is same as (66) of Art. 14.4 E with β replaced by c

Substituting the values of $\sigma_0, \sigma_1, \sigma_2$ and σ_3 as given by (97) – (98), we obtain (in physical units)

$$N = -8\pi\mu\Omega a^3 \left\{ 1 + \frac{4c^2}{15} - \frac{c^3}{3} + \frac{\varepsilon^3 A(q)}{2\pi} \right\} + O(c^4, \varepsilon^5) \quad \dots (101)$$

EXERCISE

1. Find the torque experienced by a sphere which is rotating uniformly in Oseen flow and is bounded by a pair of parallel walls $z = \pm c$. Evaluate also the velocity field.
2. Extend the analysis of the steady Oseen flow in this chapter to the case of the steady-state vibrations of axially symmetric solids in Oseen flow.

APPENDIX A

BOUNDARY VALUE PROBLEMS AND GREEN'S IDENTITIES

A.1 Some useful notations

In what follows, we shall use the following notations:

(i) If a function $f \in C^{(n)}$, then it will imply that all derivatives of order n of f are continuous. Again, if $f \in C^{(0)}$, then it will imply that f is a continuous function.

(ii) Let V denote a bounded closed region in three dimensional space. Again, let the set of all boundary points of V be denoted by S . Then, the set of all interior points of V together with the set of boundary points S is denoted by $\bar{V}$. Thus, $\bar{V} = V \cup S$.

A.2 Boundary value problems for Laplace equation

If the given problem satisfies Laplace equation in a bounded region V in three dimensions and also satisfies the prescribed boundary conditions on the boundary S of the region V , then such a problem is known as a *boundary value problem* (which is also written as BVP).

Classifications of boundary value problems for Laplace equation

We discuss three main types of boundary value problems for Laplace equation.

(i) Boundary value problem of first kind or the Dirichlet problem (Meerut 2011)

If f is a continuous function prescribed on the boundary S of some finite region V , then the problem of finding a function $u(x, y, z)$ such that $\nabla^2 u = 0$ within V and $u = f$ on S is known as the *interior Dirichlet problem*.

Similarly, if f is a continuous function prescribed on the boundary S of a finite simply connected region V , then the problem of finding a function $u(x, y, z)$ which satisfies $\nabla^2 u = 0$ outside V and is such that $u = f$ on S is known as the *exterior Dirichlet problem*.

(ii) Boundary value problem of second kind or the Neumann problem (Meerut 2011)

If f is a continuous function which is defined uniquely at each point of the boundary S of a finite region V , then the problem of finding a function $u(x, y, z)$ such that $\nabla^2 u = 0$ within V and its normal derivative $\partial u / \partial n = f$ at every point of S is known as the *interior Neumann problem*.

Similarly, if f is a continuous function prescribed at each point of the (smooth) boundary S of a bounded simply connected region V , then the problem of finding a function $u(x, y, z)$ satisfying $\nabla^2 u = 0$ outside V and $\partial u / \partial n = f$ at every point of S is known as the *exterior Neumann problem*.

(iii) Boundary value problem of third kind or the Churchill problem

If f is continuous function prescribed on the boundary S of a finite region V , then the problem of finding a function $u(x, y, z)$ such that $\nabla^2 u = 0$ within V and $\partial u / \partial n + hu = f$ at every point of S (Here, h is constant and $h \geq 0$) is known as the *interior Churchill problem*.

Similarly, if f is a continuous function prescribed at each point of S of a finite region V , then the problem of finding a function $u(x, y, z)$ such that $\nabla^2 u = 0$ outside V and $\partial u / \partial n + hu = f$ at every point of S is known as the *exterior churchill problem*.

A.3 Green's identities

(i) Green's first identity

Let S be a closed surface in the (x, y, z) space and let V be the closed region bounded by S in which $\mathbf{F}$ is a vector belonging to $C^{(1)}$ in V and continuous on V . Then, by Gauss divergence theorem, we have

$$\iiint_V \nabla \cdot \mathbf{F} dV = \iint_S \mathbf{F} \cdot \mathbf{n} dS,$$

where dV is an element of volume, dS is an element of surface area and $\mathbf{n}$ is the outward drawn normal to the surface S .

Let $F = \mathbf{f}u$, where $\mathbf{f}$ is a vector function and u is a scalar function of position. Then, (1) reduces to

$$\iiint_V \nabla \cdot (\mathbf{f}u) dV = \iint_S u \mathbf{n} \cdot \mathbf{f} dS \quad \dots(2)$$

By a vector identity, we have $\nabla \cdot (\mathbf{f}u) = \mathbf{f} \cdot \nabla u + u \nabla \cdot \mathbf{f}$... (3)

From (2) and (3), $\iiint_V (\mathbf{f} \cdot \nabla u + u \nabla \cdot \mathbf{f}) dV = \iint_S u \mathbf{n} \cdot \mathbf{f} dS$

or $\iiint_V \mathbf{f} \cdot \nabla u dV = \iint_S u \mathbf{n} \cdot \mathbf{f} dS - \iiint_V u \nabla \cdot \mathbf{f} dV$... (4)

Replacing $\mathbf{f}$ by ∇v in (4), we have

$$\iiint_V \nabla v \cdot \nabla u dV = \iint_S u \mathbf{n} \cdot \nabla v dS - \iiint_V u \nabla \cdot \nabla v dV$$

or $\iiint_V \nabla v \cdot \nabla u dV = \iint_S u \mathbf{n} \cdot \nabla v dS - \iiint_V u \nabla^2 v dV$... (5)

Since $\mathbf{n} \cdot \nabla v$ is the derivative of v in the direction of $\mathbf{n}$, it is convenient to introduce the notation $\mathbf{n} \cdot \nabla v = \partial v / \partial n$ in (5). Then, (5) reduces to

$$\iiint_V \nabla v \cdot \nabla u dV = \iint_S u \frac{\partial v}{\partial n} dS - \iiint_V u \nabla^2 v dV, \quad \dots (6)$$

which is known as *Green's first identity*.

A particular case of Green's first identity. Setting $v = u$ in (6), we have

$$\iiint_V (\nabla u)^2 dV = \iint_S u \frac{\partial u}{\partial n} dS - \iiint_V u \nabla^2 u dV \quad \dots(7)$$

(ii) **Green's second identity.** Interchanging u and v in (6), we get

$$\iiint_V \nabla u \cdot \nabla v dV = \iint_S v \frac{\partial u}{\partial n} dS - \iiint_V v \nabla^2 u dV$$

or $\iiint_V \nabla v \cdot \nabla u dV = \iint_S v \frac{\partial u}{\partial n} dS - \iiint_V v \nabla^2 u dV,$... (8)

where we have used the fact that $\nabla v \cdot \nabla u = \nabla u \cdot \nabla v$. Subtracting (8) from (6), we get

$$0 = \iint_S \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS - \iiint_V (u \nabla^2 v - v \nabla^2 u) dV$$

and hence $\iiint_V (u \nabla^2 v - v \nabla^2 u) dV = \iint_S \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS,$... (9)

which is known as *second Green's identity*. Clearly, for (9) to be true, both u and v must possess continuous second order derivatives.

APPENDIX B

TWO AND THREE DIMENSIONAL DIRAC DELTA FUNCTIONS

B.1. Introduction

We have already introduced the concept of one-dimensional Dirac delta function in chapter 10. In what follows, we propose to extend the definition to two and three dimensions.

B.2. Two dimensional Dirac delta function. Definition

Let $f(x, y)$ be an arbitrary continuous function over the region S containing the point (ξ, η) . Then, two-dimensional Dirac delta function $\delta(x - \xi, y - \eta)$ is defined as the function satisfying the following condition

$$\iint_S \delta(x - \xi, y - \eta) f(x, y) dx dy = f(\xi, \eta) \quad \dots(1)$$

Observe that $\delta(x - \xi, y - \eta)$ is a formal limit of a sequence of ordinary functions, that is,

$$\delta(x - \xi, y - \eta) = \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon(r) \quad \dots(2)$$

where

$$r^2 = (x - \xi)^2 + (y - \eta)^2.$$

Again, we have

$$\iint_S \delta(x - \xi) \delta(y - \eta) f(x, y) dx dy = f(\xi, \eta) \quad \dots(3)$$

$$\text{Comparing (1) and (2),} \quad \delta(x - \xi, y - \eta) = \delta(x - \xi) \delta(y - \eta), \quad \dots(4)$$

showing that a two-dimensional Dirac delta function can be expressed as the product of two one-dimensional Dirac delta functions.

B.3. Three-dimension Dirac delta function. Definition

Let $f(x, y, z)$ be an arbitrary continuous function in the region V containing the point (ξ, η, ζ) . Then, three dimensional Dirac delta function $\delta(x - \xi, y - \eta, z - \zeta)$ is defined as the function satisfying the following condition

$$\iiint_V \delta(x - \xi, y - \eta, z - \zeta) dx dy dz = f(\xi, \eta, \zeta) \quad \dots(1)$$

Observe that $\delta(x - \xi, y - \eta, z - \zeta)$ is a formal limit of a sequence of ordinary functions, that is,

$$\delta(x - \xi, y - \eta, z - \zeta) = \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon(r) \quad \dots(2)$$

where

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2$$

Again, we have

$$\iiint_V \delta(x - \xi, y - \eta, z - \zeta) f(x, y, z) dx dy dz = f(\xi, \eta, \zeta) \quad \dots(3)$$

$$\text{Comparing (1) and (2),} \quad \delta(x - \xi, y - \eta, z - \zeta) = \delta(x - \xi) \delta(y - \eta) \delta(z - \zeta),$$

showing that a three-dimensional Dirac Delta function can be expressed as the product of three one-dimensional Dirac delta functions.

Example. Show that the three dimensional Dirac delta function $\delta(x, y, z; \xi, \eta, \zeta)$ can be written as

$$\delta(x, y, z; \xi, \eta, \zeta) = \delta(x - \xi) \delta(y - \eta) \delta(z - \zeta).$$

Solution. Suppose that the three dimensional Dirac delta function $\delta(x, y, z; \xi, \eta, \zeta)$ is defined by

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi, \eta, \zeta) \delta(x, y, z; \xi, \eta, \zeta) d\xi d\eta d\zeta = F(x, y, z), \quad \dots(1)$$

where $F(x, y, z)$ is continuous function. If $\delta(x - \xi)$, $\delta(y - \eta)$ and $\delta(z - \zeta)$ are three one-dimensional Dirac delta functions, then we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi, \eta, \zeta) \delta(x - \xi) \delta(y - \eta) \delta(z - \zeta) d\xi d\eta d\zeta = F(x, y, z) \quad \dots(2)$$

Comparing (1) and (2), $\delta(x, y, z; \xi, \eta, \zeta) = \delta(x - \xi) \delta(y - \eta) \delta(z - \zeta)$

B.4. Dirac delta function in general curvilinear coordinates in two dimensions

Consider a transformation from cartesian coordinates x, y to curvilinear coordinates ξ, η satisfying the relations

$$x = f(\xi, \eta) \quad \text{and} \quad y = g(\xi, \eta), \quad \dots (1)$$

where f and g are single-valued, continuously differentiable functions of their arguments. Under the transformation (1), let $\xi = b_1, \eta = b_2$ correspond to $x = a_1, y = a_2$. Again, we have

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \partial f / \partial \xi & \partial f / \partial \eta \\ \partial g / \partial \xi & \partial g / \partial \eta \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \frac{\partial(f, g)}{\partial(\xi, \eta)} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = J \begin{bmatrix} d\xi \\ d\eta \end{bmatrix},$$

where J is the Jacobian of transformation. When we transform the coordinates satisfying (1), then the equation

$$\iint F(x, y) \delta(x - a_1) \delta(y - a_2) dx dy = F(a_1, a_2) \quad \dots (2)$$

takes the form
$$\iint F(f, g) \delta[f(\xi, \eta) - a_1] \delta[g(\xi, \eta) - a_2] |J| d\xi d\eta = F(a_1, a_2) \quad \dots (3)$$

From (3), it follows that the Dirac delta function $\delta[f(\xi, \eta) - a_1] \delta[g(\xi, \eta) - a_2] |J|$ assigns to any test function $F(f, g)$ the value of that test function at the points where $f = a_1, g = a_2$, i.e., at the points $\xi = b_1, \eta = b_2$. Hence, we obtain

$$\delta[f(\xi, \eta) - a_1] \delta[g(\xi, \eta) - a_2] |J| = \delta(x - a_1) \delta(y - a_2) |J| = \delta(\xi - b_1) \delta(\eta - b_2)$$

Thus,
$$\delta(x - a_1) \delta(y - a_2) = [\delta(\xi - b_1) \delta(\eta - b_2)] / |J| \quad \dots (4)$$

An important particular case : If $P(x_0, y_0)$ is a point in rectangular cartesian coordinates corresponding to the point $P(r_0, \theta_0)$ in polar coordinates, then show that

$$\delta(x - x_0) \delta(y - y_0) = [\delta(r - r_0) \delta(\theta - \theta_0)] / r$$

Solution. In polar coordinates, we have $x = r \cos \theta$ and $y = r \sin \theta$. Thus, here take $\xi = r$ and $\eta = \theta$. Also, here $a_1 = x_0, a_2 = y_0, b_1 = r_0$ and $b_2 = \theta_0$. Furthermore, here we have

$$x = f(r, \theta) = r \cos \theta \quad \text{and} \quad y = g(r, \theta) = r \sin \theta \quad \dots (1)$$

$$\text{Then, } J = \begin{vmatrix} \partial f / \partial r & \partial f / \partial \theta \\ \partial g / \partial r & \partial g / \partial \theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r, \text{ using (1)}$$

$$\text{From Art B.4, we have } \delta(x - a_1) \delta(y - a_2) = \{\delta(\xi - b_1) \delta(\eta - b_2)\} / |J| \quad \dots (2)$$

Substituting the values of $a_1 (= x_0), a_2 (= y_0), b_1 (= r_0), b_2 (= \theta_0)$ $J = r$ and noting that here $\xi = r$ and $\eta = \theta$, (2) reduces to

$$\delta(x - x_0) \delta(y - y_0) = \{\delta(r - r_0) \delta(\theta - \theta_0)\} / r$$

B.5 Dirac delta function in general curvilinear coordinates in three dimensions

Consider a transformation from cartesian coordinates x, y, z to curvilinear coordinates ξ, η, ζ satisfying the relations

$$x = f(\xi, \eta, \zeta), \quad y = g(\xi, \eta, \zeta) \quad \text{and} \quad z = h(\xi, \eta, \zeta), \quad \dots (1)$$

where f, g and h are single-valued, continuously differentiate functions of their arguments. Under the transformation (1), let $\xi = b_1, \eta = b_2, \zeta = b_3$ correspond to $x = a_1, y = a_2, z = a_3$. Again, we have

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \partial f / \partial \xi & \partial f / \partial \eta & \partial f / \partial \zeta \\ \partial g / \partial \xi & \partial g / \partial \eta & \partial g / \partial \zeta \\ \partial h / \partial \xi & \partial h / \partial \eta & \partial h / \partial \zeta \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \frac{\partial(f, g, h)}{\partial(\xi, \eta, \zeta)} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = J \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix},$$

where J is the Jacobian of transformation. When we transform the coordinates satisfying (1), then the equation

$$\iiint F(x, y, z) \delta(x - a_1) \delta(y - a_2) \delta(z - a_3) dx dy dz = F(a_1, a_2, a_3) \dots (2)$$

becomes

$$\iiint F(f, g, h) \delta[f(\xi, \eta, \zeta) - a_1] \delta[g(\xi, \eta, \zeta) - a_2] \delta[h(\xi, \eta, \zeta) - a_3] |J| d\xi d\eta d\zeta = F(a_1, a_2, a_3) \dots (3)$$

Now, as discussed in Art B.4, we have

$$\delta[f(\xi, \eta, \zeta) - a_1] \delta[g(\xi, \eta, \zeta) - a_2] \delta[h(\xi, \eta, \zeta) - a_3] |J| = \delta(x - a_1) \delta(y - a_2) \delta(z - a_3) |J| \\ = \delta(\xi - b_1) \delta(\eta - b_2) \delta(\zeta - b_3)$$

$$\text{Thus, } \delta(x - a_1) \delta(y - a_2) \delta(z - a_3) = [\delta(\xi - b_1) \delta(\eta - b_2) \delta(\zeta - b_3)] / |J| \dots (4)$$

Some important particular cases.

Case I. If $P(x_0, y_0, z_0)$ is a point in rectangular cartesian coordinates corresponding to the point $P(r_0, \theta_0, z_0)$ in cylindrical coordinates, then show that

$$\delta(x - x_0) \delta(y - y_0) \delta(z - z_0) = \{\delta(r - r_0) \delta(\theta - \theta_0) \delta(z - z_0)\} / r$$

Solution. In cylindrical polar coordinates, we have $x = r \cos \theta$, $y = r \sin \theta$ and $z = z$

Thus, here take $\xi = r$, $\eta = \theta$ and $\zeta = z$. Also here $a_1 = x_0$, $a_2 = y_0$, $a_3 = z_0$, $b_1 = r_0$, $b_2 = \theta_0$ and $b_3 = z_0$. Further, here we have

$$x = f(r, \theta, z) = r \cos \theta, \quad y = g(r, \theta, z) = r \sin \theta \quad \text{and} \quad z = h(r, \theta, z) = z$$

$$\text{Then, } J = \begin{vmatrix} \partial f / \partial r & \partial f / \partial \theta & \partial f / \partial z \\ \partial g / \partial r & \partial g / \partial \theta & \partial g / \partial z \\ \partial h / \partial r & \partial h / \partial \theta & \partial h / \partial z \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r, \text{ by (1)}$$

From Art. B.5, we have

$$\delta(x - a_1) \delta(y - a_2) \delta(z - a_3) = \{\delta(\xi - b_1) \delta(\eta - b_2) \delta(\zeta - b_3)\} / |J| \dots (2)$$

Substituting the values of $a_1 (= x_0)$, $a_2 (= y_0)$, $a_3 (= z_0)$, $b_1 (= r_0)$, $b_2 (= \theta_0)$, $b_3 (= z_0)$, $J = r$ and noting that here $\xi = r$, $\eta = \theta$ and $\zeta = z$, (2) reduce to

$$\delta(x - x_0) \delta(y - y_0) \delta(z - z_0) = \{\delta(r - r_0) \delta(\theta - \theta_0) \delta(z - z_0)\} / r$$

Case II. If $P(x_0, y_0, z_0)$ is a point in rectangular cartesian coordinates corresponding to the point $P(r_0, \theta_0, \phi_0)$ in spherical polar coordinates, then show that

$$\delta(x - x_0) \delta(y - y_0) \delta(z - z_0) = \{\delta(r - r_0) \delta(\theta - \theta_0) \delta(\phi - \phi_0)\} / r^2 \sin \theta$$

Solution. In spherical polar coordinates, $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$

Thus, here take $\xi = r$, $\eta = \theta$ and $\zeta = \phi$. Also, here $a_1 = x_0$, $a_2 = y_0$, $a_3 = z_0$, $b_1 = r_0$, $b_2 = \theta_0$ and $b_3 = \phi_0$. Further, have we have

$$x = f(r, \theta, \phi) = r \sin \theta \cos \phi, \quad y = g(r, \theta, \phi) = r \sin \theta \sin \phi, \quad z = h(r, \theta, \phi) = r \cos \theta \dots (1)$$

$$\text{Then, } J = \begin{vmatrix} \partial f / \partial r & \partial f / \partial \theta & \partial f / \partial \phi \\ \partial g / \partial r & \partial g / \partial \theta & \partial g / \partial \phi \\ \partial h / \partial r & \partial h / \partial \theta & \partial h / \partial \phi \end{vmatrix} = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}, \text{ by (1)}$$

$$= \cos \theta (r^2 \sin \theta \cos \theta \cos^2 \phi + r^2 \sin \theta \cos \theta \sin^2 \phi) + r \sin \theta (r \sin^2 \theta \cos^2 \phi + r \sin^2 \theta \sin^2 \phi)$$

$$= \cos \theta \times r^2 \sin \theta \cos \theta + r^2 \sin^3 \theta = r^2 \sin \theta (\cos^2 \theta + \sin^2 \theta) = r^2 \sin \theta$$

$$\text{From Art. B.5, } \delta(x - a_1) \delta(y - a_2) \delta(z - a_3) = \{\delta(\xi - b_1) \delta(\eta - b_2) \delta(\zeta - b_3)\} / |J| \dots (2)$$

Substituting the values of $a_1 (= x_0)$, $a_2 (= y_0)$, $a_3 (= z_0)$, $b_1 (= r_0)$, $b_2 (= \theta_0)$, $b_3 (= \phi_0)$, $J = r$ and noting that here $\xi = r$, $\eta = \theta$, $\zeta = \phi$, (2) reduce to

$$\delta(x - x_0) \delta(y - y_0) \delta(z - z_0) = \{\delta(r - r_0) \delta(\theta - \theta_0) \delta(\phi - \phi_0)\} / r^2 \sin \theta$$

APPENDIX C

ADDITIONAL TOPICS AND PROBLEMS BASED ON GREEN'S FUNCTION

[**Note:** The reader is advised to study chapter 12 before reading the matter of this Appendix C. However, references to this chapter have been given at proper places]

C.1 The eigenfunction method for computing Green's function for the given Dirichlet boundary value problem

Let D be a region in the xy -plane bounded by a simple closed curve C . Consider the Dirichlet boundary value problem described by

$$\nabla^2 u = \partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = f, \quad \forall x, y \in D \quad \dots(1)$$

$$\text{subject to the boundary condition :} \quad u = g, \quad \forall x, y \in C \quad \dots(2)$$

Now, from the definition of Green's function (Refer Art. 12.8 of chapter 12), the Green's function $G(x, y; \xi, \eta)$ for the problem given by (1) and (2) must satisfy the following relations :

$$\nabla^2 G = \delta(x - \xi, y - \eta), \quad \forall x, y \in D \quad \dots(3)$$

$$\text{and} \quad G = 0, \quad \forall x, y \in C, \quad \dots(4)$$

where $\delta(x - \xi, y - \eta)$ is two-dimensional Dirac delta function (refer Art. B. 2 in Appendix B)

Let us consider the eigen value problem associated with the operator ∇^2 in the domain D , that is,

$$\nabla^2 \phi + \lambda \phi = 0, \quad \forall x, y \in D \quad \dots(5)$$

$$\text{and} \quad \phi = 0, \quad \forall x, y \in C \quad \dots(6)$$

Let λ_{mn} be the *eigenvalues and $\phi_{mn}(x, y)$ be the corresponding eigenfunctions of the eigenvalue problem given by (1) and (2).

Assume that $G(x, y; \xi, \eta)$ and $\delta(x - \xi, y - \eta)$ possess the following Fourier series expansions in terms of the eigenfunctions $\phi_{mn}(x, y)$:

$$G(x, y; \xi, \eta) = \sum_m \sum_n A_{mn}(\xi, \eta) \phi_{mn}(x, y) \quad \dots(7)$$

$$\text{and} \quad \delta(x - \xi, y - \eta) = \sum_m \sum_n B_{mn}(\xi, \eta) \phi_{mn}(x, y) \quad \dots(8)$$

$$\text{where} \quad B_{mn} = \frac{1}{\|\phi_{mn}(x, y)\|^2} \iint_D \delta(x - \xi, y - \eta) \phi_{mn}(x, y) dx dy = \frac{\phi_{mn}(\xi, \eta)}{\|\phi_{mn}(x, y)\|} \quad \dots(9)$$

$$\text{and} \quad \|\phi_{mn}(x, y)\|^2 = \iint_D \phi_{mn}^2(x, y) dx dy \quad \dots(10)$$

Since $\phi_{mn}(x, y)$ is an eigenfunction corresponding to the eigenvalue λ_{mn} , hence by definitions of eigenfunction and eigenvalue, (5) yields

$$\nabla^2 \phi_{mn}(x, y) + \lambda_{mn} \phi_{mn}(x, y) = 0 \quad \dots(11)$$

Substituting the values of $G(x, y; \xi, \eta)$ and $\delta(x - \xi, y - \eta)$ given by (7) and (8) in (3) and (4), we have

$$\nabla^2 \sum_m \sum_n A_{mn}(\xi, \eta) \phi_{mn}(x, y) = \sum_m \sum_n B_{mn}(\xi, \eta) \phi_{mn}(x, y)$$

* Refer chapter 15 of Part I in author's "Advanced Differential equations" published by S. Chand and Co., New Delhi

or $\sum_m \sum_n A_{mn}(\xi, \eta) \nabla^2 \phi_{mn}(x, y) = \sum_m \sum_n \{\phi_{mn}(\xi, \eta) \phi_{mn}(x, y)\} / \|\phi_{mn}(x, y)\|^2$, using (9)

or $-\sum_m \sum_n \lambda_{mn} A_{mn}(\xi, \eta) \phi_{mn}(x, y) = \sum_m \sum_n \{\phi_{mn}(\xi, \eta) \phi_{mn}(x, y)\} / \|\phi_{mn}(x, y)\|^2$, using (11)

The above result yields $A_{mn}(\xi, \eta) = -\{\phi_{mn}(\xi, \eta) / \lambda_{mn} \|\phi_{mn}(x, y)\|^2\}$... (12)

Substituting the above values of $A_{mn}(\xi, \eta)$ given by (2) in (7), the required Green's function of the given Dirichlet problem is given by

$$G(x, y; \xi, \eta) = -\{\sum_m \sum_n \phi_{mn}(\xi, \eta) \phi_{mn}(x, y)\} / \lambda_{mn} \|\phi_{mn}(x, y)\|^2 \quad \dots (13)$$

An illustrative example. Find the Green's function for the Dirichlet problem on the rectangle D : $0 \leq x \leq a$, $0 \leq y \leq b$ described by the partial differential equation $(\nabla^2 + \lambda)u = 0$ in D and $u = 0$ on the boundary C of D .

Solution. Given boundary value problem is given by

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 + \lambda u = 0, \quad \forall x, y \in D \quad \dots (1)$$

subject to the boundary conditions:

$$u(x, 0) = u(x, a) = 0, \quad \forall 0 \leq x \leq a \quad \dots (2a)$$

$$\text{and} \quad u(0, y) = u(b, y) = 0, \quad \forall 0 \leq y \leq b \quad \dots (2b)$$

Now, *we proceed to find the eigenfunctions and the corresponding eigenvalues of the boundary value problem given by (1), (2a) and (2b).

Suppose that (1) has solutions of the form

$$u(x, y) = X(x) Y(y), \quad \dots (3)$$

where X is a function x alone and Y that of Y alone

Substituting this value of u in (1) we have

$$X''Y + XY'' + \lambda XY = 0 \quad \text{or} \quad X''/X = -(\lambda + Y''/Y) \quad \dots (4)$$

Since x and y are independent variables, (4) is true if each side of (4) is a constant ($= -\mu$, say) such that

$$X''/X = -\mu, \quad \text{that is,} \quad X'' + \mu X = 0 \quad \dots (5)$$

$$\text{and} \quad -(\lambda + Y''/Y) = -\mu, \quad \text{that is,} \quad Y'' + (\lambda - \mu) Y = 0 \quad \dots (6)$$

$$\text{Using (2 a), (3) yields} \quad X(0) Y(y) = 0 \quad \text{and} \quad X(a) Y(y) = 0 \quad \dots (7)$$

Since $Y(y) = 0$ leads to trivial solution $u = 0$, so we take $Y(y) \neq 0$. Then, (7) yields

$$X(0) = 0 \quad \text{and} \quad X(a) = 0 \quad \dots (8)$$

We now proceed to find the eigenfunctions and the corresponding eigenvalues of the boundary value problem given by (5) and (8). Three cases arise :

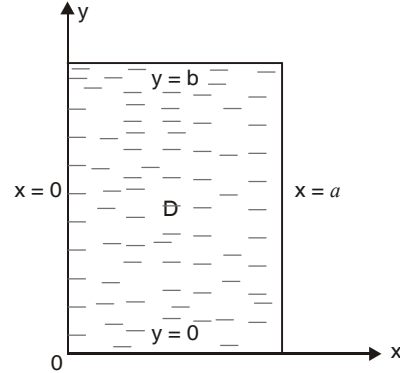
$$\textbf{Case I.} \text{ Let } \mu = 0. \text{ Then, solution of (5) is} \quad X(x) = Ax + B \quad \dots (9)$$

Using the boundary conditions (8), (10) yields $0 = B$ and $0 = Aa + B$ giving $A = B = 0$. Hence $X(x) = 0$ and so $u = 0$ which is a trivial solution. So we reject $\mu = 0$.

Case II. Let $\mu = -v^2$, where $v \neq 0$. Then, solution of (5) is

$$X(x) = A e^{vx} + B e^{-vx}, \quad A \text{ and } B \text{ being arbitrary constants} \quad \dots (10)$$

Using the boundary conditions (8), (10) yields $0 = A + B$ and $0 = A e^{av} + B e^{-av}$. Solving these equations, we get $A = B = 0$ so that $X(0) = 0$ and hence $u = 0$, which is a trivial solution. Hence, we reject $\mu = -v^2$.



* Refer Chapter 15 of part I in author's "Advanced Differential equations" published by S. Chand and Co, New Delhi

Case III. Let $\mu = v^2$, where $v \neq 0$. Then, solution of (5) is

$$X(x) = A \cos vx + B \sin vx, A \text{ and } B \text{ being arbitrary constants} \quad \dots(11)$$

Using the boundary conditions (8), (11) yields $0 = A$ and $0 = A \cos va + B \sin va$

Hence, we have $A = 0$ and $\sin va = 0$,

where we have taken $B \neq 0$, since $A = B = 0$ yields $u = 0$ which is a trivial solution, Now,

$$\sin va = 0 \Rightarrow va = m\pi, m = 1, 2, 3, \dots \Rightarrow v_m = m\pi/a, m = 1, 2, 3, \dots \quad \dots(12)$$

Corresponding to eigenvalues v_m given by (12), The desired real valued eigenfunctions are given by

$$X_m = \sin(m\pi x/a), m = 1, 2, 3, \dots$$

Next, using (2b), (3) yields $X(x) Y(0) = 0$ and $X(x) Y(b) = 0 \quad \dots(13)$

Since $X(x) = 0$ leads to trivial solution $u = 0$, so we take $X(x) \neq 0$. Then, (13) yields

$$Y(0) = 0 \quad \text{and} \quad Y(b) = 0 \quad \dots(14)$$

As before, now proceed to find the eigenfunctions and the corresponding eigenvalues of the boundary value problem given by (6) and (14). Then, we get the eigenfunctions $Y_n(y) = \sin(n\pi y/b)$, $n = 1, 2, 3, \dots$ corresponding to eigenvalues given by $\lambda - v_m^2 = (n^2\pi^2/b^2)$, $n = 1, 2, 3, \dots$

Let $\phi_{mn}(x, y)$ denote the eigenfunctions corresponding to the eigenvalues λ_{mn} for the given boundary value problem given by (1), (2a) and (2b). Then, we have

$$\phi_{mn}(x, y) = X_m(x) Y_n(y) = \sin(m\pi x/a) \sin(n\pi y/b), m = 1, 2, 3, \dots, n = 1, 2, 3, \dots \quad \dots(15)$$

$$\lambda_{mn} = v_m^2 + n^2\pi^2/b^2 = m^2\pi^2/a^2 + n^2\pi^2/b^2 = \pi^2(m^2/a^2 + n^2/b^2), m = 1, 2, 3, \dots, n = 1, 2, 3, \dots(16)$$

Here, the norm of $\phi_{mn}(x, y)$, i.e., $\|\phi_{mn}(x, y)\|$, is given by

$$\begin{aligned} \|\phi_{mn}(x, y)\|^2 &= \int_{x=0}^a \int_{y=0}^b \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} dx dy = \left(\int_0^a \sin^2 \frac{m\pi x}{a} dx \right) \times \left(\int_0^b \sin^2 \frac{n\pi y}{b} dy \right) \\ &= \frac{1}{2} \int_0^a \left(1 - \cos \frac{2m\pi x}{a} \right) dx \times \frac{1}{2} \int_0^b \left(1 - \cos \frac{2n\pi y}{b} \right) dy = \frac{1}{4} \left[x - \frac{\sin(2m\pi x/a)}{(2m\pi/a)} \right]_0^a \times \left[y - \frac{\sin(2n\pi y/b)}{(2n\pi/b)} \right]_0^b \\ \text{Thus,} \quad \|\phi_{mn}(x, y)\|^2 &= (a/2) \times (b/2) = ab/4 \quad \dots(17) \end{aligned}$$

Now, from Art. C.1, the Green's function $G(x, y; \xi, \eta)$ is given by

$$G(x, y; \xi, \eta) = - \left\{ \sum_m \sum_n \phi_{mn}(\xi, \eta) \phi_{mn}(x, y) \right\} / \lambda_{mn} \|\phi_{mn}(x, y)\|^2 \quad \dots(18)$$

Substituting the values of $\phi_{mn}(\xi, \eta)$, $\phi_{mn}(x, y)$, λ_{mn} and $\|\phi_{mn}(x, y)\|$ with help of (15), (16) and (17) in (18), the required Green's function $G(x, y; \xi, \eta)$ for the given Dirichlet problem is given by

$$G(x, y; \xi, \eta) = - \frac{4ab}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(m\pi x/a) \sin(n\pi y/b) \sin(m\pi \xi/a) \sin(n\pi \eta/b)}{(m^2/a^2 + n^2/b^2)}$$

C.2 The space form of the wave equation or Helmholtz equation

Consider the wave equation in three dimensions

$$\partial^2 u / \partial t^2 = c^2 \nabla^2 u, \quad \text{or} \quad \partial^2 u / \partial t^2 = c^2 (\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 + \partial^2 u / \partial z^2) \quad \dots(1)$$

Suppose that (1) has solutions of the form

$$u(x, y, z, t) = \phi(x, y, z) T(t), \quad \dots(2)$$

where $\phi(x, y, z)$ is function of x, y, z only whereas $T(t)$ is function of t alone. Substituting the above value of u in (1), we have

$$\phi(d^2 T / dt^2) = c^2 T (\partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial y^2 + \partial^2 \phi / \partial z^2) \quad \text{or} \quad \phi T'' = c^2 T \nabla^2 \phi$$

Then,
$$T'' / c^2 T = \nabla^2 \phi / \phi, \quad \text{where} \quad T'' = d^2 T / dt^2 \quad \dots(3)$$

Now, the L.H.S. of (3) is a function of t alone whereas the R.H.S. of (3) is function of x, y and z . Hence (3) is true only if its each side is a constant ($= -\lambda$, say). Then, (3) yields

$$\nabla^2 \phi / \phi = -\lambda \quad \text{giving} \quad \nabla^2 \phi + \lambda \phi = 0, \quad \dots(4)$$

which is known as the *space form of the wave equation* or *Helmholtz equation*.

C.3 Helmholtz's Theorem

Let $\psi(\mathbf{r})$ be a solution of the space form of the wave equation $\nabla^2 \psi + k^2 \psi = 0$ whose partial derivatives of the first and second orders are continuous within the volume V and on the closed surface S bounding V . Again, let $\mathbf{r}$ be the position vector of any point P outside the region V and let $\mathbf{r}'$ be the position vector of a point P' lying in the region V . Then,

$$\frac{1}{4\pi} \iint_S \left(\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \psi(\mathbf{r}')}{\partial n} - \psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) dS = \begin{cases} \psi(\mathbf{r}), & \text{if } \mathbf{r} \in V \\ 0, & \text{if } \mathbf{r} \notin V \end{cases}$$

where $\mathbf{n}$ is the outward normal to S .

Proof Let u be a solution of the Helmholtz equation

$$\nabla^2 u + k^2 u = 0 \quad \dots(1)$$

in the region V and let all the singularities of u lie outside the closed region V bounded

by S as shown in the adjoining figure (i). Consider the singularity solution u' of (1) given by

$$u' = \{e^{ik|\mathbf{r}-\mathbf{r}'|}\} / |\mathbf{r}-\mathbf{r}'| \quad \dots(2)$$

From second Green's identity (refer Art. A. 3 in Appendix A), we have

$$\iint_V (u \nabla^2 u' - u' \nabla^2 u) dV = \iint_S \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) dS \quad \dots(3)$$

Since $P(\mathbf{r})$ lies outside the region V , $|\mathbf{r}-\mathbf{r}'| \neq 0$, and hence from (2), it follows that we can find $\nabla^2 u'$ for all $\mathbf{r}' \in V$.

Setting $u = \psi$ and $u' = \psi'$ in (3), we have

$$\iiint_V (\psi \nabla^2 \psi' - \psi' \nabla^2 \psi) dV = \iint_S \left(\psi \frac{\partial \psi'}{\partial n} - \psi' \frac{\partial \psi}{\partial n} \right) dS \quad \dots(4)$$

Putting $u = \psi$ in (1) gives $\nabla^2 \psi = -k^2 \psi$. Again, from (2), $\psi' = u' = \{e^{ik|\mathbf{r}-\mathbf{r}'|}\} / |\mathbf{r}-\mathbf{r}'|$, Substituting these values in (4), we obtain

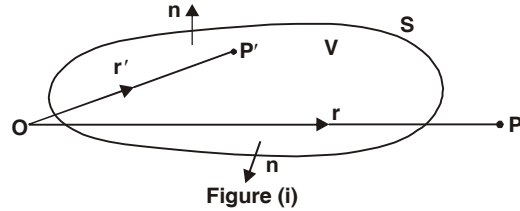
$$\iiint_V \left\{ \psi \nabla^2 \left(\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} (-k^2 \psi) \right\} dV = \iint_S \left\{ \psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \psi(\mathbf{r}')}{\partial n} \right\} dS \quad \dots(5)$$

$$\text{Now,} \quad \nabla^2 \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = (ik)^2 \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = -k^2 \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \quad \dots(6)$$

Using (6), we see that L.H.S. of (5) vanishes. Hence, (5) reduces to

$$\iint_S \left\{ \psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \psi(\mathbf{r}')}{\partial n} \right\} dS = 0, \quad \dots(7)$$

which is true when the point $P(\mathbf{r})$ lies outside the surface S as shown in figure (i).



We now proceed with the other situation when $P(\mathbf{r})$ lies inside the surface S as shown in adjoining figure (ii). We draw a small sphere with centre at $P(\mathbf{r})$ and radius ε . Let R denote the region which is exterior to C and interior to S . Proceeding as before, on applying second Green's identity to the region R and noting that $|\mathbf{r} - \mathbf{r}'| \neq 0$, we obtain

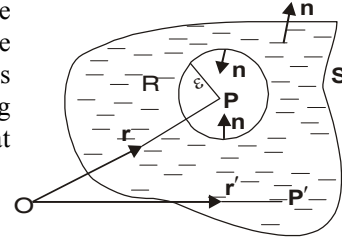


Figure (ii)

$$\iint_R \left\{ \psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \psi(\mathbf{r}')}{\partial n} \right\} dS = 0$$

or $\iint_C \left\{ \psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \psi(\mathbf{r}')}{\partial n} \right\} dS + \iint_S \left\{ \psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \psi(\mathbf{r}')}{\partial n} \right\} dS = 0$... (18)

Note that when the point $P'(\mathbf{r}')$ lies on the surface of the sphere C , then $|\mathbf{r} - \mathbf{r}'| = \varepsilon$. The direction of normals $\mathbf{n}$ is as shown in figure (ii).

Now, on the surface of the sphere C , we have

$$\left. \begin{aligned} |\mathbf{r} - \mathbf{r}'| &= \varepsilon, & \psi(\mathbf{r}') &= \psi(\mathbf{r}) + O(\varepsilon), & dS &= \varepsilon^2 \sin \theta d\theta d\phi \\ \frac{\partial \psi(\mathbf{r}')}{\partial n} &= \left[\frac{\partial \psi(\mathbf{r})}{\partial n} \right]_P + O(\varepsilon), & \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} &= \frac{e^{ik\varepsilon}}{\varepsilon}, & \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} &= -\frac{\partial}{\partial \varepsilon} \left(\frac{e^{ik\varepsilon}}{\varepsilon} \right) = \left(\frac{1}{\varepsilon^2} - \frac{e^{ik}}{\varepsilon} \right) e^{ik\varepsilon} \end{aligned} \right\} \dots (9)$$

Using (9), the first on the L.H.S. of (8)

$$\begin{aligned} &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left\{ \psi(\mathbf{r}) + O(\varepsilon) \right\} \left(\frac{1}{\varepsilon^2} - \frac{ik}{\varepsilon} \right) e^{ik\varepsilon} - \frac{e^{ik\varepsilon}}{\varepsilon} \left\{ \left(\frac{\partial \psi(\mathbf{r})}{\partial n} \right)_P + O(\varepsilon) \right\} \right\} \varepsilon^2 \sin \theta d\theta d\phi \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left\{ \psi(\mathbf{r}) + O(\varepsilon) \right\} (1 - ik\varepsilon) e^{ik\varepsilon} - \varepsilon e^{ik\varepsilon} \left\{ \left(\frac{\partial \psi(\mathbf{r})}{\partial n} \right)_P + O(\varepsilon) \right\} \right\} \sin \theta d\theta d\phi \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \psi(\mathbf{r}) \sin \theta d\theta d\phi, \text{ on letting } \varepsilon \rightarrow 0 \\ &= \psi(\mathbf{r}) \times [\cos \theta]_0^{\pi} \times [\phi]_0^{2\pi} = 4\pi \psi(\mathbf{r}) \end{aligned}$$

Substituting the above value of the L.H.S. of (8) in complete equation (8), we obtain

$$4\pi \psi(\mathbf{r}) + \iint_S \left\{ \psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \psi(\mathbf{r}')}{\partial n} \right\} dS = 0$$

$$\Rightarrow \frac{1}{4\pi} \iint_S \left\{ \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \psi(\mathbf{r}')}{\partial n} - \psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right\} dS = \psi(\mathbf{r}) \dots (10)$$

Now, on combining the results (7) and (10), we have

$$\frac{1}{4\pi} \iint_S \left\{ \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \psi(\mathbf{r}')}{\partial n} - \psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right\} dS = \begin{cases} \psi(\mathbf{r}), & \text{if } \mathbf{r} \in V \\ 0, & \text{if } \mathbf{r} \notin V \end{cases} \dots (11)$$

C.4 Application of Green's function in determining the solution of the wave equation.

Consider the space form of the wave equation, namely,

$$\nabla^2 \psi + k^2 \psi = 0 \dots (1)$$

In what follows, we propose to show how the solution of (1) under certain boundary conditions can be made to depend on the determination of an appropriate Green's function $G(\mathbf{r}, \mathbf{r}')$.

Let $G(\mathbf{r}, \mathbf{r}')$ be a Green's function such that

$$(i) \quad G(\mathbf{r}, \mathbf{r}') \text{ satisfies (1), i.e.,} \quad \nabla^2 G(\mathbf{r}, \mathbf{r}') + k^2 G(\mathbf{r}, \mathbf{r}') = 0 \quad \dots (2)$$

(ii) $G(\mathbf{r}, \mathbf{r}')$ is finite and continuous with respect to either the variables x, y, z or to the variables x', y', z' for the points $\mathbf{r}, \mathbf{r}'$ belonging to a region V which is bounded by a closed surface S except in the neighbourhood of the point $\mathbf{r}$, where it has a singularity of the same type as $\{e^{ik|\mathbf{r}-\mathbf{r}'|}\}/|\mathbf{r}-\mathbf{r}'|$ as $\mathbf{r}' \rightarrow \mathbf{r}$.

Now, proceeding as in the derivation of equation (10) of Art. C.3, we can prove that, if $\mathbf{r}$ is the position vector of a point within V , then

$$\psi(\mathbf{r}) = \frac{1}{4\pi} \iint_S \left\{ G(\mathbf{r}, \mathbf{r}') \frac{\partial \psi(\mathbf{r}')}{\partial n} - \psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \right\} dS, \quad \dots (3)$$

where $\mathbf{n}$ is the outward-drawn normal to the surface S .

We now proceed to discuss two important particular cases of the above result (3).

Particular Case 1: When $G(\mathbf{r}, \mathbf{r}') = G_1(\mathbf{r}, \mathbf{r}')$, where $G_1(\mathbf{r}, \mathbf{r}')$ satisfies the boundary condition $G_1(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r}' \in S$, then (3) takes the form

$$\psi(\mathbf{r}) = -\frac{1}{4\pi} \iint_S \psi(\mathbf{r}') \frac{\partial G_1(\mathbf{r}, \mathbf{r}')}{\partial n} dS \quad \dots (4)$$

Using formula (4), the value of ψ at any point $\mathbf{r}$ within S can be calculated in terms of the values of ψ on the boundary S .

Particular case 2 : When $G(\mathbf{r}, \mathbf{r}') = G_2(\mathbf{r}, \mathbf{r}')$, where $G_2(\mathbf{r}, \mathbf{r}')$ satisfies the boundary condition $\partial G_2(\mathbf{r}, \mathbf{r}') / \partial n = 0$ for $\mathbf{r}' \in S$, then (3) takes the form

$$\psi(\mathbf{r}) = \frac{1}{4\pi} \iint_S \frac{\partial \psi(\mathbf{r}')}{\partial n} G_2(\mathbf{r}, \mathbf{r}') dS \quad \dots (5)$$

Using formula (5), the value of ψ at any point $\mathbf{r}$ within S can be calculated in term of the value of $\partial \psi / \partial n$ on the boundary S .

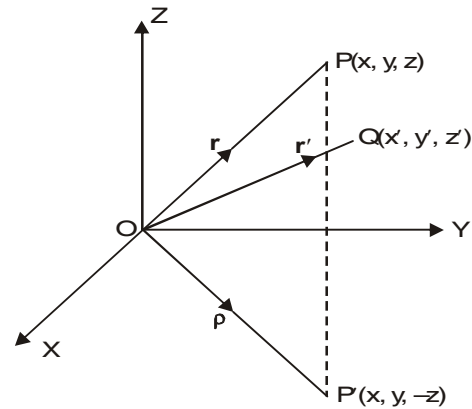
C.5 Determination of the Green's function for the Helmholtz equation for the half-space $z \geq 0$ (Meerut 2006)

In the given problem, the boundary is the xy -plane. Let $\mathbf{p}$ be the position vector of the image $P'(x, y, -z)$ in the xy -plane of the point $P(x, y, z)$ with position vector $\mathbf{r}$. Again, let $\mathbf{r}'$ be the position vector of any point $Q(x', y', z')$ of the half-space $z \geq 0$. When Q lies on the boundary, then clearly, $QP = QP'$ so that $|\mathbf{r} - \mathbf{r}'| = |\mathbf{p} - \mathbf{r}'|$. Thus, for the present problem, we take $G_1(\mathbf{r}, \mathbf{r}') = \{e^{ik|\mathbf{r}-\mathbf{r}'|}\}/|\mathbf{r}-\mathbf{r}'|$

$$- \{e^{ik|\mathbf{p}-\mathbf{r}'|}\}/|\mathbf{p}-\mathbf{r}'| \quad \dots (1)$$

We now proceed to verify that $G_1(\mathbf{r}, \mathbf{r}')$ satisfies the Helmholtz equation $\nabla^2 \psi + k^2 \psi = 0$. Thus, we wish to prove that

$$(\partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2) G_1(\mathbf{r}, \mathbf{r}') + k^2 G_1(\mathbf{r}, \mathbf{r}') = 0 \quad \dots (2)$$



Let $R_1 = |\mathbf{r} - \mathbf{r}'| = \{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{1/2}$... (3)

$R_2 = |\boldsymbol{\rho} - \mathbf{r}'| = \{(x-x')^2 + (y-y')^2 + (z+z')^2\}^{1/2}$... (4)

Using (3) and (4), (1) reduce to $G_1(\mathbf{r}, \mathbf{r}') = (1/R_1) \times e^{ikR_1} - (1/R_2) \times e^{ikR_2}$... (5)

From (5), $\partial R_1 / \partial x' = -(x-x')/R_1$ and $\partial R_2 / \partial x' = -(x-x')/R_2$... (6)

Differentiating (5) partially w.r.to x' , we have

$$\begin{aligned} \frac{\partial G_1(\mathbf{r}, \mathbf{r}')}{\partial x'} &= \frac{ik}{R_1} e^{ikR_1} \frac{\partial R_1}{\partial x'} + \left(-\frac{1}{R_1^2}\right) e^{ikR_1} \frac{\partial R_1}{\partial x'} - \left[\frac{ik}{R_2} e^{ikR_2} \frac{\partial R_2}{\partial x'} + \left(-\frac{1}{R_2^2}\right) e^{ikR_2} \frac{\partial R_2}{\partial x'}\right] \\ &= -\left(\frac{ik}{R_1} - \frac{1}{R_1^2}\right) e^{ikR_1} \frac{x-x'}{R_1} + \left(\frac{ik}{R_2} - \frac{1}{R_2^2}\right) e^{ikR_2} \frac{x-x'}{R_2}, \text{ using (6)} \end{aligned}$$

Thus, $\frac{\partial G_1(\mathbf{r}, \mathbf{r}')}{\partial x'} = -\left(\frac{ik}{R_1^2} - \frac{1}{R_1^3}\right) e^{ikR_1} (x-x') + \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3}\right) e^{ikR_2} (x-x')$... (7)

Now, differentiating (7) partially w.r.to x' , we have

$$\begin{aligned} \frac{\partial^2 G_1(\mathbf{r}, \mathbf{r}')}{\partial x'^2} &= -ik \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3}\right) e^{ikR_1} \frac{\partial R_1}{\partial x'} (x-x') - \left(-\frac{2ik}{R_1^3} + \frac{3}{R_1^4}\right) \frac{\partial R_1}{\partial x'} e^{ikR_1} (x-x') \\ &\quad + \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3}\right) e^{ikR_1} + ik \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3}\right) e^{ikR_2} \frac{\partial R_2}{\partial x'} (x-x') \\ &\quad + \left(-\frac{2ik}{R_2^3} + \frac{3}{R_2^4}\right) \frac{\partial R_2}{\partial x'} e^{ikR_2} (x-x') - \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3}\right) e^{ikR_2} \\ \text{or } \frac{\partial^2 G_1(\mathbf{r}, \mathbf{r}')}{\partial x'^2} &= \left\{ ik \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3}\right) \frac{(x-x')^2}{R_1} + \left(-\frac{2ik}{R_1^3} + \frac{3}{R_1^4}\right) \frac{(x-x')^2}{R_1} + \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3}\right) \right\} e^{ikR_1} \\ &\quad + \left\{ -ik \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3}\right) \frac{(x-x')^2}{R_2} - \left(-\frac{2ik}{R_2^3} + \frac{3}{R_2^4}\right) \frac{(x-x')^2}{R_2} - \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3}\right) \right\} e^{ikR_2} \end{aligned} \quad \dots (8)$$

$$\begin{aligned} \text{Similarly, } \frac{\partial^2 G_2(\mathbf{r}, \mathbf{r}')}{\partial y'^2} &= \left\{ ik \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3}\right) \frac{(y-y')^2}{R_1} + \left(-\frac{2ik}{R_1^3} + \frac{3}{R_1^4}\right) \frac{(y-y')^2}{R_1} + \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3}\right) \right\} e^{ikR_1} \\ &\quad + \left\{ -ik \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3}\right) \frac{(y-y')^2}{R_2} - \left(-\frac{2ik}{R_2^3} + \frac{3}{R_2^4}\right) \frac{(y-y')^2}{R_2} - \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3}\right) \right\} e^{ikR_2} \end{aligned} \quad \dots (9)$$

Now, from (3) and (4), $\partial R_1 / \partial z' = -(z-z')/R_1$ and $\partial R_2 / \partial z' = (z+z')/R_2$... (10)

Differentiating (5) partially w.r.to z' , we have

$$\begin{aligned} \frac{\partial G_1(\mathbf{r}, \mathbf{r}')}{\partial z'} &= \frac{ik}{R_1} e^{ikR_1} \frac{\partial R_1}{\partial z'} + \left(-\frac{1}{R_1^2}\right) e^{ikR_1} \frac{\partial R_1}{\partial z'} - \frac{ik}{R_2} e^{ikR_2} \frac{\partial R_2}{\partial z'} + \frac{1}{R_2^2} e^{ikR_2} \frac{\partial R_2}{\partial z'} \\ \text{or } \frac{\partial G_1(\mathbf{r}, \mathbf{r}')}{\partial z'} &= -\left(\frac{ik}{R_1^2} - \frac{1}{R_1^3}\right) e^{ikR_1} (z-z') - \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3}\right) e^{ikR_2} (z+z'), \text{ using (10)} \end{aligned} \quad \dots (11)$$

Now, differentiating (11) partially w.r.t z' and simplifying, we get

$$\frac{\partial^2 G_1(\mathbf{r}, \mathbf{r}')}{\partial z'^2} = \left\{ ik \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3} \right) \frac{(z-z')^2}{R_1} + \left(-\frac{2ik}{R_1^3} + \frac{3}{R_1^4} \right) \frac{(z-z')^2}{R_1} + \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3} \right) \right\} e^{ikR_1} \\ - \left\{ ik \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3} \right) \frac{(z+z')^2}{R_2} + \left(-\frac{2ik}{R_2^3} + \frac{3}{R_2^4} \right) \frac{(z+z')^2}{R_2} + \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3} \right) \right\} e^{ikR_2} \quad \dots (12)$$

Now, $(\partial^2 / \partial x'^2 + \partial / \partial y'^2 + \partial / \partial z'^2) G_1(\mathbf{r}, \mathbf{r}') = \partial^2 G_1(\mathbf{r}, \mathbf{r}') / \partial x'^2 + \partial^2 G_1(\mathbf{r}, \mathbf{r}') / \partial y'^2 + \partial^2 G_1(\mathbf{r}, \mathbf{r}') / \partial z'^2$

$$= \left\{ ik \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3} \right) \frac{(x-x')^2 + (y-y')^2 + (z-z')^2}{R_1} \right. \\ \left. + \left(-\frac{2ik}{R_1^3} + \frac{3}{R_1^4} \right) \frac{(x-x')^2 + (y-y')^2 + (z-z')^2}{R_1} + 3 \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3} \right) \right\} e^{ikR_1} \\ - \left\{ ik \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3} \right) \frac{(x-x')^2 + (y-y')^2 + (z+z')^2}{R_2} \right. \\ \left. + \left(-\frac{2ik}{R_2^3} + \frac{3}{R_2^4} \right) \frac{(x-x')^2 + (y-y')^2 + (z+z')^2}{R_2} + 3 \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3} \right) \right\} e^{ikR_2} \\ \text{[Using (8), (9) and (12)]}$$

$$= \left\{ ik \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3} \right) \frac{R_1^2}{R_1} + \left(-\frac{2ik}{R_1^3} + \frac{3}{R_1^4} \right) \frac{R_1^2}{R_1} + 3 \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3} \right) \right\} e^{ikR_1} \\ - \left\{ ik \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3} \right) \frac{R_2^2}{R_2} + \left(-\frac{2ik}{R_2^3} + \frac{3}{R_2^4} \right) \frac{R_2^2}{R_2} + 3 \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3} \right) \right\} e^{ikR_2}, \text{ using (3) and (4)} \\ = -k^2 \{ (1/R_1) \times e^{ikR_1} - (1/R_2) \times e^{ikR_2} \} = -k^2 G_1(\mathbf{r}, \mathbf{r}'), \text{ using (5)}$$

Thus, $(\partial^2 / \partial x'^2 + \partial^2 / \partial y'^2 + \partial^2 / \partial z'^2) G_1(\mathbf{r}, \mathbf{r}') + k^2 G_1(\mathbf{r}, \mathbf{r}') = 0$, i.e., $\nabla^2 G_1(\mathbf{r}, \mathbf{r}') + k^2 G_1(\mathbf{r}, \mathbf{r}') = 0$, showing that $G_1(\mathbf{r}, \mathbf{r}')$ satisfies the Helmholtz equation.

Again, as already discussed, when $Q(\mathbf{r}')$ lies on the boundary S of the half-space $z \geq 0$, we have $|\mathbf{r} - \mathbf{r}'| = |\boldsymbol{\rho} - \mathbf{r}'|$ and hence, we have

$$G_1(\mathbf{r}, \mathbf{r}') = 0 \quad \text{for} \quad \mathbf{r}' \in S$$

Then, from particular case 1 of Art. C.4, we have

$$\psi(\mathbf{r}) = -\frac{1}{2\pi} \iint_S \psi(\mathbf{r}') \frac{\partial G_1(\mathbf{r}, \mathbf{r}')}{\partial n} dS, \quad \dots (13)$$

where we have taken 2π in place of 4π in the result of Art C.4. for the present problem.

Here bounding surface S is the xy -plane and so here, we have

$$dS = dx' dy' \quad \dots (14)$$

$$\text{Also, } \frac{\partial G_1(\mathbf{r}, \mathbf{r}')}{\partial n} = - \left[\frac{\partial G_1(\mathbf{r}, \mathbf{r}')}{\partial z'} \right]_{z'=0} = - \left(\frac{ik}{R_1^2} - \frac{1}{R_1^3} \right) e^{ikR_1} z - \left(\frac{ik}{R_2^2} - \frac{1}{R_2^3} \right) e^{ikR_2} z, \text{ using (11)} \\ = \frac{2z e^{ikR}}{R^2} \left(ik - \frac{1}{R} \right), \text{ as on boundary } xy\text{-plane, } R_1 = R_2 = R, \text{ say}$$

$$\text{Thus, } \frac{\partial G_1(\mathbf{r}, \mathbf{r}')}{\partial n} = 2 \frac{\partial}{\partial z} \left(\frac{e^{ikR}}{R} \right) \quad \dots (15)$$

Let $f(x', y')$ be the value of ψ on the boundary xy -plane, Then, using (14) and (15), we obtain

$$\psi(r) = -\frac{1}{2\pi} \int_{x'=-\infty}^{\infty} \int_{y'=-\infty}^{\infty} f(x', y') \frac{\partial}{\partial z} \left(\frac{e^{ikR}}{R} \right) dx' dy'$$

or
$$\psi(r) = -\frac{1}{2\pi} \frac{\partial}{\partial z} \int_{x'=-\infty}^{\infty} \int_{y'=-\infty}^{\infty} f(x', y') \frac{e^{ikR}}{R} dx' dy', \text{ where } z > 0 \quad \dots (16)$$

Re-wring (16),
$$\psi(x, y, z) = -\frac{1}{2\pi} \frac{\partial}{\partial z} \int_{x'=-\infty}^{\infty} \int_{y'=-\infty}^{\infty} f(x', y') \frac{e^{ikR}}{R} dx' dy'$$

Exercise. Find the Green's function for the Helmholtz equation for the half plane $x \geq 0$ and hence solve the Helmholtz equation.

Hint. Proceed as in Art C.5. Then, solution of the Helmholtz equation is given by

$$\psi(x, y, z) = \psi(r) = -\frac{1}{2\pi} \frac{\partial}{\partial x} \int_{y'=-\infty}^{\infty} \int_{z'=-\infty}^{\infty} f(y', z') \frac{e^{ikR}}{R} dy' dz'$$

where $\psi = f(y', z')$ on the boundary plane $x = 0$

C.6. Solution of one-dimensional wave equation using the Green's function technique

Consider one-dimensional wave equation

$$\partial^2 u / \partial x^2 = (1/c^2) \times (\partial^2 u / \partial t^2), \quad 0 \leq x \leq a, \quad t \geq 0, \quad \dots (1)$$

where $u(x, t)$ is the deflection of the string. We now solve (1) with help of *method of separation of variables (or product method) under the following boundary and initial conditions :

Boundary conditions : $u(0, t) = u(a, t) = 0, t \geq 0 \quad \dots (2)$

Initial conditions : $u(x, 0) = f(x), 0 \leq x \leq a \quad \dots (3a)$

$$u_t(x, 0) = (\partial u / \partial t)_{t=0} = g(x), 0 \leq x \leq a \quad \dots (3b)$$

Suppose that (1) has solutions of the form $u(x, t) = X(x) T(t) \quad \dots (4)$

where $X(x)$ is function of x alone and $T(t)$ is function of t alone.

Substituting the value of u given by (4) in (1) we have

$$X''T = (1/c^2) \times XT'' \quad \text{or} \quad X''/X = T''/c^2T \quad \dots (5)$$

Since x and t are independent variables, (5) can be true if each side is equal to the same constant, say μ . Then, (5) yields

$$X'' - \mu X = 0 \quad \dots (6)$$

and

$$T'' - \mu c^2 T = 0 \quad \dots (7)$$

Using (2), (4) yields $X(0) T(t) = 0$ and $X(a) T(t) = 0 \quad \dots (8)$

Since $T(t) = 0$ yields to $u = 0$, so we suppose that $T(t) \neq 0$. Then, (8) gives

$$X(0) = 0 \quad \text{and} \quad X(a) = 0 \quad \dots (9)$$

We now solve (6) under the boundary conditions (9). Three cases arise:

Case I. Let $\mu = 0$, Then, solution of (6) is $X(x) = Ax + B, \quad \dots (10)$

where A and B are arbitrary constants. Using the boundary conditions (9), (10) gives $0 = B$ and $0 = Aa + B$. These equations give $A = B = 0$ and so $X(x) = 0$. This leads to $u = 0$ which does not satisfy the initial conditions (3a) and (3b). So we reject $\mu = 0$

* Refer Art. 1.9 and Art 2.10 A in part III of author's "Advanced Differential equations", published by S. Chand and Co., New Delhi

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then, solution of (6) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}, A \text{ and } B \text{ being arbitrary constants} \quad \dots(11)$$

Using the boundary conditions (9), (11) gives $0 = A + B$ and $0 = Ae^{a\lambda} + Be^{-a\lambda}$ $\dots(12)$

Solving (12), $A = B = 0$ so that $X(x) = 0$. This leads to $u = 0$, which does not satisfy (3a) and (3b). So we reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2$, where $\lambda \neq 0$. Then, solution of (1) is given by

$$X(x) = A \cos \lambda x + B \sin \lambda x, A \text{ and } B \text{ being arbitrary constants} \quad \dots(13)$$

Using the boundary conditions (9), (13) yields $0 = A$ and $0 = A \cos \lambda a + B \sin \lambda a$ from which we have $A = 0$ and $\sin \lambda a = 0$,

where we have taken $B \neq 0$, since otherwise $X(x) = 0$ and so $u = 0$ which does not satisfy (3a) and (3b).

$$\text{Now, } \sin \lambda a = 0 \Rightarrow \lambda a = n\pi \Rightarrow \lambda = n\pi / a, \quad n = 1, 2, 3, \dots \quad \dots(14)$$

$$\text{Hence non-zero solution } X_n(x) \text{ of (10) are given by } X_n(x) = B_n \sin (n\pi x / a) \quad \dots(15)$$

Using (14), we have $\mu = -\lambda^2 = -(n^2\pi^2) / a^2$. Hence, (7) reduces to

$$T'' + (n^2\pi^2 c^2 / a^2) T = 0, \text{ whose general solution is given by}$$

$$T_n(t) = C_n \cos (n\pi ct/a) + D_n \sin(n\pi ct/a), C_n \text{ and } D_n \text{ being arbitrary constants} \quad \dots(16)$$

$$\text{Thus, } u_n(x, t) = X_n(x) T_n(t) = \{E_n \cos (n\pi ct/a) + F_n \sin (n\pi ct/a)\} \sin (n\pi x/a),$$

are solutions of (1) satisfying the boundary condition (2). Here $E_n (= C_n B_n)$ and $F_n (= B_n D_n)$ are new arbitrary constants. In order to obtain a solution also satisfying the initial conditions (3a) and (3b), we consider more general solution $u(x, t)$ by the method of superposition

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) \quad \text{or} \quad u(x, t) = \sum_{n=1}^{\infty} \left(E_n \cos \frac{n\pi ct}{a} + F_n \sin \frac{n\pi ct}{a} \right) \sin \frac{n\pi x}{a} \quad \dots(17)$$

Differentiating (17) partially w.r. 't', we have

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-\frac{n\pi c E_n}{a} \sin \frac{n\pi ct}{a} + \frac{n\pi c F_n}{a} \cos \frac{n\pi ct}{a} \right) \sin \frac{n\pi x}{a} \quad \dots(18)$$

Putting $t = 0$ in (17) and (18) and using initial conditions (3a) and (3b), we get

$$f(x) = \sum_{n=1}^{\infty} E_n \sin(n\pi x / a) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} (n\pi c F_n / a) \sin(n\pi x / a), \quad \dots(19)$$

which are Fourier sine series expansions for $f(x)$ and $g(x)$ respectively. Hence, we get

$$E_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx \quad \dots(20)$$

$$\text{and } \frac{n\pi c F_n}{a} = \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi x}{a} dx \quad \text{so that} \quad F_n = \frac{2}{n\pi c} \int_0^a g(x) \sin \frac{n\pi x}{a} dx \quad \dots(21)$$

Thus, the general solution of (1) is given by (17) and E_n and F_n are given by (20) and (21).

$$\text{From (20) and (21), } E_n = \frac{2}{a} \int_0^a f(\xi) \sin \frac{n\pi \xi}{a} d\xi \quad \text{and} \quad F_n = \frac{2}{n\pi c} \int_0^a g(\xi) \sin \frac{n\pi \xi}{a} d\xi \quad \dots(22)$$

Substituting the values of E_n and F_n given by (22) in (17), we obtain

$$u(x, t) = \sum_{n=1}^{\infty} \left[\left\{ \frac{2}{a} \int_0^a f(\xi) \sin \frac{n\pi \xi}{a} d\xi \right\} \cos \frac{n\pi ct}{a} + \left\{ \frac{2}{n\pi c} \int_0^a g(\xi) \sin \frac{n\pi \xi}{a} d\xi \right\} \sin \frac{n\pi ct}{a} \right] \sin \frac{n\pi x}{a}$$

Interchanging the operations of summation and integration, the above equation reduces to

$$u(x, t) = \int_0^a \left(\frac{2}{a} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \sin \frac{n\pi \xi}{a} \cos \frac{n\pi c t}{a} \right) f(\xi) d\xi + \int_0^a \left[\frac{2}{\pi c} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{a} \sin \frac{n\pi \xi}{a} \sin \frac{n\pi c t}{a} \right] g(\xi) d\xi \quad \dots (23)$$

We now define Green's function $G(x, t; \xi, \tau)$ as follows:

$$G(x, t; \xi, \tau) = \frac{2}{\pi c} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{a} \sin \frac{n\pi \xi}{a} \sin \frac{n\pi c(t - \tau)}{a} \quad \dots (24)$$

It can be easily verified that the above series converges for all values of x , t , ξ and τ .

$$\text{From (24),} \quad G_t(x, t; \xi, \tau) = \partial G(x, t; \xi, \tau) / \partial t = \frac{2}{a} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \sin \frac{n\pi \xi}{a} \cos \frac{n\pi c(t - \tau)}{a} \quad \dots (25)$$

Putting $\tau = 0$ in (24) and (25), we have

$$G(x, t; \xi, 0) = \frac{2}{\pi c} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{a} \sin \frac{n\pi \xi}{a} \sin \frac{n\pi c t}{a}, \quad G_t(x, t; \xi, 0) = \frac{2}{a} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \sin \frac{n\pi \xi}{a} \cos \frac{n\pi c t}{a}$$

Using the above results, the series solution (17) of the given initial boundary value problem can be re-written in terms of Green's function as

$$u(x, t) = \int_0^a G_t(x, t; \xi, 0) f(\xi) d\xi + \int_0^a G(x, t; \xi, 0) g(\xi) d\xi \quad \dots (26)$$

Observe that $G(x, t; \xi, \tau)$ as a function of x satisfies the boundary conditions. That is, we have

$$G(0, t; \xi, \tau) = 0 \quad \text{and} \quad G(a, t; \xi, \tau) = 0, \quad t \geq 0$$

$$\begin{aligned} \text{Further,} \quad & G(x, t; \xi, \tau) = 0, \quad 0 \leq x \leq a \\ & G(\xi, t; x, \tau) = G(x, t; \xi, \tau) \quad \text{for all } x \text{ and } \xi \\ & G(x, \tau; \xi, t) = -G(x, t; \xi, \tau) \quad \text{for all } t \text{ and } \tau \end{aligned}$$

The function $G(x, t; \xi, \tau)$ defined by (24) is known as the *Green's function* of the given initial boundary value problem given by (1), (2), (3a) and (3b).

C.7. Solution of one-dimensional inhomogeneous wave equation using the Green's function technique

In many physical problems we obtain an inhomogeneous wave equation when there is a force function operating on the medium. Suppose that an external force density (force per unit length of the vibrating string) $\rho f(x, t)$ is applied to the string which is independent of the deflection $u(x, t)$ of the vibrating string. In such a situation we arrive at an equation containing a source term. This equation is said to be an *inhomogeneous wave equation*.

Consider one-dimensional inhomogeneous wave equation

$$\partial^2 u / \partial t^2 = c^2 (\partial^2 u / \partial x^2) + f(x, t), \quad 0 \leq x \leq a, \quad t \geq 0 \quad \dots (1)$$

where $u(x, t)$ represents the deflection of the string. We now solve (1) under the following boundary and initial conditions:

$$\text{Boundary conditions:} \quad u(0, t) = u(a, t), \quad t > 0 \quad \dots (2)$$

$$\text{Initial conditions :} \quad u(x, 0) = g(x), \quad 0 \leq x \leq a \quad \dots (3a)$$

$$\text{and} \quad u_t(x, 0) = h(x), \quad 0 \leq x \leq a \quad \dots (3b)$$

We now proceed to express the solution of the above problem as Fourier sine series in x , that is,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x / a), \quad \dots (4)$$

where we have taken undetermined functions $u_n(t)$ of time as multiplying constants. We have selected the above form for $u(x, t)$, since this form automatically satisfies both the boundary conditions given by (2). We now proceed to determine the unknown functions $u_n(t)$ in such a manner so that the resulting value of $u(x, t)$ given by (4) may satisfy (1) and initial conditions (3a) and (3b). To this end, we assume the Fourier sine series expansions of $f(x, t)$, $g(x)$ and $h(x)$ given by

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x / a), \quad \text{where} \quad \dots (5a)$$

$$f_n(t) = \frac{2}{a} \int_0^a f(\xi, t) \sin(n\pi \xi / a) d\xi \quad \dots (5b)$$

$$g(x) = \sum_{n=1}^{\infty} g_n \sin(n\pi x / a), \quad \text{where} \quad \dots (6a)$$

$$g_n = \frac{2}{a} \int_0^a g(\xi) \sin(n\pi \xi / a) d\xi \quad \dots (6b)$$

$$h(x) = \sum_{n=1}^{\infty} h_n \sin(n\pi x / a), \quad \text{where} \quad \dots (7a)$$

$$h_n = \frac{2}{a} \int_0^a h(\xi) \sin(n\pi \xi / a) d\xi \quad \dots (7b)$$

$$\text{From (4),} \quad \left. \begin{aligned} \frac{\partial u}{\partial t} &= \sum_{n=1}^{\infty} u'_n(t) \sin \frac{n\pi x}{a}, & \frac{\partial^2 u}{\partial t^2} &= \sum_{n=1}^{\infty} u''_n(t) \sin \frac{n\pi x}{a}, \\ \frac{\partial u}{\partial x} &= \sum_{n=1}^{\infty} \left(\frac{n\pi}{a} \right) u_n(t) \cos \frac{n\pi x}{a} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} \left(-\frac{n^2 \pi^2}{a^2} \right) u_n(t) \sin \frac{n\pi x}{a} \end{aligned} \right\} \quad \dots (8)$$

Substituting the values of $f(x, t)$, $\partial^2 u / \partial x^2$ and $\partial^2 u / \partial t^2$ given by (5a) and (8) in (1), we get

$$\sum_{n=1}^{\infty} u''_n(t) \sin \frac{n\pi x}{a} = - \sum_{n=1}^{\infty} \left(\frac{n^2 \pi^2 c^2}{a^2} \right) u_n(t) \sin \frac{n\pi x}{a} + \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{a}$$

$$\text{or} \quad \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \left\{ u''_n(t) + \left(\frac{n^2 \pi^2 c^2}{a^2} \right) u_n(t) - f_n(t) \right\} = 0 \quad \dots (9)$$

Since $\sin(n\pi x / a)$ are linearly independent functions, (9) is satisfied only if, the expression in parenthesis vanishes $\forall n \in N$, that is,

$$u''_n(t) + (n^2 \pi^2 c^2 / a^2) u_n(t) - f_n(t) = 0 \quad \text{or} \quad u''_n(t) + (n^2 \pi^2 c^2 / a^2) u_n(t) = f_n(t), \quad \dots (10)$$

which is a second order differential equation whose general solution will contain two arbitrary constants. In order to determine these two arbitrary constants, we require two initial conditions. These conditions can be determined with the help of the initial conditions (3a) and (3b).

Putting $t = 0$ in (4) and using, the initial condition (3a), we have

$$g(x) = \sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi x}{a} \quad \text{or} \quad \sum_{n=1}^{\infty} g_n \sin \frac{n\pi x}{a} = \sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi x}{a}, \text{ using (6a)} \quad \dots (11)$$

$$\text{Comparing the coefficients of } \sin(n\pi x / a), \quad (11) \quad \Rightarrow \quad u_n(0) = g_n \quad \dots (12)$$

Differentiating both sides of (4) partially w.r.t. 't', we have

$$\frac{\partial u(x, t)}{\partial t} = u_t(x, t) = \sum_{n=1}^{\infty} u'_n(t) \sin \frac{n\pi x}{a} \quad \dots (13)$$

Putting $t = 0$ in (13) and using the initial condition (3b), we have

$$h(x) = \sum_{n=1}^{\infty} u'_n(0) \sin \frac{n\pi x}{a} \quad \text{or} \quad \sum_{n=1}^{\infty} h_n \sin \frac{n\pi x}{a} = \sum_{n=1}^{\infty} u'_n(0) \sin \frac{n\pi x}{a}, \text{ using (7a)} \quad \dots (14)$$

$$\text{Comparing the coefficients of } \sin(n\pi x / a), \quad (14) \quad \Rightarrow \quad u'_n(0) = h_n \quad \dots (15)$$

Thus, the original problem of solving partial differential equation (1) with initial conditions (3a) and (3b) has been reduced to that of solving an ordinary differential equation (10) with initial conditions (12) and (15).

We now propose to use the superposition principle. Accordingly, we adopt the following procedure to solve (10) subject to the initial conditions (12) and (15). We now split the solution $u_n(t)$ into two parts $b_n(t)$ and $c_n(t)$, i.e., we assume that

$$u_n(t) = b_n(t) + c_n(t), \quad \dots (16)$$

$$\text{where } b_n(t) \text{ is the solution of} \quad b''_n(t) + (n^2 \pi^2 c^2 / a^2) b_n(t) = f_n(t), \quad \dots (17)$$

$$\text{subject to the initial conditions :} \quad b_n(0) = b'_n(0) = 0 \quad \dots (18)$$

$$\text{while } c_n(t) \text{ is the solution of} \quad c''_n(t) + (n^2 \pi^2 c^2 / a^2) c_n(t) = 0, \quad \dots (19)$$

$$\text{subject to the initial conditions:} \quad c_n(0) = c'_n(0) = 0 \quad \dots (20)$$

We now proceed to determine the proposed values of $b_n(t)$ and $c_n(t)$ one by one:

Determination of solution $b_n(t)$ of initial value problem given by (17) and (18)

In what follows, the results of Art. 9.2 of chapter 9 will be used.

Taking Laplace transform of both sides of (17), we obtain

$$L \{ b''_n(t) \} + (n^2 \pi^2 c^2 / a^2) \times L \{ b_n(t) \} = L \{ f_n(t) \}$$

$$\text{or} \quad p^2 L \{ b_n(t) \} - p b_n(0) - b'_n(0) + (n\pi c / a)^2 \times L \{ b_n(t) \} = L \{ f_n(t) \}, \text{ using result 8 of page 9.2}$$

$$\text{or} \quad \{ p^2 + (n\pi c / a)^2 \} \times L \{ b_n(t) \} = L \{ f_n(t) \}, \text{ using (18)}$$

$$\text{or} \quad L \{ b_n(t) \} = \frac{1}{p^2 + (n\pi c / a)^2} L \{ f_n(t) \} = L \left\{ \frac{\sin(n\pi c t / a)}{(n\pi c / a)} \right\} L \{ f_n(t) \}$$

[Using result 5 of table given on page 9.3]

Using the convolution theorem (refer result (24) on page 9.4), the above equation yields

$$b_n(t) = \int_0^t \frac{\sin\{n\pi c(t-\tau)/a\}}{(n\pi c/a)} f_n(\tau) d\tau = \frac{a}{n\pi c} \int_0^t \sin \frac{n\pi c(t-\tau)}{a} f_n(\tau) d\tau \quad \dots(21)$$

Determination of solution $c_n(t)$ of initial value problem given by (19) and (20)

$$\text{Let } D \equiv d/dt. \text{ Then, (20) becomes } \{D^2 + (n^2\pi^2 c^2/a^2)\} c_n(t) = 0, \quad \dots(22)$$

whose auxiliary equation is $D^2 + n^2\pi^2 c^2/a^2 = 0$ giving $D = \pm i(n\pi c/a)$.

Hence, the general solution of (22) is given by

$$c_n(t) = A \cos(n\pi c t/a) + B \sin(n\pi c t/a), \text{ } A \text{ and } B \text{ being arbitrary constants} \quad \dots(23)$$

$$\text{From (23), } c'_n(t) = -(An\pi c/a) \sin(n\pi c t/a) + (Bn\pi c/a) \cos(n\pi c t/a) \quad \dots(24)$$

Putting $t = 0$ in (23) and (24) and using initial conditions given by (20), we get $A = g_n$ and $B = (h_n a / n\pi c)$. Hence, (23) reduces to

$$c_n(t) = g_n \cos(n\pi c t/a) + (h_n a / n\pi c) \sin(n\pi c t/a) \quad \dots(25)$$

$$\text{From (4) and (16), } u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{a} + \sum_{n=1}^{\infty} c_n(t) \cos \frac{n\pi x}{a} \quad \dots(26)$$

Substituting the values of $b_n(t)$ and $c_n(t)$ given by (21) and (25) in (26), we have

$$u(x, t) = \sum_{n=1}^{\infty} \frac{a}{n\pi c} \int_0^t \sin \frac{n\pi c(t-\tau)}{a} \sin \frac{n\pi x}{a} f_n(\tau) d\tau + \sum_{n=1}^{\infty} \left[g_n \cos \frac{n\pi c t}{a} + \frac{h_n a}{n\pi c} \sin \frac{n\pi c t}{a} \right] \sin \frac{n\pi x}{a} \quad \dots(27)$$

Let $u_1(x, t)$ and $u_2(x, t)$ denote respectively the first and second terms on R.H.S. of (27). Thus,

$$u_1(x, t) = \sum_{n=1}^{\infty} \frac{a}{n\pi c} \int_0^t \sin \frac{n\pi c(t-\tau)}{a} \sin \frac{n\pi x}{a} f_n(\tau) d\tau \quad \dots(28)$$

$$\text{and } u_2(x, t) = \sum_{n=1}^{\infty} \left[g_n \cos \frac{n\pi c t}{a} + \frac{h_n a}{n\pi c} \sin \frac{n\pi c t}{a} \right] \sin \frac{n\pi x}{a} \quad \dots(29)$$

Thus, $u_1(x, t)$ represents the forced vibrations of the string under the influence of an external force whereas $u_2(x, t)$ represents the solution of the problem of freely vibrating string with the given initial conditions.

$$\text{Re-writing (5b), we have } f_n(\tau) = \frac{2}{a} \int_0^a f(\xi, \tau) \sin \frac{n\pi \xi}{a} d\xi \quad \dots(30)$$

Substituting the value of $f_n(\tau)$ given by (30) in (28), we have

$$u_1(x, t) = \int_0^t \int_0^l \left\{ \frac{2}{a} \sum_{n=1}^{\infty} \frac{a}{n\pi c} \sin \frac{n\pi c(t-\tau)}{a} \sin \frac{n\pi x}{a} \sin \frac{n\pi \xi}{a} \right\} f(\xi, \tau) d\xi d\tau \quad \dots(31)$$

We now define the Green's function as

$$G(x, \xi, t-\tau) = \frac{2}{\pi c} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi c(t-\tau)}{a} \sin \frac{n\pi x}{a} \sin \frac{n\pi \xi}{a} \quad \dots(32)$$

$$\text{Then, (31) reduces to } u_1(x, t) = \int_0^t \int_0^a G(x, \xi, t-\tau) f(\xi, \tau) d\xi d\tau \quad \dots(33)$$

Physical interpretation of the Green's function $G(x, \xi, t - \tau)$

Recall that $\rho f(x, t)$ represents the external force per unit length of the vibrating string, ρ being the mass per unit length of the string. In what follows, suppose that the external force $f(x, t)$ is non zero only in the small neighbourhood $\Delta\xi$ of the point ξ_0 and also during the small time interval $\Delta\tau$ around the time point τ_0 and zero otherwise. Then, the force $F(\tau)$ acting on the string in the interval $\Delta\xi$ is given by

$$F(\tau) = \rho \int_{\xi_0 - \Delta\xi/2}^{\xi_0 + \Delta\xi/2} f(\xi, \tau) d\xi \quad \dots (34)$$

Let I denote the impulse due to the concentrated force $F(\tau)$. Recall that impulse I of this localised force may be obtained by an integral of the force with time. Since the force is non zero only over a short duration of time $\Delta\tau$, we have

$$I = \int_{\tau_0 - \Delta\tau/2}^{\tau_0 + \Delta\tau/2} F(\tau) d\tau = \rho \int_{\tau_0 - \Delta\tau/2}^{\tau_0 + \Delta\tau/2} d\tau \int_{\xi_0 - \Delta\xi/2}^{\xi_0 + \Delta\xi/2} f(\xi, \tau) d\xi, \text{ using (34)} \quad \dots (35)$$

Since $u_1(x, t)$ represents the part of the solution of (1) due to the external force, hence the displacement $u_1(x, t)$ due to impulse I is given by (33).

Recall that we have already assumed that the external force $f(x, t)$ is non zero only in the small neighbourhood $\Delta\xi$ of the point ξ_0 and also during the small time interval $\Delta\tau$ around the time point τ_0 . Again, note that the function $G(x, \xi, t - \tau)$ is a well defined function in the neighbourhood of the point (ξ_0, τ_0) . In view of these two facts, $G(x, \xi_0, t - \tau_0)$ can be taken outside the integral for very small values of $\Delta\xi$ and $\Delta\tau$, by applying the mean value theorem. Thus, using mean value theorem, (33) yields approximate displacement given by

$$u_1(x, t) \cong G(x, \xi_0, t - \tau_0) \int_{\tau_0 - \Delta\tau/2}^{\tau_0 + \Delta\tau/2} \int_{\xi_0 - \Delta\xi/2}^{\xi_0 + \Delta\xi/2} f(\xi, \tau) d\xi d\tau = G(x, \xi_0, t - \tau_0) \frac{I}{\rho}, \text{ using (35)} \quad \dots (36)$$

Hence, it follows that the Green's function $G(x, \xi_0, t - \tau_0)$ can be interpreted as the displacement at x at time t due to a unit impulse (measured in the units of ρ) applied at the point ξ_0 at the time τ_0 . In this context, note that $G(x, \xi_0, t - \tau_0) = 0$ for $t < \tau_0$, because there cannot be any displacement even before the external force is applied.

From the above discussion it is obvious that the effect of a disturbed force of density $\rho f(x, t)$ per unit length of the string can be obtained by integrating the small displacements due to applied force over the entire length of the string and over the entire duration of the applied force, that is,

$$u_1(x, t) = \int_0^t \int_0^a G(x, \xi, t - \tau) f(\xi, \tau) d\xi d\tau, \quad \dots (37)$$

which is exactly the equation (33).

An important particular case. Fundamental solution of the given wave equation.

Let $f(x, t)$ be a two-dimensional Dirac delta function, i.e., let

$$f(x, t) = \delta(x - \xi_0) \delta(t - \tau_0)$$

so that

$$f(\xi, \tau) = \delta(\xi - \xi_0) \delta(\tau - \tau_0) \quad \dots (38)$$

Substituting the above value of $f(\xi, \tau)$ given by (38) in (33), we get

$$u(x, t) = \int_0^t \int_0^a G(x, \xi, t - \tau) \delta(\xi - \xi_0) \delta(\tau - \tau_0) d\xi d\tau = G(x, \xi_0, t - \tau_0),$$

[Using result (3) of Art. B.2 of Appendix B]

showing that the Green's function can be regarded as a solution of the given equation (1) for the two dimensional Dirac delta source function. In other words, the Green's function can be treated as solution of the following initial boundary value problem

$$\partial^2 u / \partial t^2 = c^2 (\partial^2 u / \partial x^2) + \delta(x - \xi) \delta(t - \tau)$$

with boundary conditions

$$u(0, t) = u(a, t) = 0$$

and initial conditions

$$u(x, 0) = u_t(x, 0) = 0, \text{ for } 0 < x < a$$

In view of the above fact, Green's function is also known as the *fundamental solution* of the given wave equation for the given boundary conditions. The advantage of finding the fundamental solution or the Green's function for a particular set of boundary conditions lies in the fact that solution of the inhomogeneous equation for any source function $f(x, t)$ can be found by simply evaluating the integral of type involved in (33), and not solve the problem for a different source. In this context, we must make sure that the new problem under consideration must have the same geometry and the same boundary conditions, because the Green's function is determined for a particular geometry of the problem and the boundary conditions.

An illustrative solved example. Solve the following inhomogeneous wave equation with a time dependent external force:

$u_{tt} = c^2 u_{xx} + g(x) \cos kt, 0 < x < a, t > 0$ with homogenous initial and boundary conditions, namely, $u(0, t) = u(a, t) = 0, t \geq 0; u(x, 0) = u_t(x, 0) = 0, 0 \leq x \leq a$. Solve for $u(x, t)$ assuming $k \neq (n\pi c / a) \forall n \in \mathbb{N}$.

Solution. Given $\partial^2 u / \partial t^2 = c^2 (\partial^2 u / \partial x^2) + g(x) \cos kt, 0 < x < a, t > 0$... (1)

with boundary conditions:

$$u(0, t) = u(a, t) = 0, t \geq 0 \quad \dots (2)$$

initial conditions:

$$u(x, 0) = 0, 0 \leq x \leq a \quad \dots (3a)$$

$$u_t(x, 0) = 0, 0 \leq x \leq a \quad \dots (3b)$$

Let the required solution be expressed as a Fourier sine series in x , that is,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{a}, \quad \dots (4)$$

where we have taken undetermined functions $u_n(t)$ of time as multiplying constants. We have selected the above form for $u(x, t)$ because this form automatically satisfies both the prescribed boundary conditions (2). We now proceed to determine $u_n(t)$ so that the resulting solution $u(x, t)$ given by (4) may also satisfy (1) and the prescribed initial conditions. To this end we assume the Fourier sine series expansion of the external force $f(x, t) (= g(x) \cos kt)$ given by

$$f(x, t) = g(x) \cos kt = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{a} \quad \dots (5)$$

where

$$f_n(t) = \frac{2}{a} \int_0^a g(\xi) \cos kt \sin \frac{n\pi \xi}{a} d\xi = \frac{2 \cos kt}{a} \int_0^a g(\xi) \sin \frac{n\pi \xi}{a} d\xi = g_n \cos kt \quad \dots (6)$$

where

$$g_n = \frac{2}{a} \int_0^a g(\xi) \sin \frac{n\pi \xi}{a} d\xi \quad \dots (7)$$

$$\left. \begin{aligned} \text{From (4), } \frac{\partial u}{\partial t} &= \sum_{n=1}^{\infty} u'_n(t) \sin \frac{n\pi x}{a}, & \frac{\partial^2 u}{\partial t^2} &= \sum_{n=1}^{\infty} u''_n(t) \sin \frac{n\pi x}{a}, \\ \frac{\partial u}{\partial x} &= \sum_{n=1}^{\infty} \left(\frac{n\pi}{a} \right) u_n(t) \cos \frac{n\pi x}{a} \text{ and } \frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} \left(-\frac{n^2 \pi^2}{a^2} \right) u_n(t) \sin \frac{n\pi x}{a} \end{aligned} \right\} \quad \dots (8)$$

Substituting the values of $g(x) \cos kt$, $\partial^2 u / \partial t^2$ and $\partial^2 u / \partial x^2$ given by (5) and (8) in (1), we get

$$\begin{aligned} \sum_{n=1}^{\infty} u''_n(t) \sin \frac{n\pi x}{a} &= -\sum_{n=1}^{\infty} \left(\frac{n^2 \pi^2 c^2}{a^2} \right) u_n(t) \sin \frac{n\pi x}{a} + \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{a} \\ \text{or } \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \left\{ u''_n(t) + \left(\frac{n^2 \pi^2 c^2}{a^2} \right) u_n(t) - g_n \cos kt \right\} &= 0, \text{ using (6)} \quad \dots (9) \end{aligned}$$

Since $\sin(n\pi x / a)$ are linearly independent functions, (9) is satisfied only if, the expression in parenthesis vanishes $\forall n \in N$, that is,

$$u''_n(t) + (n^2 \pi^2 c^2 / a^2) u_n(t) - g_n \cos kt = 0 \quad \text{or} \quad u''_n(t) + (n\pi c / a)^2 u_n(t) = g_n \cos kt \quad \dots (10)$$

$$\text{Let } D \equiv d / dt. \text{ Then, (4) becomes} \quad \{D^2 + (n\pi c / a)^2\} u_n(t) = g_n \cos kt \quad \dots (11)$$

$$\text{The auxiliary equation of (11) is} \quad D^2 + (n\pi c / a)^2 = 0 \quad \text{giving} \quad D = \pm (n\pi c / a)$$

Hence, C.F. of (11) = $c_1 \cos(n\pi c t / a) + c_2 \sin(n\pi c t / a)$, c_1 and c_2 being arbitrary constants.

Particular integral (P.I.) or particular solution of (11) is given by

$$P.I. = \frac{1}{D^2 + (n\pi c / a)^2} g_n \cos kt = g_n \frac{1}{(n\pi c / a)^2 - k^2} \cos kt.$$

Hence, the general solution of (11) is given by $u_n(t) = \text{C.F.} + \text{P.I.}$, i.e.,

$$u_n(t) = C_n \cos \left(\frac{n\pi c t}{a} \right) + D_n \sin \left(\frac{n\pi c t}{a} \right) + \frac{g_n \cos kt}{(n\pi c / a)^2 - k^2} \quad \dots (12)$$

In order to determine c_1 and c_2 , we require two initial conditions. This can be done by using initial conditions (3a) and (3b) as follows.

Putting $t = 0$ in (4) and using the initial condition (3a), we have

$$0 = \sum_{n=1}^{\infty} u_n(0) \sin(n\pi x / a) \quad \text{giving} \quad u_n(0) = 0 \quad \dots (13)$$

$$\text{From (4),} \quad u_t(x, t) = \partial u / \partial t = \sum_{n=1}^{\infty} u'_n(t) \sin(n\pi x / a) \quad \dots (14)$$

Putting $t = 0$ in (14) and using the initial condition (3b), we have

$$0 = \sum_{n=1}^{\infty} u'_n(0) \sin(n\pi x / a) \quad \text{giving} \quad u'_n(0) = 0 \quad \dots (15)$$

$$\text{From (12), } u'_n(t) = -\left(\frac{C_n n\pi c}{a} \right) \sin \frac{n\pi c t}{a} + \left(\frac{D_n n\pi c}{a} \right) \cos \frac{n\pi c t}{a} - \frac{g_n k \sin kt}{(n\pi c / a)^2 - k^2} \quad \dots (16)$$

Putting $t = 0$ in (16) and using initial condition (15), we get $D_n = 0$.

Next, putting $t = 0$ and $D_n = 0$ in (12) and using initial condition (12), we get

$$0 = C_n + g_n / \{(n\pi c / a)^2 - k^2\} \quad \text{so that} \quad C_n = -[g_n / \{(n\pi c / a)^2 - k^2\}]$$

Substituting the above values of C_n and D_n in (12), we have

$$u_n(t) = \frac{g_n}{(n\pi c / a)^2 - k^2} \left(\cos kt - \cos \frac{n\pi ct}{a} \right), \quad \dots (17)$$

where g_n is the n th Fourier coefficient of $g(x)$ given by (7).

C.8. Solution of one-dimensional heat equation using the Green's function technique.

Consider the heat flow problem in a finite rod of length a described by the partial differential equation

$$\partial u / \partial t = k(\partial^2 u / \partial x^2), \quad 0 \leq x \leq a, \quad t > 0 \quad \dots (1)$$

where $k = K / \rho\sigma$ is the thermal diffusivity of the material of the rod. Also, K , ρ , σ are respectively, thermal conductivity, density and specific heat of the material of the rod

$$\text{with boundary conditions:} \quad u(0, t) = u(a, t) = 0, \quad t > 0 \quad \dots (2)$$

$$\text{and initial condition:} \quad u(x, 0) = f(x), \quad 0 \leq x \leq a \quad \dots (3)$$

We now proceed to solve the above problem with help of *method of separation (or product method).

$$\text{Suppose that (1) has solutions of the form} \quad u(x, t) = X(x) T(t), \quad \dots (4)$$

where $X(x)$ is a function of x alone and $T(t)$ that of t alone. Substituting this value of u in (1) gives

$$XT' = kX''T \quad \text{so that} \quad X''/X = T'/kT \quad \dots (5)$$

Since x and t are independent variables, (5) can be true only if each side is equal to the same constant, say μ . Hence, (5) leads to the following two ordinary differential equations:

$$X'' - \mu X = 0 \quad \dots (6)$$

$$\text{and} \quad T' = \mu kT \quad \dots (7)$$

$$\text{Using (2), (4) gives} \quad X(0) T(t) = 0 \quad \text{and} \quad X(a) T(t) = 0 \quad \dots (8)$$

Since $T(t) = 0$ leads to $u = 0$ which does not satisfy (3). Hence, we take $T(t) \neq 0$. Then, by (8)

$$X(0) = 0 \quad \text{and} \quad X(a) = 0 \quad \dots (9)$$

We now solve (6) under the boundary conditions (9). Three cases arise:

$$\text{Case I: Let } \mu = 0. \text{ Then, solution of (6) is} \quad X(x) = Ax + B \quad \dots (10)$$

Using the boundary conditions (9), (10) yields $0 = B$ and $0 = Aa + B$. Solving these equations, we get $A = B = 0$. Hence $X(x) = 0$ and so $u = 0$ which does not satisfy (3). So we reject $\mu = 0$.

Case II: Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then, the general solution of (6) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}, \quad A \text{ and } B \text{ being arbitrary constants} \quad \dots (11)$$

Using the boundary conditions (9), (11) yields $0 = A + B$ and $0 = Ae^{\lambda a} + Be^{-\lambda a}$. Solving these equations, we get $A = B = 0$. Hence $X(x) = 0$ and so $u = 0$ which does not satisfy (3). So reject $\mu = 0$.

Case III: Let $\mu = -\lambda^2$, where $\lambda \neq 0$. Then, the solution of (6) is

$$X(x) = A \cos \lambda x + B \sin \lambda x, \quad A \text{ and } B \text{ being arbitrary constants} \quad \dots (12)$$

* Refer Art. 2.3 A in part III in author's "Advanced Differential Equations", published by S. Chand and Co., New Delhi

Using the boundary conditions (9), (12) yields $A = 0$ and $0 = A \cos \lambda a + B \sin \lambda a$ so that $A = 0$ and $\sin \lambda a = 0$, ... (13)
 which is obtained by assuming that $B \neq 0$ because if $A = B = 0$, then $X(x) = 0$ and so $u = 0$ which does not satisfy (3).

$$\text{Now, } \sin \lambda a = 0 \Rightarrow \lambda a = n\pi, n = 1, 2, 3 \dots \Rightarrow \lambda = n\pi/a, n = 1, 2, 3 \dots (14)$$

Hence nonzero solutions $X_n(x)$ of (6) are given by ... (15)

$$X_n(x) = B_n \sin(n\pi x/a), n = 1, 2, 3, \dots$$

$$\text{Using (14), (7) yields } \frac{dT}{T} = -\frac{n^2 \pi^2 k}{a^2} dt, \quad \text{since } \mu = -\lambda^2 = -\frac{n^2 \pi^2}{a^2}$$

whose general solution is given by

$$T_n = C_n e^{-(n^2 \pi^2 k/a^2)t}, n = 1, 2, 3, \dots (16)$$

From (15) and (16), $u_n(x, t) = X_n(t) T_n(t) = E_n \sin(n\pi x/a) e^{-(n^2 \pi^2 k/a^2)t}$, $n = 1, 2, 3, \dots$ are solutions of (1), satisfying the boundary conditions (2). Here $E_n (= B_n C_n)$ are new arbitrary constants.

Using the principle of superposition, the general solution of (1) is given by

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a) e^{-(n^2 \pi^2 k/a^2)t} \dots (17)$$

The series represented by (17) can be shown to be uniformly convergent, and it is termwise differentiable.

Substituting $t = 0$ in (17) and using the initial condition (3), we have

$$f(x) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a),$$

which is Fourier sine series. Hence, the constants E_n are given by

$$E_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx = \frac{2}{a} \int_0^a f(y) \sin \frac{n\pi y}{a} dy, \quad n = 1, 2, 3, \dots (18)$$

Substituting the above value of E_n in (17), we obtain the following representation of the solution of (1)

$$u(x, t) = \sum_{n=1}^{\infty} \left[\frac{2}{a} \int_0^a f(y) \sin \frac{n\pi y}{a} dy \right] \sin \frac{n\pi x}{a} e^{-(n^2 \pi^2 k/a^2)t}$$

or

$$u(x, t) = \int_0^a \left[\frac{2}{a} \sum_{n=1}^{\infty} e^{-(n^2 \pi^2 k/a^2)t} \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{a} \right] f(y) dy \dots (19)$$

Let us define a function $G(x, y, t)$ as follows :

$$G(x, y, t) = \frac{2}{a} \sum_{n=1}^{\infty} e^{-(n^2 \pi^2 k/a^2)t} \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{a}, \quad 0 \leq x \leq a, \quad 0 \leq y \leq a, \quad t > 0 \dots (20)$$

Using (20), the solution of the given boundary value problem is given by

$$u(x, t) = \int_0^a G(x, y, t) f(y) dy, \quad 0 \leq x \leq a, \quad t > 0 \dots (21)$$

The function $G(x, y, t)$ defined by (20) is known as the Green's function for the given heat equation.

It can be easily seen that the Green's function $G(x, y, t)$ possess the following properties :

$$(i) \frac{\partial G(x, y, t)}{\partial t} = k \frac{\partial^2 G(x, y, t)}{\partial x^2} = k \frac{\partial^2 G(x, y, t)}{\partial y^2}, \quad 0 \leq x \leq a, \quad 0 \leq y \leq a, \quad t > 0 \quad \dots (22)$$

$$(ii) G(0, y, t) = G(a, y, t) \quad \text{for} \quad 0 \leq y \leq a, \quad t > 0 \quad \dots (23)$$

$$(iii) G(x, y, t) = G(y, x, t) \quad \text{for} \quad 0 \leq x \leq a, \quad 0 \leq y \leq a, \quad t > 0 \quad \dots (24)$$

Observe that the equation (22) can be derived from equation (20) by term-by-term differentiation. Hence, it follows that the Green's function $G(x, y, t)$ satisfies the heat equation, Since $G(x, y, t)$ is symmetric, it also satisfies the boundary conditions.

Physical interpretation of the Green's function $G(x, y, t)$

The Green's function $G(x, y, t)$ can be interpreted as the effect on the temperature at x at time t due to a heat source of unit magnitude (measured in unit of $\rho\sigma$) applied instantaneously at point y at time $t = 0$. It follows that the effect of the initial condition at $t = 0$ may be regarded as equivalent to that of an instantaneous heat source.

C.9. Solution of one-dimensional inhomogeneous heat equation involving an external heat source using the Green's function technique

We propose to solve one-dimensional inhomogeneous heat equation

$$\partial u / \partial t = k(\partial^2 u / \partial x^2) + f(x, t), \quad 0 \leq x \leq a, \quad t > 0 \quad \dots (1)$$

$$\text{with the boundary conditions:} \quad u(0, t) = u(a, t) = 0, \quad t > 0 \quad \dots (2)$$

$$\text{and the initial condition:} \quad u(x, 0) = 0, \quad 0 \leq x \leq a \quad \dots (3)$$

Here $k = K / \rho\sigma$ is the thermal diffusivity of the material of the rod. Also, K, ρ, σ are respectively, thermal conductivity, density and specific heat of the material of the rod. Again, here $f(x, t)$ is proportional to the volumetric heat source.

We now proceed to express the solution of the above problem as Fourier sine series in x , that is,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{a}, \quad \dots (4)$$

where we have taken undetermined functions $u_n(t)$ of time as multiplying constants. We have selected the above form for $u(x, t)$, since this form automatically satisfies both the boundary conditions given by (2). We now proceed to determine the unknown functions $u_n(t)$ in such a manner so that the resulting value of $u(x, t)$ given by (4) may satisfy (1) and initial condition (3). To this end, we assume the Fourier sine series expansions of $f(x, t)$ given by

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x / a), \quad \text{where} \quad \dots (5)$$

$$f_n(t) = \frac{2}{a} \int_0^a f(\xi, t) \sin(n\pi \xi / a) d\xi \quad \dots (6)$$

$$\text{From (4), } \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} u'_n(t) \sin \frac{n\pi x}{a}, \quad \frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} u_n(t) \left(\frac{n\pi}{a} \right) \cos \frac{n\pi x}{a}, \quad \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} u_n(t) \left(-\frac{n^2 \pi^2}{a^2} \right) \sin \frac{n\pi x}{a}$$

Substituting the above values of $f(x, t)$, $\partial u / \partial t$ and $\partial^2 u / \partial x^2$ in (1), we get

$$\sum_{n=1}^{\infty} u'_n(t) \sin \frac{n\pi x}{a} = -k \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{a^2} u_n(t) \sin \frac{n\pi x}{a} + \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{a}$$

or
$$\sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \left\{ u'_n(t) + (kn^2\pi^2 / a^2)u_n(t) - f_n(t) \right\} = 0 \quad \dots (7)$$

Since $\sin(n\pi x / a)$ are linearly independent functions, the above equation (7) is satisfied only if, the expression in parenthesis vanishes $\forall n \in N$ that is,

$$u'_n(t) + (kn^2\pi^2 / a^2)u_n(t) - f_n(t) = 0 \quad \text{or} \quad du_n(t) / dt + (kn^2\pi^2 / a^2)u_n(t) = f_n(t), \quad \dots (8)$$
 which is a linear differential equation whose integrating factor (I.F.) is given by

$$e^{\int (kn^2\pi^2 / a^2) dt} \text{ i.e., } e^{(kn^2\pi^2 t / a^2)} \text{ and hence the general solution of (8) is given by}$$

$$u_n(t) e^{(kn^2\pi^2 t / a^2)} = \int_0^t f_n(\tau) e^{(kn^2\pi^2 \tau / a^2)} d\tau + c_n, c_n \text{ being an arbitrary constant} \quad \dots (9)$$

In order to determine the value of c_n , we require one initial condition. This condition can be determined with the help of the initial condition (3). Putting $t = 0$ in (4) and using the initial condition (3), we have

$$0 = \sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi x}{a} \quad \Rightarrow \quad u_n(0) = 0, \quad \forall n \in N \quad \dots (10)$$

Now putting $t = 0$ in (9) and using (10), we get $c = 0$. Then, (9) gives

$$u_n(t) = e^{-(kn^2\pi^2 / a^2)t} \int_0^t f(\tau) e^{-(kn^2\pi^2 / a^2)\tau} d\tau \quad \text{or} \quad u_n(t) = \int_0^t e^{-(kn^2\pi^2 / a^2)(t-\tau)} f_n(\tau) d\tau \quad \dots (11)$$

Substituting the above value of $u_n(0)$ in (4), the required solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} \left[\int_0^t e^{-(kn^2\pi^2 / a^2)(t-\tau)} f_n(\tau) d\tau \right] \sin \frac{n\pi x}{a} \quad \dots (12)$$

From (6),
$$f_n(\tau) = \frac{2}{a} \int_0^a f(y, \tau) \sin \frac{n\pi y}{a} dy \quad \dots (13)$$

Using (13), (12) yields
$$u(x, t) = \sum_{n=1}^{\infty} \left[\int_0^t e^{-(kn^2\pi^2 / a^2)(t-\tau)} \left\{ \frac{2}{a} \int_0^a f(y, \tau) \sin \frac{n\pi y}{a} dy \right\} d\tau \right] \sin \frac{n\pi x}{a}$$

Interchanging the order of summation and integration, the above equation reduces to

$$u(x, t) = \int_0^t \int_0^a \left[\frac{2}{a} \sum_{n=1}^{\infty} e^{-(kn^2\pi^2 / a^2)(t-\tau)} \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{a} \right] f(y, \tau) dy d\tau \quad \dots (14)$$

We now define the Green's function as

$$G(x, y, t - \tau) = \frac{2}{a} \sum_{n=1}^{\infty} e^{-(kn^2\pi^2 / a^2)(t-\tau)} \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{a}, \quad 0 \leq x \leq a, 0 \leq y \leq a, t \geq 0 \quad \dots (15)$$

Using the above definition (15), the required solution in terms of Green's function can be re-written as

$$u(x, t) = \int_0^t \int_0^a G(x, y, t - \tau) f(y, \tau) dy d\tau \quad \dots (16)$$

Remarks. Observe that equation (1) of Art. C.8 can be obtained from equation (1) of Art. C.9 by setting $f(x, t) = 0$. Again, we observe that equation (3) of Art. C.9 can be obtained from equation of (3) of Art. C.8 by setting $f(x) = 0$. Furthermore, we observe that the Green's function defined by (15) in Art. C.9 can be obtained from the Green's function defined by (20) in Art. C.8 by replacing t by $t - \tau$. Thus, it follows that the Green's functions for both the problems, i.e. a source free problem with initial

condition (See Art. C.8), and that involving an external heat source, i.e., inhomogeneous equation with zero initial condition (See Art. C.9), are identical. Therefore, it follows that the initial condition can be regarded as equivalent to an instantaneous heat source at time $\tau = 0$.

Physical interpretation of the Green's function $G(x, y, t - \tau)$

In what follows, suppose that the external source function $f(y, \tau)$ is non zero only in the small neighbourhood Δy of the point y_0 and also during the small time interval $\Delta \tau$ around the time point τ_0 and zero otherwise. It follows that the heat source is switched on only in small neighbourhood Δy of the point y_0 and also during the small time interval $\Delta \tau$ around the time τ_0 . We know that the function $f(y, \tau)$ represents the volumetric heat source in units of $\rho\sigma$. In other words, $f(y, \tau) = F(y, \tau) / \rho\sigma$, where $F(y, \tau)$ represents the volumetric heat source. Let Q denote the total heat which is pumped into the system. Then, we have

$$Q = \int_0^t d\tau \int_0^a F(y, \tau) dy = \rho\sigma \int_{\tau_0 - \Delta\tau/2}^{\tau_0 + \Delta\tau/2} d\tau \int_{y_0 - \Delta y/2}^{y_0 + \Delta y/2} f(y, \tau) dy \quad \dots (17)$$

Using (16), the temperature $u(x, t)$ for $t > \tau_0 + \Delta\tau/2$ is given by

$$u(x, t) = \int_{\tau_0 - \Delta\tau/2}^{\tau_0 + \Delta\tau/2} d\tau \int_{y_0 - \Delta y/2}^{y_0 + \Delta y/2} G(x, y, t - \tau) f(y, \tau) dy \quad \dots (18)$$

Recall that we have assumed that $f(y, \tau)$ is non zero only in the small neighbourhood Δy of the point y_0 and also during the small time interval $\Delta \tau$ around the time point τ_0 and zero otherwise. Furthermore, the Green's function $G(x, y, t - \tau)$ is a well defined function in a very small neighbourhood of the point (y_0, τ_0) . Hence, using mean value theorem, we can pull out $G(x, y_0, t - \tau_0)$ from the integral for very small values of Δy and $\Delta \tau$. Thus, from (18), an approximate temperature is given by

$$u(x, t) \cong G(x, y_0, t - \tau_0) \int_{\tau_0 - \Delta\tau/2}^{\tau_0 + \Delta\tau/2} d\tau \int_{y_0 - \Delta y/2}^{y_0 + \Delta y/2} f(y, \tau) dy = G(x, y_0, t - \tau_0) \times \frac{Q}{\rho\sigma}, \text{ using (17)}$$

This can be interpreted as the influence of an instantaneous heat source at time τ_0 at point y_0 . Hence, it follows that the Green's function $G(x, y_0, t - \tau_0)$ represents the temperature at point x at time t due to an instantaneous heat source of unit magnitude (measured in units of $\rho\sigma$, i.e., $Q/\rho\sigma = 1$) applied at point y_0 at time τ_0 .

Remark. Observe that (1) is a linear partial differential equation. Hence, if the magnitude of the external source function is doubled, then the temperature $u(x, t)$ is also doubled everywhere, and so on.

C.10. The use of Green's function in the determination of the solution of heat equation (or the diffusion equation)

Here we propose to find the solution $u(\mathbf{r}, t)$ of the heat equation

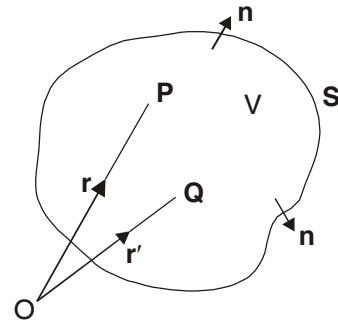
$$\partial u / \partial t = k \nabla^2 u \quad \dots (1)$$

in the volume V , which is bounded by the simple surface S , subject to boundary condition

$$u(\mathbf{r}, t) = f(\mathbf{r}, t), \text{ if } \mathbf{r} \in S \quad \dots (2)$$

$$\text{and initial condition } u(\mathbf{r}, 0) = g(\mathbf{r}), \text{ if } \mathbf{r} \in V \quad \dots (3)$$

where $\mathbf{r}$ is the position vector of a point P in volume V and $\mathbf{r}'$ is the position vector of another point Q in V as shown in the adjoining figure.



For the given initial boundary value problem given by (1), (2) and (3), we define the Green's function $G(\mathbf{r}, \mathbf{r}', t - t')$, $t > t'$ which satisfies the equation

$$\partial G(\mathbf{r}, \mathbf{r}', t - t') / \partial t = k \nabla^2 G(\mathbf{r}, \mathbf{r}', t - t') \quad \dots (4)$$

$$\text{the boundary condition} \quad G(\mathbf{r}, \mathbf{r}', t - t') = 0, \quad \text{if } \mathbf{r}' \in S \quad \dots (5)$$

$$\text{and the initial condition that} \quad \lim_{t \rightarrow t'} G(\mathbf{r}, \mathbf{r}', t - t') = 0$$

at all points of V except at the point $\mathbf{r}$ where $G(\mathbf{r}, \mathbf{r}', t - t')$ takes the form

$$\frac{1}{8\{\pi k(t - t')\}^{3/2}} \exp\left\{-\frac{|\mathbf{r} - \mathbf{r}'|^2}{4k(t - t')}\right\} \quad \dots (6)$$

Since $G(\mathbf{r}, \mathbf{r}', t - t')$ depends only on t , it follows that it can be treated as a function of $t - t'$. Therefore, equation (4) can be re-written in the following equivalent form :

$$\partial G(\mathbf{r}, \mathbf{r}', t - t') / \partial t' + k \nabla^2 G(\mathbf{r}, \mathbf{r}', t - t') = 0 \quad \dots (7)$$

From these equations, the physical interpretation of Green's function $G(\mathbf{r}, \mathbf{r}', t - t')$ is as follows: $G(\mathbf{r}, \mathbf{r}', t - t')$ is the temperature at the point $P'(\mathbf{r}')$ at time t due to an instantaneous point source of unit strength generated at time t' at the point $P(\mathbf{r})$, the solid being initially at zero temperature, and its surface being maintained at zero temperature.

Observe the time t' lies within the interval of t for which equations (1) and (2) are valid. Therefore, (1) and (2) may be re-written in the form

$$\partial u / \partial t' = k \nabla^2 u, \quad t' < t \quad \dots (8)$$

$$u(\mathbf{r}', t') = f(\mathbf{r}', t'), \quad \text{if } \mathbf{r}' \in S \quad \dots (9)$$

$$\text{Now, we have} \quad \frac{\partial(uG)}{\partial t'} = u \frac{\partial G}{\partial t'} + G \frac{\partial u}{\partial t'} = k(G \nabla^2 u - u \nabla^2 G), \text{ by (7) and (8)} \quad \dots (10)$$

Let ε be an arbitrary positive constant. Now, from (10), we have

$$\int_0^{t-\varepsilon} \left\{ \iiint_V \frac{\partial(uG)}{\partial t'} dV \right\} dt' = k \int_0^{t-\varepsilon} \left\{ \iiint_V (G \nabla^2 u - u \nabla^2 G) dV \right\} dt' \quad \dots (11)$$

Now, interchanging the order of integration on the left hand side of (11) and applying second Green's identity (refer Art. A.3 in Appendix A) on the right hand side of (11), we have

$$\iiint_V \left\{ \int_0^{t-\varepsilon} \frac{d(uG)}{dt} dt' \right\} dV = k \int_0^{t-\varepsilon} \left\{ \iint_S \left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n} \right) dS \right\} dt' \quad \dots (12)$$

where $\partial / \partial n$ denotes differentiation along the outward-drawn normal to S .

From (2) and (5), we have $G = 0$ on S and $u(\mathbf{r}, t) = f(\mathbf{r}, t)$ on S .

$$\therefore \text{R.H.S of (12)} = -k \int_0^t \left\{ \iint_S f(\mathbf{r}', t) \frac{\partial G}{\partial n} dS \right\} dt' \quad \dots (13)$$

$$\begin{aligned} \text{Now, L.H.S. of (12)} &= \iiint_V [uG]_{t'=0}^{t-\varepsilon} dV = \iiint_V [uG]_{t'=t-\varepsilon} dV - \iiint_V [uG]_{t'=0} dV \\ &= u(\mathbf{r}, t) \iiint_V [G(\mathbf{r}, \mathbf{r}', t - t')]_{t'=t-\varepsilon} dV - \iiint_V G(\mathbf{r}, \mathbf{r}', t) g(\mathbf{r}') dV, \text{ by (3)} \end{aligned}$$

Using the expression (6) for $G(\mathbf{r}, \mathbf{r}', t - t')$, we can easily verify that

$$\iiint_V [G(\mathbf{r}, \mathbf{r}', t - t')]_{t'=t-\varepsilon} dV = \iiint_V \frac{1}{8(\pi k \varepsilon)^{3/2}} \exp\left\{-\frac{|\mathbf{r} - \mathbf{r}'|^2}{4\varepsilon}\right\} dV = 1, \text{ as } \varepsilon \rightarrow 0$$

Thus, the L.H.S of (12) = $u(\mathbf{r}, t) - \iiint_V G(\mathbf{r}, \mathbf{r}', t) g(\mathbf{r}') dV$... (14)

Now, from (12), (13) and (14), we obtain

$$u(\mathbf{r}, t) - \iiint_V G(\mathbf{r}, \mathbf{r}', t) g(\mathbf{r}') dV = -k \int_0^t \left\{ \iint_S f(\mathbf{r}', t') \frac{\partial G}{\partial n} dS \right\} dt'$$

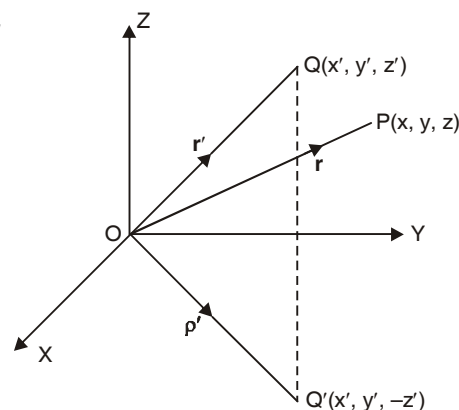
giving $u(\mathbf{r}, t) = \iiint_V g(\mathbf{r}') G(\mathbf{r}, \mathbf{r}', t) dV - k \int_0^t \left\{ \iint_S f(\mathbf{r}', t') \frac{\partial G}{\partial n} dS \right\} dt'$, ... (15)

which is the solution of the boundary value problem given by (1), (2) and (3)

An illustrative example. If the surface $z = 0$ of the semi-infinite solid $z \geq 0$ is maintained at temperature $f(x, y, t)$ for $t > 0$, and if the initial temperature of the solid is $g(x, y, z)$, determine the distribution of temperature in the solid.

Hint. Let V denote the half space $z \geq 0$ and let S denote the entire xy -plane. Let $P(\mathbf{r})$ be any point in the semi-infinite solid $z \geq 0$ and let $Q(\mathbf{r}')$ be another point in the same solid. Let $Q'(\mathbf{r}')$ be the image of Q in the xy -plane. Then, the appropriate Green's function for the present problem can be taken as

$$G(\mathbf{r}, \mathbf{r}', t - t') = \frac{1}{8\{\pi k(t - t')\}^{3/2}} \left[\exp\left\{-\frac{|\mathbf{r} - \mathbf{r}'|^2}{4k(t - t')}\right\} - \exp\left\{-\frac{|\mathbf{r} - \mathbf{r}''|^2}{4k(t - t')}\right\} \right]$$



For this Green's function, we have

$$\frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial z'}\right)_{z'=0} = -\frac{z}{8\pi^{3/2}k^{5/2}(t - t')^{3/2}} \exp\left\{-\frac{(x - x')^2 + (y - y')^2 + z^2}{4k(t - t')}\right\}$$

Hence, using result (15) of Art. C. 7, the required solution is given by

$$u(\mathbf{r}, t) = \frac{1}{8(\pi k)^{3/2}} \iiint_V g(\mathbf{r}') (e^{-|\mathbf{r} - \mathbf{r}'|^2/4kt} - e^{-|\mathbf{r} - \mathbf{r}''|^2/4kt}) dV + \frac{z}{8(\pi k)^{3/2}} \int_0^t \iint_S \frac{f(x', y', t')}{(t - t')^{5/2}} \exp\left\{-\frac{(x - x')^2 + (y - y')^2 + z^2}{4k(t - t')}\right\} dx' dy' dt'$$

C.11. The use of Green's function in the determination or the solution of heat equation (or diffusion equation) for infinite rod

Determine Green's function for the problem of heat flow in an infinite rod described by partial differential equation $\partial u / \partial t = k(\partial^2 u / \partial x^2)$, $-\infty < x < \infty$, $t > 0$ subject to the initial condition $u(x, 0) = f(x)$, $-\infty < x < \infty$. Hence, find the solution of the given problem in terms of Green's function.

Solution. Given $\partial u / \partial t = k(\partial^2 u / \partial x^2)$, $-\infty < x < \infty$, $t > 0$... (1)

and the initial condition $u(x, 0) = f(x)$, $-\infty < x < \infty$ (2)

* Suppose that (1) has solution of the form $u(x, t) = X(x) T(t)$, ... (3)

where $X(x)$ is a function of x alone and $T(t)$ that of t alone. Substituting this value of u in (1) yields

$$XT' = k X'' T \quad \text{or} \quad X''/X = T'/kT \quad \dots (4)$$

* For more details refers chapters 1 and 2 in part III of author's "Advanced Differential Equation", published by S. Chand and Co, Neal Delhi.

Since x and t are independent variables, (4) can only be true if its each side is equal to the same constant, say μ . Hence, (4) yields the following two ordinary differential equations ;

$$X'' - \mu X = 0 \quad \dots (5)$$

and

$$T' = \mu k T \quad \dots (6)$$

$$\text{Solving (1),} \quad T = C e^{k\mu t}, \quad C \text{ being an arbitrary constant} \quad \dots (7)$$

When $\mu > 0$, (7) shows that T and hence $u(x, t)$ grows exponentially with time. In view of realistic physical considerations, it is appropriate to suppose that $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$, while $|u(x, t)| < M$ as $x \rightarrow \infty$. Therefore, in order that $u(x, t)$ remains bounded, μ must be negative and so we take $\mu = -\lambda^2$, where $\lambda \geq 0$. Thus, (5) takes the form $X'' + \lambda^2 X = 0$ whose general solution is

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x, \quad C_1 \text{ and } C_2 \text{ being arbitrary constants} \quad \dots (8)$$

$$\text{Now, setting } \mu = -\lambda^2, (7) \text{ yields} \quad T = C e^{-k\lambda^2 t} \quad \dots (9)$$

Therefore a solution $u(x, t, \lambda)$ of (1) is given by

$$u(x, t, \lambda) = XT = (A \cos \lambda x + B \sin \lambda x) e^{-k\lambda^2 t}, \text{ by (3) (8) and (9)} \quad \dots (10)$$

where $A (= C_1 C)$ and $B (= C_2 C)$ are new arbitrary constants

Since $f(x)$ is not necessarily a periodic function, we have to use Fourier integral in place of Fourier series in our given boundary value problem. Again, since A and B are arbitrary, we may treat them as functions of λ and write $A = A(\lambda)$ and $B = B(\lambda)$. In the given problem, since no boundary condition is provided to limit our choice of λ , hence we must consider all possible values. Using the principle of superposition, this summation of the product solutions leads to the relation

$$u(x, t) = \int_0^\infty u(x, t, \lambda) d\lambda = \int_0^\infty \{A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x\} e^{-k\lambda^2 t} d\lambda \quad \dots (11)$$

which is the general solution of (1).

Putting $t = 0$ in (11) and using the initial condition (2), we have

$$f(x) = \int_0^\infty \{A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x\} d\lambda \quad \dots (12)$$

From the **Fourier integral theorem, we have

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(y) \cos \lambda(x-y) dy \right] d\lambda$$

$$\text{or} \quad f(x) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(y) (\cos \lambda x \cos \lambda y + \sin \lambda x \sin \lambda y) dy \right] d\lambda$$

$$\text{or} \quad f(x) = \frac{1}{\pi} \int_0^\infty \left[\cos \lambda x \int_{-\infty}^\infty f(y) \cos \lambda y dy + \sin \lambda x \int_{-\infty}^\infty f(y) \sin \lambda y dy \right] d\lambda \quad \dots (13)$$

$$\text{Let } A(\lambda) = \frac{1}{\pi} \int_{-\infty}^\infty f(y) \cos \lambda y dy \quad \text{and} \quad B(\lambda) = \frac{1}{\pi} \int_{-\infty}^\infty f(y) \sin \lambda y dy \quad \dots (14)$$

Using (14), (13) may be re-written as

$$f(x) = \int_0^\infty \{A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x\} d\lambda \quad \dots (15)$$

Comparing equations (12) and (15), equation (12) can be re-written as

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(y) \cos \lambda(x-y) dy \right] d\lambda \quad \dots (16)$$

** Refer Art. 1.3 in part IV B in author's "Advanced Differential Equations", published by S. Chand and Co., New Delhi

Then, from equation (11), we have

$$u(x, t) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(y) \cos \lambda(x-y) e^{-k\lambda^2 t} dy \right] d\lambda \quad \dots (17)$$

Changing the orders of integration on R.H.S of (17), we have

$$u(x, t) = \frac{1}{\pi} \int_{-\infty}^\infty f(y) \left\{ \int_0^\infty e^{-k\lambda^2 t} \cos \lambda(x-y) d\lambda \right\} dy \quad \dots (18)$$

From Integral Calculus,
$$\int_0^\infty e^{-z^2} \cos 2bz \, dz = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad \dots (19)$$

Assume that $z = \lambda\sqrt{kt}$, $dz = \sqrt{kt} \, d\lambda$ and $2bz = \lambda(x-y)$ $\dots (20)$

Then, from (20), $2b\lambda\sqrt{kt} = \lambda(x-y)$ so that $b = (x-y) / 2\sqrt{kt}$ $\dots (21)$

Using (20) and (21), (19) reduces to

so that
$$\int_0^\infty e^{-k\lambda^2 t} \cos \lambda(x-y) \cdot \sqrt{kt} \, d\lambda = \frac{\sqrt{\pi}}{2} e^{-\{(x-y)^2 / 4kt\}}$$

$$\int_0^\infty e^{-k\lambda^2 t} \cos \lambda(x-y) d\lambda = \frac{\sqrt{\lambda}}{(4kt)^{1/2}} e^{-\{(x-y)^2 / 4kt\}} \quad \dots (22)$$

Using (22), (18) yields
$$u(x, t) = \frac{1}{(4k\pi t)^{1/2}} \int_{-\infty}^\infty f(y) e^{-\{(x-y)^2 / 4kt\}} dy \quad \dots (23)$$

Hence, if $f(y)$ is bounded for all real values of y , then the required solution of the boundary value problem is given by (23)

Let
$$G(x-y, kt) = \frac{1}{(4k\pi t)^{1/2}} e^{-\{(x-y)^2 / 4kt\}} \quad \dots (24)$$

Using (24), (23) may be re-written as

$$u(x, t) = \int_{-\infty}^\infty G(x-y, kt) f(y) dy, \quad \dots (25)$$

The function $G(x-y, kt)$ defined by (24) is known as the Green's function of heat transfer in an infinite rod.

Exercise

1. Use the method of images and prove that the harmonic Green's function for the half-space $z \geq 0$ is $G(\mathbf{r}, \mathbf{r}') = 1/(4\pi r) - 1/(4\pi r')$, where $r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$ and $r'^2 = (x-x')^2 + (y-y')^2 + (z+z')^2$.

2. Show that the Green's function $G(x, y, t)$ for the heat flow problem in semi-infinite rod described by the partial differential equation $\partial u / \partial t = k(\partial^2 u / \partial x^2)$, $x > 0$, $t > 0$ subject to boundary condition $u(0, t) = 0$, $t > 0$ and initial condition $u(x, 0) = f(x)$, $x > 0$ is given in the form

$$G(x, y, t) = [e^{-\{(x-y)^2 / 4t\}} - e^{-\{(x+y)^2 / 4t\}}] / (4\pi t)^{1/2}$$

3. Using Green's function technique solve $\nabla^2 u = 0$ in the circle $|r| < 3$ subject to the condition $u = f(\theta)$ on the circle $|r| = 3$.

Ans. Green's function $= G(\mathbf{r}, \mathbf{r}') = \frac{1}{2} \log_e \frac{81 + r^2 r'^2 - 18rr' \cos(\theta' - \theta)}{9\{r^2 + r'^2 - 2rr' \cos(\theta' - \theta)\}}$

and solution is given by

$$u(r, \theta) = \frac{9-r^2}{2\pi} \int_0^{2\pi} \frac{f(\theta')}{9+r^2-6r\cos(\theta'-\theta)} d\theta'$$

4. Find the Green's function of the Dirichlet problem for semi-infinite space $y \geq 0$, $-\infty < x < \infty$, $-\infty < z < \infty$ and $u = f(x, z)$ on $y = 0$ and solve this problem.

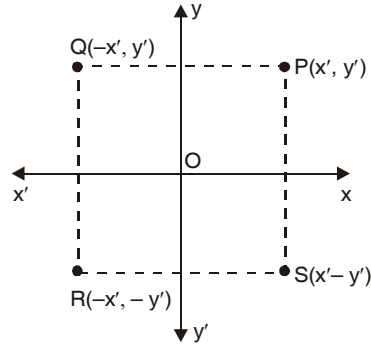
Ans. Green's function = $G(\mathbf{r}, \mathbf{r}') = \{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{-1/2} - \{(x-x')^2 + (y+y')^2 + (z-z')^2\}^{-1/2}$ and solution is given by $u(x, y, z) = \frac{y}{2\pi} \int_{x'=-\infty}^{\infty} \int_{z'=-\infty}^{\infty} \frac{f(x', z') dx' dz'}{\{(x-x')^2 + y^2 + (z-z')^2\}^{3/2}}$

5. Find the Green's function for the following initial boundary value problem:

$\partial^2 u / \partial t^2 = c^2 (\partial^2 u / \partial x^2) + f(x, t)$ $0 < x < a$, $t > 0$ with initial conditions $u(x, 0) = u_t(x, 0) = 0$, $0 < x < a$ and boundary conditions $u(0, t) = u_x(a, t) = 0$, $t > 0$.

6. Determine the Green's function for the Robin's problem on the quarter infinite plane described by $\nabla^2 u = f(x, y)$, $x > 0$, $y > 0$ subject to the conditions $u = g(y)$ on $x = 0$ and $\partial u / \partial n = h(y)$ on $y = 0$

Solution. In xy -plane, let us consider a singularity at (x, y) . Then, we consider its image over one side, again reflect the image over the image of that side, and so on. This process is stopped after an even number of reflections because then we come back to the original domain. Let $Q(-x', y')$ be the image of $P(x', y')$ in the y -axis and let $R(-x', -y')$ and $S(x', -y')$ be the images of Q and P in the x -axis. Then the Green's function $G(x, y)$ is obtained by placing three unit charges at proper places. We place, a positive charge at $(-x', y')$ and negative charges at $(-x', -y')$ and $(x', -y')$. Then, we obtain



$$G(x, y) = \frac{1}{4\pi} \log_e \frac{\{(x-x')^2 + (y-y')^2\} \{(x-x')^2 + (y+y')^2\}}{\{(x+x')^2 + (y-y')^2\} \{(x+x')^2 + (y+y')^2\}} \quad \dots (2)$$

It can easily be verified that (i)

$\nabla^2 G = 0$ except at the source point (x', y')

(ii) $G = 0$ on $x = 0$

(iii) $\partial G / \partial n = 0$ on $y = 0$

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$$\text{From (4), } \left. \begin{aligned} \frac{\partial u}{\partial t} &= \sum_{n=1}^{\infty} u'_n(t) \sin \frac{n\pi x}{a}, & \frac{\partial^2 u}{\partial t^2} &= \sum_{n=1}^{\infty} u''_n(t) \sin \frac{n\pi x}{a}, \\ \frac{\partial u}{\partial x} &= \sum_{n=1}^{\infty} \left(\frac{n\pi}{a} \right) u_n(t) \cos \frac{n\pi x}{a} & \text{and } \frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} \left(-\frac{n^2 \pi^2}{a^2} \right) u_n(t) \sin \frac{n\pi x}{a} \end{aligned} \right\} \dots (8)$$

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$$\text{From (4), } \left. \begin{aligned} \frac{\partial u}{\partial t} &= \sum_{n=1}^{\infty} u'_n(t) \sin \frac{n\pi x}{a}, & \frac{\partial^2 u}{\partial t^2} &= \sum_{n=1}^{\infty} u''_n(t) \sin \frac{n\pi x}{a}, \\ \frac{\partial u}{\partial x} &= \sum_{n=1}^{\infty} \left(\frac{n\pi}{a} \right) u_n(t) \cos \frac{n\pi x}{a} & \text{and } \frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} \left(-\frac{n^2 \pi^2}{a^2} \right) u_n(t) \sin \frac{n\pi x}{a} \end{aligned} \right\} \dots (8)$$

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We now define the Green's function as

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$f(y, \tau) = F(y, \tau) / \rho\sigma$, where $F(y, \tau)$ represents the volumetric heat source. Let Q denote the total heat which is pumped into the system. Then, we have

Page D.1

$$y(x) = \int_0^l G_M(x, t) \phi(t) dt + w(x) \int_0^l w(x) y(x) dx$$

or

$$y(x) = \int_0^l G_m(x, t) \{f(t) + \lambda y(t)\} dt + \frac{1}{\sqrt{l}} \int_0^l \frac{1}{\sqrt{l}} y(x) dx$$

[Note that from Ex. 1, page 11.52, $w(x) = 1/\sqrt{l}$]

or

$$y(x) = \int_0^l G_M(x, t) f(t) dt + \lambda \int_0^l G_M(x, t) y(t) dt + C,$$

where $C \left(= \frac{1}{l} \int_0^l y(x) dx \right)$ is an arbitrary constant.

Page D.4

$$y(x) = \int_0^l G_M(x, t) \phi(x) dt + C w(x) \quad \text{or} \quad y(x) = \int_0^l G_M(x, t) + f(t) dt + \frac{C}{\sqrt{2}}$$

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$$\int_0^\pi w(x) \phi(x) dx = 0 \quad \text{or} \quad \int_0^\pi \sin x f(x) dx = 0 \quad \text{or} \quad \int_0^\pi \sin t f(t) dt = 0]$$

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For $x \neq t$, (6) reduces to $LG_M = -w_1(x) w_1(t) - w_2(x) w_2(t) \quad \dots(7)$

Page D.10

or $y(x) = (E_1 / \sqrt{2}) \times \cos kx + (F_1 / \sqrt{2}) \times \sin kx - \frac{\sin kx}{2\pi k} \int_0^{2\pi} t \cos kt f(t) dt$

where $a = (E_1 / \sqrt{2} + E_2 / 2\pi k)$ and $b (= F / \sqrt{2} - F_2 / 2\pi k)$ are new arbitrary constants

APPENDIX D

ADDITIONAL PROBLEMS BASED ON MODIFIED (OR GENERALIZED) GREEN'S FUNCTION

[**Note:** The reader is advised to study the entire matter given on pages 11.48–11.62 of chapter 11 before reading the matter of this Appendix D. However, references to the above mentioned matter is given at proper places]

D.1 Additional problems based on Art. 11.12, 11.13 and 11.14 of Chapter 11

Ex.1. Obtain the generalized Green's function for the boundary value problem $y'' = 0$, $y'(0) = y'(l) = 0$, $0 \leq x \leq l$.

Sol. Given $-(d^2y/dx^2) = 0$, $y'(0) = y'(l) = 0$, $0 \leq x \leq l$... (1)

Here $-(d^2/dx^2)$ is a self adjoint operator and hence (1) is a self adjoint system. Refer Ex. 1, page 11.52. Start from equation (2) and proceed upto equation (17). Thus, the required generalized Green's function $G_M(x, t)$ is given by

$$G_M(x, t) = \frac{l}{3} + \frac{x^2 + l^2}{2l} - \begin{cases} l, & \text{if } 0 \leq x < t \\ x, & \text{if } t < x \leq l \end{cases}$$

Ex.2. Transform the boundary value problem $-(d^2y/dx^2 + \lambda y) = f(x)$, $y(0) = y'(l) = 0$, $0 \leq x \leq l$ into an integral equation.

Sol. Given $-(d^2y/dx^2) = f(x) + \lambda y$, $y'(0) = y'(l) = 0$, $0 \leq x \leq l$... (1)

Here $-(d^2y/dx^2)$ is a self adjoint operator. Consider the associated self adjoint system

$$-y'' = 0, \quad 0 \leq x \leq l \quad \dots (2)$$

with the boundary conditions : $y'(0) = 0$... (3a)

and $y'(l) = 0$... (3b)

Now, proceed exactly as in Ex. 1, page 11.52 starting with equation (2) upto equation (17). Then, the modified Green's function $G_M(x, t)$ of the given boundary value problem is given by

$$G_M(x, t) = \frac{l}{3} + \frac{x^2 + l^2}{2l} - \begin{cases} l, & \text{if } 0 \leq x < t \\ x, & \text{if } t < x \leq l \end{cases}$$

Comparing (1) with $Ly = \phi(x)$, here $\phi(x) = f(x) + \lambda y(x)$. Hence, using result (11b) of Art. 11.13, the required integral equation is given by

$$y(x) = \int_0^l G_M(x, t) \phi(t) dt + w(x) \int_0^l w(x) y(x) dx$$

or
$$y(x) = \int_0^l G_m(x, t) \{f(t) + \lambda y(t)\} dt + \frac{1}{\sqrt{l}} \int_0^l \frac{1}{\sqrt{l}} y(x) dx$$

[Note that from Ex. 1, page 11.52, $w(x) = 1/\sqrt{l}$]

or
$$y(x) = \int_0^l G_M(x, t) f(t) dt + \lambda \int_0^l G_M(x, t) y(t) dt + C,$$

where $C \left(= \frac{1}{l} \int_0^l y(x) dx \right)$ is an arbitrary constant.

Ex.3. Determine the modified Green's function of the system $\frac{d}{dx} \left\{ (1-x^2) \frac{dy}{dx} \right\} = f(x)$, $-1 \leq x \leq 1$ with boundary conditions that $y(x)$ is finite at $x = 1$ and $x = -1$. Hence transform the given boundary value problem into an integral equation.

Sol. Given
$$-\frac{d}{dx} \left\{ (1-x^2) \frac{dy}{dx} \right\} = -f(x), \quad -1 \leq x \leq 1 \quad \dots (1)$$

with boundary conditions: $y(1) = \text{finite}$ and $y(-1) = \text{finite}$... (2)

Here $-(d/dx)\{(1-x^2)(d/dx)\}$ is a self adjoint operator

Consider the associated self adjoint system
$$-\frac{d}{dx} \left\{ (1-x^2) \frac{dy}{dx} \right\} = 0, \quad -1 \leq x \leq 1 \quad \dots (3)$$

with boundary conditions: $y(-1) = \text{finite}$... (4a)

and $y(1) = \text{finite}$... (4b)

Now, proceed as in Ex. 3, page 11.56 starting with (3) and stop at (15). Thus, the modified Green's function $G_M(x, t)$ of the given boundary value problem is given by

$$G_M(x, t) = \log 2 - \frac{1}{2} - \begin{cases} (1/2) \times \log \{(1-x)/(1+t)\}, & -1 < x \leq t \\ (1/2) \times \log \{(1+x)/(1-t)\}, & t < x \leq 1 \end{cases}$$

Comparing (1) with $Ly = \phi(x)$, we have $\phi(x) = -f(x)$. Hence, using result (11b) of Art. 11.13, the required integral equation is given by

$$y(x) = \int_{-1}^1 G_m(x, t) \phi(t) dt + w(x) \int_{-1}^1 w(x) y(x) dx$$

or
$$y(x) = -\int_{-1}^1 G_m(x, t) f(t) dt + \frac{1}{2} \int_{-1}^1 y(x) dx, \text{ as } w(x) = \frac{1}{\sqrt{2}}$$

or
$$y(x) = C - \int_{-1}^1 G_M(x, t) f(t) dt, \text{ where } C = \frac{1}{2} \int_{-1}^1 y(x) dx$$

Ex.4. Find the generalized Green's function of the boundary value problem $-k(d^2y/dx^2) = f(x)$ with $y'(0) = y'(l) = 0$, $0 \leq x \leq l$. Hence transform the given boundary value problem into an integral equation.

Sol. Given
$$-k(d^2y/dx^2) = f(x), \quad y'(0) = y'(l) = 0, \quad 0 \leq x \leq l \quad \dots (1)$$

Here $-k(d^2/dx^2)$ is a self adjoint operator. Consider the associated self adjoint system :

$$-ky'' = 0, \quad 0 \leq x \leq l \quad \dots (2)$$

with boundary conditions: $y'(0) = 0$... (3a)

and $y'(l) = 0$... (3b)

The general solution of (2) is
$$y(x) = Ax + B \quad \dots (4)$$

From (4),
$$y'(x) = A \quad \dots (5)$$

Putting $x = 0$ and $x = l$ in (5) and using (3a) and (3b), we get $A = 0$. Hence the boundary value problem given by (2), (3a) and (3b) has a non-zero solution $y(x) = B$, where B is an arbitrary constant

Here, $\|y(x)\| = \text{norm of } y(x) = \left\{ \int_0^l [y(x)]^2 dx \right\}^{1/2} = B\sqrt{l}$

Let $w(x) = y(x) / \|y(x)\| = B / (B\sqrt{l}) = 1/\sqrt{l}$... (6)

Thus, $w(x)$ is a non-zero normalized solution of the boundary value problem given by (2), (3a) and (3b). Clearly $\int_0^l [w(x)]^2 dx = 1$... (7)

Then, for $x \neq t$, the required generalized Green's function $G_M(x, t)$ must satisfy

$$-k(d^2 G_M / dx^2) = -w(x) w(t) \quad \text{or} \quad d^2 G_M / dx^2 = 1/l \quad \dots (8)$$

The general solution of (8) is of the form $G_M(x, t) = Ax + B + x^2 / 2kl$

Hence, we take $G_M(x, t) = \begin{cases} a_1 x + a_2 + x^2 / 2kl, & 0 \leq x < t \\ b_1 x + b_2 + x^2 / 2kl, & t < x \leq l \end{cases}$... (9)

From (9), $\frac{\partial G_M}{\partial x} = \begin{cases} a_1 + x / kl, & 0 \leq x < t \\ b_1 + x / kl, & t < x \leq l \end{cases}$... (10)

In addition to the above property (9), the proposed generalized Green's function must satisfy the following properties.

(i) Since $G_M(x, t)$ must satisfy the boundary conditions (3a) and (3b), (10) gives

$$(\partial G_M / \partial x)_{x=0} = 0 \quad \text{and} \quad (\partial G_M / \partial x)_{x=l} = 0$$

Hence, $a_1 = 0$ and $b_1 + (1/k) = 0$ so that $a_1 = 0$ and $b_1 = -(1/k)$... (11)

(ii) $G_M(x, t)$ is continuous at $x = t$, that is,

$$a_1 t + a_2 + t^2 / 2kl = b_1 t + b_2 + t^2 / 2kl \quad \text{or} \quad b_2 = a_2 + (t/k), \text{ using (11)} \quad \dots (12)$$

(iii) $(\partial G_M / \partial x)_{x=t+0} - (\partial G_M / \partial x)_{x=t-0} = 1/p(t)$, where $p(x) = -k$... (13)

or $b_1 + t/kl - (a_1 + t/kl) = -(1/k)$, using (10)

or $-(1/k) = -(1/k)$, using (11). Thus, (13) is identically satisfied.

Substituting the values of a_1 , b_1 and b_2 given by (11) and (12) in (9), we get

$$G_M(x, t) = \begin{cases} a_2 + x^2 / 2kl, & 0 \leq x < t \\ a_2 + (t-x)/k + x^2 / 2kl, & t < x \leq l \end{cases} \quad \dots (14)$$

(iv) In order that $G_M(x, t)$ may be symmetric, we must have

$$\int_0^l G_M(x, t) w(x) dx = 0 \quad \text{or} \quad \int_0^l G_M(x, t) dx = 0, \text{ as } w(x) = \frac{1}{\sqrt{l}}$$

or $\int_0^t G_M(x, t) dx + \int_t^l G_M(x, t) dx = 0$ or $\int_0^t \left(a_2 + \frac{x^2}{2kl} \right) dx + \int_t^l \left(a_2 + \frac{t}{k} - \frac{x}{k} + \frac{x^2}{2kl} \right) dx$, using (14)

$$\left[a_2 x + \frac{x^3}{6kl} \right]_0^t + \left[a_2 x + \frac{tx}{k} - \frac{x^2}{2k} + \frac{x^3}{6kl} \right]_t^l = 0$$

$$\text{or} \quad a_2 l + \frac{t^2}{6kl} + a_2 t + \frac{tl}{k} - \frac{l^2}{2k} + \frac{l^2}{6k} - a_2 t - \frac{t^2}{k} + \frac{t^2}{2k} - \frac{t^3}{6kl} = 0$$

or $a_2 l = \frac{t^2}{2k} - \frac{tl}{k} + \frac{l^2}{3k}$ so that $a_2 = \frac{1}{kl} \left(\frac{t^2}{2} - tl + \frac{l^2}{3} \right)$

Substituting the above value of a_2 in (14), we have

$$G_M(x, t) = \begin{cases} \frac{1}{kl} \left(\frac{t^2}{2} - tl + \frac{l^2}{3} \right) + \frac{x^2}{2kl}, & 0 \leq x < t \\ \frac{1}{kl} \left(\frac{t^2}{2} - tl + \frac{l^2}{3} \right) + \frac{t-x}{k} + \frac{x^2}{2kl}, & t < x \leq l \end{cases}$$

or $G_M(x, t) = \begin{cases} (x^2 + t^2) / 2kl - (t/k) + (l/3k), & 0 \leq x \leq t \\ (x^2 + t^2) / 2kl - (x/k) + (l/3k), & t \leq x \leq l \end{cases} \quad \dots(15)$

Using result (11a) of Art. 11.13, the required integral equation is given by

$$y(x) = \int_0^l G_M(x, t) \phi(x) dt + C w(x) \quad \text{or} \quad y(x) = \int_0^l G_M(x, t) + f(t) dt + \frac{C}{\sqrt{2}}$$

or $y(x) = A + \int_0^l G_M(x, t) f(t) dt$, A being an arbitrary constant

Ex.5. Transform the boundary value problem $y'' + y = f(x)$, $y(0) = y(\pi) = 0$, into an integral equation.

Sol. Given $-(d^2 y / dx^2 + y) = -f(x)$, $y(0) = y(\pi) = 0$, $0 \leq x \leq \pi$...(1)

Here $-(d^2 y / dx^2 + 1)$ is a self adjoint operator.

Consider the associated self adjoint system $-(y'' + y) = 0$, $0 \leq x \leq \pi$...(2)

with the boundary conditions: $y(0) = 0$...(3a)

and $y(\pi) = 0$...(3b)

The general solution of (2) is $y(x) = A \cos x + B \sin x$...(4)

Putting $x = 0$ and $x = \pi$ in (4) and using (3a) and (3b), we get $A = 0$. Hence, the boundary value problem given by (2), (3a) and (3b) has a non-trivial solution $y(x) = B \sin x$, B being an arbitrary constant

$$\begin{aligned} \text{Here, } \|y(x)\| &= \left\{ \int_0^\pi [(y(x))]^2 dx \right\}^{1/2} = \left\{ \int_0^\pi (B^2 \sin^2 x) dx \right\}^{1/2} = B \left\{ \int_0^\pi \frac{1 - \cos 2x}{2} dx \right\}^{1/2} \\ &= B \left\{ \left[x/2 - (\sin 2x)/4 \right]_0^\pi \right\}^{1/2} = B \times (\pi/2)^{1/2} \end{aligned}$$

Let $w(x) = y(x) / \|y(x)\| = (2/\pi)^{1/2} \times \sin x$...(5)

so that $w(x)$ is a non-zero normalized solution of the problem given by (2), (3a) and (3b). Clearly,

$$\int_0^\pi [w(x)]^2 dx = 1 \quad \dots(6)$$

Then for $x \neq t$, the modified Green's function of the given boundary value problem must satisfy

$$-(d^2 G_M / dx^2 + G_M) = -w(x) w(t) \quad \text{or} \quad (D^2 + 1) G_M = (2/\pi) \times \sin x \sin t, \text{ using (6)} \quad \dots(7)$$

where $D \equiv d/dx$. The complementary function (= C.F.) of (7) is $C \sin x + D \cos x$

Again, the particular integral (P.I.) of (7)

$$= \frac{1}{D^2 + 1} \frac{2}{\pi} \sin x \sin t = \frac{2 \sin t}{\pi} \frac{1}{D^2 + 1} \sin x = \frac{2 \sin t}{\pi} \times \left(-\frac{x \cos x}{2} \right) = -\frac{x}{\pi} \sin t \cos x$$

Thus, the general solution of (7) is of the form

$$G_M(x, t) = C \sin x + D \cos x - (x / \pi) \times \sin t \cos x$$

Hence, we take
$$G_M(x, t) = -\frac{x}{\pi} \sin t \cos x + \begin{cases} C_1 \sin x + D_1 \cos x, & 0 \leq x < t \\ C_2 \sin x + D_2 \cos x, & t < x \leq \pi \end{cases} \quad \dots (8)$$

In addition to the above property (8), the proposed modified Green's function must satisfy the following properties.

(i) Since $G_M(x, t)$ must satisfy the boundary conditions (3a) and (3b), (8) gives

$$G(0, t) = 0 \quad \text{and} \quad G(\pi, t) = 0$$

Hence, $D_1 = 0$ and $\sin t - D_2 = 0$ so that $D_1 = 0$ and $D_2 = \sin t$. So, (8) yields

$$G_M(x, t) = -\frac{x}{\pi} \sin t \cos x + \begin{cases} C_1 \sin x, & 0 \leq x < t \\ C_2 \sin x + \cos x \sin t, & t < x \leq \pi \end{cases} \quad \dots (9)$$

(ii) $G_M(x, t)$ must be continuous at $x = t$. Hence, (9) gives

$$C_1 \sin t = C_2 \sin t + \cos t \sin t \quad \text{so that} \quad C_2 = C_1 - \cos t \quad \dots (10)$$

(iii) We must have $(\partial G_M / \partial x)_{x=t+0} - (\partial G_M / \partial x)_{x=t-0} = 1 / p(t)$, where $p(t) = -1$ $\dots (11)$

From (9),
$$\frac{\partial G_M}{\partial x} = -\frac{1}{\pi} \sin t \cos x + \frac{x}{\pi} \sin t \sin x + \begin{cases} C_1 \cos x, & 0 \leq x < t \\ C_2 \cos x - \sin x \sin t, & t < x \leq \pi \end{cases}$$

Hence, (11) reduces to
$$C_2 \cos t - \sin^2 t - C_1 \cos t = -1 \quad \text{or} \quad (C_1 - C_2) \cos t = \cos^2 t$$

Thus,
$$C_1 - C_2 = \cos t \quad \text{so that} \quad C_2 = C_1 - \cos t \quad \dots (12)$$

Observe that relations (10) and (12) are identical. Using (12), (9) reduces to

$$G_M(x, t) = -\frac{x}{\pi} \sin t \cos x + \begin{cases} C_1 \sin x, & 0 \leq x < t \\ (C_1 - \cos t) \sin x + \cos x \sin t, & t < x \leq \pi \end{cases}$$

or
$$G_M(x, t) = -\frac{x}{\pi} \sin t \cos x + C_1 \sin x + \begin{cases} 0, & 0 \leq x < t \\ -\sin(x-t), & t < x \leq \pi \end{cases}$$

or
$$G_M(x, t) = -(x / \pi) \times \sin t \cos x + C_1 \sin x - \sin(x-t) H(x-t), \quad \dots (13)$$

where $H(x-t)$ is the usual Heaviside unit function which is defined as

$$H(x-t) = \begin{cases} 0, & \text{if } x < t \\ 1, & \text{if } x > t \end{cases} \quad \dots (14)$$

In order that $G_M(x, t)$ may be symmetric, we must have

$$\int_0^\pi G_M(x, t) w(x) dx = 0 \quad \dots (15)$$

The unknown constant C_1 occurring in (13) can be determined with help of (5), (13) and (15).

Comparing (1) with $Ly = \phi(x)$, we have $\phi(x) = -f(x)$. Hence using result (11a) of Art. 11.13, the required integral equation is given by

$$y(x) = E w(x) + \int_0^\pi G_M(x, t) \phi(t) dt, \quad E \text{ being an arbitrary constant}$$

$$\text{or } y(x) = E \times (2/\pi)^{1/2} \times \sin x - \int_0^\pi G_M(t, x) f(t) dt, \quad \text{using (5)} \quad \dots(16)$$

$$[\because G_M(x, t) \text{ is symmetric} \Rightarrow G_M(x, t) = G_M(t, x)]$$

$$\text{From (13), } G_M(t, x) = -(t/\pi) \times \sin x \cos t + C_1 \sin t - \sin(t-x) H(t-x) \quad \dots(17)$$

Substituting the above value of $G_M(t, x)$ in (16), we have

$$y(x) = E \times (2/\pi)^{1/2} \times \sin x - \int_0^\pi \{-(t/\pi) \times \sin x \cos t + C_1 \sin t - \sin(t-x) H(t-x)\} f(t) dt$$

$$\text{or } y(x) = E \times (2/\pi)^{1/2} \times \sin x + \frac{\sin x}{\pi} \int_0^\pi t \cos t f(t) dt - C_1 \int_0^\pi \sin t f(t) dt$$

$$+ \int_0^x f(t) \sin(t-x) H(t-x) dt + \int_x^\pi f(t) \sin(t-x) H(t-x) dt$$

$$\text{or } y(x) = E \times (2/\pi)^{1/2} \times \sin x + \{(\sin x)/\pi\} \times F - C_1 \times 0 + 0 + \int_x^\pi \sin(t-x) f(t) dt \quad \dots(18)$$

$[\because \int_0^\pi t \cos t f(t) dt = \text{constant} = F]$, say and for the existence of integral equation, the following consistency condition must be satisfied:

$$\int_0^\pi w(x) \phi(x) dx = 0 \quad \text{or} \quad \int_0^\pi \sin x f(x) dx = 0 \quad \text{or} \quad \int_0^\pi \sin t f(t) dt = 0]$$

Let $E \times (2/\pi)^{1/2} + F/\pi = k$. Then, (18) may be re-written as

$$y(x) = k \sin x + \int_x^\pi \sin(t-x) f(t) dt, \quad k \text{ being an arbitrary constant}$$

D.2 Extension of the theory of Art. 11.13 of chapter 11 to the case when the associated adjoint system has two linearly independent solutions $y_1(x)$ and $y_2(x)$ in place of exactly one non-zero solution

Revised working rule of construction of modified (or generalized) Green's function

Consider an inhomogeneous equation with boundary conditions:

$$Ly = \phi(x), \quad \alpha_1 y(a) + \beta_1 y'(a) = 0, \quad \alpha_2 y(b) + \beta_2 y'(b) = 0 \quad \dots(1)$$

Here L is a self adjoint system. Consider the following associated self adjoint system:

$$Ly = 0, \quad \text{where} \quad L \equiv p(x) \frac{d^2}{dx^2} + \frac{dp}{dx} \frac{d}{dx} + q(x) \quad \dots(2)$$

$$\text{with boundary conditions: } \alpha_1 y(a) + \beta_1 y'(a) = 0 \quad \dots(3a)$$

$$\text{and } \alpha_2 y(b) + \beta_2 y'(b) = 0 \quad \dots(3b)$$

with the usual assumption that one of α_1 and β_1 and one of α_2 and β_2 are non-zero.

Suppose that the homogeneous boundary value problem given by (2), (3a) and (3b) possess two non-trivial linearly independent solutions $y_1(x)$ and $y_2(x)$.

$$\text{Then,} \quad \|y_1(x)\| = \text{norm of } y_1(x) = \left\{ \int_a^b [y_1(x)]^2 dx \right\}^{1/2}$$

$$\text{and} \quad \|y_2(x)\| = \text{norm of } y_2(x) = \left\{ \int_a^b [y_2(x)]^2 dx \right\}^{1/2}$$

$$\text{Let} \quad w_1(x) = y_1(x) / \|y_1(x)\| \quad \text{and} \quad w_2(x) = y_2(x) / \|y_2(x)\|, \quad \dots(4)$$

so that $w_1(x)$ and $w_2(x)$ are two non-trivial normalized solutions of the boundary value given by (2), (3a) and (3b). Clearly, we have

$$\int_a^b [w_1(x)]^2 dx = 1 \quad \text{and} \quad \int_a^b [w_2(x)]^2 dx = 1 \quad \dots(5)$$

Then, by definition of modified Green's function $G_M(x, t)$ of the given boundary value problem, $G_M(x, t)$ satisfies the differential equation

$$LG_M = \delta(x-t) - w_1(x) w_1(t) - w_2(x) w_2(t) \quad \dots(6)$$

$$\text{For } x \neq t, (6) \text{ reduces to} \quad LG_M = -w_1(x) w_1(t) - w_2(x) w_2(t) \quad \dots(7)$$

$$\text{For a given } t, \text{ let} \quad G_M(x, t) = \begin{cases} G_1(x, t), & \text{if } a \leq x < t \\ G_2(x, t), & \text{if } t < x \leq b \end{cases} \quad \dots(8)$$

where G_1 and G_2 are such that

(i) The functions G_1 and G_2 satisfy the equations in their respective intervals of definition, that is,

$$LG_1 = -w_1(x) w_1(t) - w_2(x) w_2(t), \quad a \leq x < t \quad \dots(9a)$$

$$LG_2 = -w_1(x) w_1(t) - w_2(x) w_2(t), \quad t < x \leq b \quad \dots(9b)$$

(ii) G_1 satisfies the boundary conditions (3a) whereas G_2 satisfies the boundary condition (3b)

(iii) $(\partial G_M / \partial x)_{n=t+0} - (\partial G_M / \partial x)_{n=t-0} = 1 / p(t)$ must be satisfied

(iv) In order that $G_M(x, t)$ may be symmetric, we must have

$$\int_a^b G_M(x, t) w_1(x) dx = 0 \quad \text{and} \quad \int_a^b G_M(x, t) w_2(x) dx = 0 \quad \dots(9)$$

(v) The given boundary value problem can be reduced to an integral equation only if the following so-called consistency conditions are satisfied:

$$\int_a^b \phi(x) w_1(x) dx = 0 \quad \text{and} \quad \int_a^b \phi(x) w_2(x) dx = 0 \quad \dots(10)$$

The required integral equation is given by

$$y(x) = \int_a^b G_M(x, t) \phi(t) dt + C_1 w_1(x) + C_2 w_2(x) \quad \dots(11)$$

$$\text{or} \quad y(x) = \int_a^b G_M(x, t) \phi(t) dt + w_1(x) \int_a^b w_1(x) y(x) dx + w_2(x) \int_a^b w_2(x) y(x) dx \quad \dots(12)$$

An illustrative example. Obtain the modified Green's function of the boundary value problem $d^2 y / dx^2 + k^2 y = f(x)$ subject to the boundary conditions $y(0) = y(2\pi)$ and $y'(0) = y'(2\pi)$, $0 \leq x \leq 2\pi$, k being a non-zero integral. Hence, convert the given boundary value problem into an integral equation. [Kanpur 2011]

Sol. Given $-(d^2 y / dx^2 + k^2 y) = -f(x)$, $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$, $0 \leq x \leq 2\pi$... (1)

Here $-(d^2/dx^2 + k^2)$ is a self adjoint operator

Consider the associated self adjoint system

$$-(d^2 y / dx^2 + k^2 y) = 0 \quad \text{i.e.} \quad (D^2 + k^2)y = 0, \quad \text{where } D \equiv d/dx, \quad 0 \leq x \leq 2\pi \quad \dots(2)$$

$$\text{subject to the boundary condition:} \quad y(0) = y(2\pi) \quad \dots(3a)$$

$$\text{and} \quad y'(0) = y'(2\pi) \quad \dots(3b)$$

We easily verify that $y_1(x) = \cos kx$ and $y_2(x) = \sin kx$ are two linearly independent non-zero solutions of the boundary value problem given by (2), (3a) and (3b). Then, we have

$$\|y_1(x)\| = \left\{ \int_0^{2\pi} [y_1(x)]^2 dx \right\}^{1/2} = \left\{ \int_0^{2\pi} \cos^2 kx dx \right\}^{1/2} = \left\{ \int_0^{2\pi} \frac{1 + \cos 2kx}{2} dx \right\}^{1/2} = \left\{ \left[\frac{x}{2} + \frac{\sin 2kx}{4} \right]_0^{2\pi} \right\}^{1/2}$$

$$\text{Thus,} \quad \|y_1(x)\| = \sqrt{\pi}. \quad \text{Similarly,} \quad \|y_2(x)\| = \sqrt{\pi}$$

Let $w_1(x) = y_1(x) / \|y_1(x)\|$ and $w_2(x) = y_2(x) / \|y_2(x)\|$. Then, we have

$$w_1(x) = (\cos kx) / \sqrt{2} \quad \text{and} \quad w_2(x) = (\sin kx) / \sqrt{2} \quad \dots(4)$$

For $x \neq t$, $G_M(x, t)$ must satisfy $-(d^2 G_M / dx^2 + k^2 G_M) = -w_1(x) w_1(t) - w_2(x) w_2(t)$

$$\text{or} \quad (D^2 + k^2) G_M = (1/\pi) \times \cos kx \cos kt + (1/\pi) \times \sin kx \sin kt \quad \dots(5)$$

C.F. of (5) = $a \cos kx + b \sin kx$, a and b being arbitrary constants

$$\begin{aligned} \text{and P.I. (5)} &= \frac{1}{\pi} \frac{1}{D^2 + k^2} [\cos kx \cos kt + \sin kx \sin kt] = \frac{1}{\pi} \times \cos kt \times \frac{x}{2k} \sin kx + \frac{1}{\pi} \times \sin kt \times \frac{x}{2k} \times (-\cos kt) \\ &= (x / 2\pi k) \times \sin k(x-t) \end{aligned}$$

$$\text{We take} \quad G_M(x, t) = \frac{x}{2\pi k} \sin k(x-t) + \begin{cases} A_1 \cos kx + B_1 \sin kx, & 0 \leq x < t \\ C_1 \cos kx + D_1 \sin kx, & t < x \leq 2\pi \end{cases} \quad \dots(6)$$

$$\text{From (6),} \quad \frac{\partial G_M}{\partial x} = \frac{\sin k(x-t)}{2\pi k} + \frac{x \cos k(x-t)}{2\pi} + \begin{cases} -A_1 k \sin kx + B_1 k \cos kx, & 0 \leq x < t \\ -C_1 k \sin kx + D_1 k \cos kx, & t < x \leq 2\pi \end{cases} \quad \dots(7)$$

The desired modified Green's function $G_M(x, t)$ must satisfy the following properties:

$$\begin{aligned} (i) \quad G_M(x, t) \text{ satisfies (3a)} &\Rightarrow G_M(0, t) = G_M(2\pi, t) \\ &\Rightarrow A_1 = -(1/k) \times \sin kt + C \quad \text{so that} \quad A_1 - C_1 = -(1/k) \times \sin kt \quad \dots(8) \end{aligned}$$

$$\begin{aligned} G_M(x, t) \text{ satisfies (3b)} &\Rightarrow (\partial G_M / \partial x)_{x=0} = (\partial G_M / \partial x)_{x=2\pi} \\ &\Rightarrow B_1 k = \cos kt + D_1 k \Rightarrow B_1 = (1/k) \times \cos kt + D_1 \Rightarrow B_1 - D_1 = (1/k) \times \cos kt \quad \dots(9) \end{aligned}$$

(ii) $G_M(x, t)$ is continuous at $x = t$. Hence, we have

$$A_1 \cos kt + B_1 \sin kt = C_1 \cos kt + D_1 \sin kt \Rightarrow (A_1 - C_1) \cos kt + (B_1 - D_1) \sin kt = 0$$

$$\Rightarrow -(1/k) \times \sin kt \cos kt + (1/k) \times \cos kt \sin kt = 0, \text{ using (8) and (9)}$$

which is identically satisfied.

$$\begin{aligned}
(iii) \quad G_M(x, t) \text{ satisfies} \quad & (\partial G_M / \partial x)_{x=t+0} - (\partial G_M / \partial x)_{x=t-0} = 1/p(t) \\
\Rightarrow \quad & -C_1 k \sin kt + D_1 k \cos kt - (-A_1 k \sin kt + B_1 k \cos kt) = -1 \\
\Rightarrow \quad & (A_1 - C_1) \sin kt - (B_1 - D_1) \cos kt = -(1/k) \\
\Rightarrow \quad & -(1/k) \sin^2 kt - (1/k) \times \cos^2 kt = -(1/k) \Rightarrow -(1/k) = -(1/k), \text{ using (8) and (9)}
\end{aligned}$$

which is identically satisfied.

Substituting the values of A_1 and B_1 given by (8) and (9) in (6), we have

$$\begin{aligned}
G_M(x, t) &= \frac{x \sin k(x-t)}{2\pi k} + \begin{cases} \{C_1 - (1/k) \times \sin kt\} \cos kx + \{D_1 + (1/k) \times \cos kt\} \sin kx, & 0 \leq x < t \\ C_1 \cos kx + D_1 \sin kx, & t < x \leq 2\pi \end{cases} \\
&= \frac{x \sin k(x-t)}{2\pi k} + \begin{cases} C_1 \cos kx + D_1 \sin kx + (1/k) \times \sin k(x-t), & 0 \leq x < t \\ C_1 \cos kx + D_1 \sin kx, & t < x \leq 2\pi \end{cases} \\
&= C_1 \cos kx + D_1 \sin kx + \frac{x \sin k(x-t)}{2\pi k} + \begin{cases} (1/k) \times \sin k(x-t), & 0 \leq x < t \\ 0, & t < x \leq 2\pi \end{cases} \\
&= C_1 \cos kx + D_1 \sin kx + \frac{\sin k(x-t)}{k} \begin{cases} (x/2\pi + 1), & 0 \leq x < t \\ x/2\pi, & t < x \leq 2\pi \end{cases}
\end{aligned}$$

$$\text{Thus,} \quad G_M(x, t) = C_1 \cos kx + D_1 \sin kx + (1/k) \times \sin k(x-t) \{x/2\pi + H(x-t)\} \quad \dots(10)$$

where $H(x-t)$ is Heavide unit function defined by

$$H(x-t) = \begin{cases} 0, & \text{if } x < t \\ 1, & \text{if } x > t \end{cases} \quad \dots(11)$$

We know that $G_M(x, t)$ satisfies the following conditions.

$$\int_0^{2\pi} G_M(x, t) w_1(x) dx = 0 \quad \text{and} \quad \int_0^{2\pi} G_M(x, t) w_2(x) dx = 0 \quad \dots(12)$$

Using (10), (11) and (12), we can determine C_1 and D_1 occurring in (11).

Comparing (1) with $Ly = \phi(x)$, we have $\phi(x) = -f(x)$. Hence, using the result (11) of Art. D.2, the required integral equation is given by

$$y(x) = E_1 w_1(x) + F_1 w_2(x) + \int_0^{2\pi} G_M(x, t) \phi(t) dt, \quad E_1 \text{ and } F_1 \text{ being arbitrary constants}$$

$$\text{or} \quad y(x) = (E_1 / \sqrt{2}) \times \cos kx + (F_1 / \sqrt{2}) \times \sin kx - \int_0^{2\pi} G_M(t, x) f(t) dt, \text{ using (4)} \quad \dots(13)$$

$$[\because G_M(x, t) \text{ is symmetric} \Rightarrow G_M(x, t) = G_M(t, x)]$$

$$\text{From (10),} \quad G_M(t, x) = C_1 \cos kt + D_1 \sin kt + (1/k) \times \sin k(t-x) \{t/2\pi + H(t-x)\}$$

Substituting the above value of $G_M(t, x)$ in (13), we have

$$y(x) = (E_1 / \sqrt{2}) \times \cos kx + (F_1 / \sqrt{2}) \times \sin kx - \int_0^{2\pi} \{C_1 \cos kt + D_1 \sin kt + (1/k) \times \sin k(t-x) \{t/2\pi + H(t-x)\} f(t) dt$$

$$\text{or } y(x) = (E_1 / \sqrt{2}) \times \cos kx + (F_1 / \sqrt{2}) \times \sin kx - C_1 \int_0^{2\pi} \cos kt f(t) dt - D_1 \int_0^{2\pi} \sin kt f(t) dt \\ - \frac{1}{2\pi k} \int_0^{2\pi} t \sin k(x-t) f(t) dt - \frac{1}{k} \int_0^{2\pi} \sin k(x-t) H(t-x) f(t) dt$$

The given boundary value problem can be reduced to the above integral equation only if the following consistency conditions are satisfied :

$$\int_0^{2\pi} w_1(x) \phi(x) dx = \int_0^{2\pi} w_2(x) \phi(x) dx = 0, \quad \text{i.e.,} \quad \int_0^{2\pi} \cos kx \phi(x) dx = \int_0^{2\pi} \sin kx \phi(x) dx = 0 \quad \dots (14)$$

Using (14), the above integral equation reduces to

$$y(x) = (E_1 / \sqrt{2}) \times \cos kx + (E_1 / \sqrt{2}) \times \sin kx - \frac{1}{2\pi k} \int_0^{2\pi} t (\sin kx \cos kt - \cos kx \sin kt) f(t) dt \\ - \frac{1}{k} \left[\int_0^x \sin k(x-t) H(t-x) f(t) dt + \int_x^{2\pi} \sin k(x-t) H(t-x) f(t) dt \right]$$

$$\text{or } y(x) = (E_1 / \sqrt{2}) \times \cos kx + (F_1 / \sqrt{2}) \times \sin kx - \frac{\sin kx}{2\pi k} \int_0^{2\pi} t \cos kt f(t) dt \\ + \frac{\cos kx}{2\pi k} \int_0^{2\pi} t \sin kt f(t) dt - \frac{1}{k} \left[0 + \int_x^{2\pi} \sin k(x-t) f(t) dt \right]$$

$$\text{or } y(x) = (E_1 / \sqrt{2}) \times \cos kx + (F_1 / \sqrt{2}) \times \sin kx - (F_2 / 2\pi k) \times \sin kx + (E_2 / 2\pi k) \times \cos kx \\ - \frac{1}{k} \int_x^{2\pi} \sin k(x-t) f(t) dt, \quad \dots (15)$$

where $\int_0^{2\pi} t \cos kt f(t) dt = \text{constant} = F_2$ say and $\int_0^{2\pi} t \sin kt f(t) dt = \text{constant} = E_2$, say

Thus, the required integral equation (15) reduces to

$$\text{or } y(x) = a \cos kx + b \sin kx - \frac{1}{k} \int_x^{2\pi} \sin k(x-t) f(t) dt,$$

where $a = (E_1 / \sqrt{2} + E_2 / 2\pi k)$ and $b = (F_1 / \sqrt{2} - F_2 / 2\pi k)$ are new arbitrary constants

MISCELLANEOUS PROBLEMS ON THE ENTIRE BOOK

1. Solution of the initial value problem $(d^2y/dx^2) + a_1(x) (dy/dx) + a_2(x) y = F(x)$, $0 \leq x \leq 1$, $y(0) = C_0$, $(dy/dx)_{x=0} = C_1$, where $a_1(x)$, $a_2(x)$ and $F(x)$ are continuous functions on $[0, 1]$ may be reduced, in general, to a solution of

(a) Fredholm integral equation of first kind (b) Volterra integral equation of first kind
(c) Fredholm integral equation of second kind (d) Volterra integral equation of second kind

Hint : Ans. (d). Refer Art. 2.4.

[Gate 2006]

2. Using method of degenerate kernels, determine the solution of the integral equation

$$\phi(x) = (1-x^2)^{-1/2} + \lambda \int_0^1 \cos^{-1} \xi \phi(\xi) d\xi \quad [\text{Kanpur 2006}]$$

Solution. Let $C = \int_0^1 \cos^{-1} \xi \phi(\xi) d\xi \quad \dots(1)$

Then the given integral equation reduces to $\phi(x) = (1-x^2)^{-1/2} + \lambda C \quad \dots(2)$

From (2), $\phi(\xi) = (1-\xi^2)^{-1/2} + \lambda C \quad \dots(3)$

Using (3), (1) yields $C = \int_0^1 \{(1-\xi^2)^{-1/2} + \lambda C\} \cos^{-1} \xi d\xi$

or $C = \int_0^1 \cos^{-1} \xi (1-\xi^2)^{-1/2} d\xi + \lambda C \int_0^1 (1 \cdot \cos^{-1} \xi) d\xi$

or $C = \left[(-1/2) \times (\cos^{-1} \xi)^2 \right]_0^1 + \lambda C \left\{ \left[\xi \cos^{-1} \xi \right]_0^1 + \int_0^1 \xi (1-\xi^2)^{-1/2} d\xi \right\}$

or $C = \pi^2/8 + \lambda C \left\{ 0 - \left[(1-\xi^2)^{1/2} \right]_0^1 \right\} = \pi^2/8 + \lambda C$

or $C(1-\lambda) = \pi^2/8$ so that $C = \pi^2/8(1-\lambda)$, $\lambda \neq 1$

Hence, from (2), the required solution is $\phi(x) = (1-x^2)^{-1/2} + \left\{ \pi^2/8(1-\lambda) \right\}$

3. Solve $\phi(x) = 2x + \lambda \int_0^1 \sin \log x \phi(t) dt \quad [\text{Kanpur 2006}]$

Solution. Re-writing, the given-equation is given by

$$\phi(x) = 2x + \lambda \sin \log x \int_0^1 \phi(t) dt \quad \dots(1)$$

Let $C = \int_0^1 \phi(t) dt$... (2)

Using (2), (1) yields $\phi(x) = 2x + \lambda C \sin \log x$... (3)

From (3), $\phi(t) = 2t + C\lambda \sin \log t$... (4)

Using (3), (2) yields $C = \int_0^1 (2t + C\lambda \sin \log t) dt$

or $C = \left[t^2 \right]_0^1 + C\lambda \int_0^1 \sin \log t dt = 1 + C\lambda I$, ... (5)

where $I = \int_0^1 \sin \log t dt = \left[t \sin \log t \right]_0^1 - \int_0^1 \left\{ t \times \cos \log t \times \frac{1}{t} \right\} dt$

or $-I = \int_0^1 \cos \log t dt = \left[t \cos \log t \right]_0^1 - \int_0^1 \left\{ t \times (-\sin \log t) \times \frac{1}{t} \right\} dt$

or $-I = 1 + I$ or $2I = -1$ or $I = -1/2$... (16)

Using (6), (5) yields $C = 1 - (C\lambda)/2$ giving $C = 2/(2 + \lambda)$

Hence, from (3), the required solution is given by

$$\phi(x) = 2x + \{2\lambda / (2 + \lambda)\} \sin \log x, \lambda \neq -2$$

4. Which of the following function is a solution of Fredholm type integral equation

$$f(x) = x + \int_0^x x + f(t) dt? \quad (a) 2x/3 \quad (b) 3x/3 \quad (c) 3x/4 \quad (d) 4x/3.$$

[Ans. (b)] [Gate 2006]

5. Which of the following function is a solution of Volterra type integral equation

$$f(x) = x + \int_0^x \sin(x-t) f(t) dt. \quad (a) x + x^3/3 \quad (b) x - x^3/3 \quad (c) x + x^3/6 \quad (d) x - x^3/6$$

[Ans. (c)] [Gate 2006]

6. Show that $R(x, \xi; \lambda) = K(x, \xi) + \lambda \int_{\xi}^x K(x, \xi) R(x, \xi; \lambda) dz$ is the solution of a non-

homogenous Volterra integral equation of second kind $\phi(x) = f(x) + \lambda \int_0^x K(x, \xi) \phi(\xi) d\xi$,

where $R(x, \xi; \lambda)$ is the resolvent kernel of the equation. [Kanpur 2007]

Hint : Refer Art. 5.12. page 5.37.

7. If $u(x)$ has continuous first and second order derivatives and satisfies the boundary value problem $d^2u/dx^2 + \lambda u = 0$ with $u(0) = u(l) = 0$, then $u(x)$ is continuous and satisfies the homogenous linear integral equation $u(x) = \lambda \int_0^1 G(x, \xi) u(\xi) d\xi$, where

$$G(x, \xi) = \begin{cases} (\xi/l) \times (1-x), & 0 < \xi < x \\ (x/l) \times (1-\xi), & x < \xi < l \end{cases} \quad [\text{Kanpur 2007}]$$

Hint : Refer solved Ex. 1 (a), page 2.14.

8. Solved the following integral equations :

$$(i) \phi(x) = e^x + \lambda \int_0^1 (5x^2 - 3) \xi^2 \phi(\xi) d\xi \quad [\text{Kanpur 2007}]$$

$$(ii) e^{-x^2/2} = \left(1/\sqrt{2\pi}\right) \int_{-\infty}^{\infty} e^{-1x-t^2} \phi(t) dt \quad [\text{Meerut 2007}]$$

Ans. (i) $\phi(x) = e^x + \lambda(e-2)(5x^2-3)$.

9. Find the iterated kernels for the following kernel : $K(x, \xi) = x + \sin \xi$; $a = -\pi, b = \pi$, where a and b are the limits of the integral; [Kanpur 2007]

Hint : Refer part (iii) of Ex. 1, page 5.12.

10. The value of α for which the integral equation $u(x) = \alpha \int_0^1 e^{x-t} u(t) dt$, has a non-trivial solution is (a) -2 (b) -1 (c) 1 (d) 2 [GATE 2007]

Hint. Ans. (c). Proceed as in solved Ex. 1, page 3.3.

11. The integral equation $x(t) - \int_0^1 \cos t \sec s ds = \sinh t, 0 < t \leq 1$ has

- (a) no solution (b) a unique solution
(c) more than one but finitely many solutions (d) infinitely many solutions

[GATE 2008]

12. Suppose $y(x) = \lambda \int_0^{2\pi} y(t) \sin(x+t) dt, x \in [0, 2\pi]$ has eigenvalues $\lambda = 1/\pi$ and $\lambda = -1/\pi$

with corresponding eigenfunctions $y_1(x) = \sin x + \cos x$ and $y_2(x) = \sin x - \cos x$

respectively. Then the integral equation $y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} y(t) \sin(x+t) dt, x \in [0, 2\pi]$ has a solution when $f(x) =$ (a) 1 (b) $\cos x$ (c) $\sin x$ (d) $1 + \sin x + \cos x$. [GATE 2007]

13. The integral equation $x(t) = \sin t + \lambda \int_0^1 (s^2 t^3 + e^{s^2+t^3}) x(s) ds, 0 \leq t \leq 1, \lambda \in \mathbf{R}, \lambda \neq 0$

has a solution for (a) all non zero values of λ (b) no value of λ
(c) only countables many + ve values of λ (d) only countables many-ve values of λ

[GATE 2008]

14. Define (i) Integral equation (ii) Singular integral equation (iii) Integro-differential equation.

[Meerut 2008]

15. Obtain the Fredholm integral equation of second kind corresponding to boundary value problems $d^2u/dx^2 + \lambda u = x, u(0) = 0, u'(1) = 1$ [Kanpur 2008]

16. Show that $R(x, \xi; \lambda) = K(x, \xi) + \lambda \int_{\xi}^x K(x, z) R(z, \xi; \lambda) dz$ is the solution of a non-homogeneous Volterra integral equation of second kind $\phi(x) = f(x) + \lambda \int_0^x K(x, \xi) \phi(\xi) d\xi$, where $R(x, \xi, \lambda)$ is the resolvent kernel of the equation. [Kanpur 2007]

17. Solve the integral equation $\phi(x) = e^x + \lambda \int_0^1 (5x^2 - 3) \xi^2 \phi(\xi) d\xi$ [Kanpur 2007]

18. Find the first two iterated kernels of the kernel $K(x, \xi) = (x - \xi)^2$; $a = -1$, $b = -1$.

Hint. Refer Ex. 1 (iii), page 5.33

[Kanpur 2008, 09]

19. Solve the integral equation $\phi(x) = 1 + \int_0^x (x - \xi) \phi(\xi) d\xi$ with $\phi_0(x) = 0$ [Kanpur 2008]

20. Find the iterated kernels of the function upto third order : $K(x, t) = e^{x+t}$, $a = 0$, $b = 1$

Hint: Do as in Ex. 2(i), page 5.16 upto equation (6).

(Kanpur 2009)

Ex. 21. Show that the longitudinal vibrations of a rod fixed at one end and free at the other represent the Fredholm integral equation with Schmidt kernel. [Meerut 2005, 07, 09]

[Sol.] The longitudinal vibrations of a rod fixed at one end and free at the other are governed by the following boundary value problem;

$$y''(x) = F(x), \quad \dots (1)$$

where $F(x)$ is a known continuous function satisfying the boundary conditions :

$$y(a) = 0 \quad \dots (2a)$$

and

$$y'(b) = 0 \quad \dots (2b)$$

Integrating both sides of (1) w.r.t. 'x' from a to x , we get

$$[y'(x)]_a^x = \int_a^x F(x) dx \quad \text{or} \quad y'(x) - y'(a) = \int_a^x F(x) dx$$

or

$$y'(x) = \int_a^x F(x) dx + c_1 \quad \text{where } c_1 = y'(a) \quad \dots (3)$$

Integrating both sides of (3) w.r.t. 'x' from a to x , we get

$$[y(x)]_a^x = \int_a^x \int_a^x F(x) (dx)^2 + [c_1 x]_a^x \quad \text{or} \quad y(x) - y(a) = \int_a^x \int_a^x F(t) (dt)^2 + c_1 (x - a)$$

or

$$y(x) = \int_a^x (x - t) F(t) dt + c_1 (x - a) \quad \dots (4)$$

[on using the boundary condition (2a) and the result of Art. 1.14]

Re-writing (3), we have

$$y'(x) = \int_a^x F(t) dt + c_1 \quad \dots (5)$$

Putting $x = b$ in (5) and using the boundary condition (2b), we get

$$0 = \int_a^b F(t) dt + c_1 \quad \text{so that} \quad c_1 = -\int_a^b F(t) dt \quad \dots (6)$$

Substituting the value of c_1 given by (6) in (4), we have

$$y(x) = \int_a^x (x - t) F(t) dt - (x - a) \int_a^b F(t) dt$$

or

$$y(x) = \int_a^x (x - t) F(t) dt + \int_a^b (a - x) F(t) dt$$

or

$$y(x) = \int_a^x (x - t) F(t) dt + \int_a^x (a - x) F(t) dt + \int_x^b (a - x) F(t) dt$$

or

$$y(x) = \int_a^x \{(x - t) + (a - x)\} F(t) dt + \int_x^b (a - x) F(t) dt$$

or

$$y(x) = \int_a^x (a - t) F(t) dt + \int_x^b (a - x) F(t) dt \quad \dots (7)$$

which can be re-written as

$$y(x) = \int_a^b G(x, t) F(t) dt, \quad \dots (8)$$

where

$$G(x, t) = \begin{cases} a - x, & \text{if } x \leq t \\ a - t & \text{if } x \geq t \end{cases} \quad \dots (9)$$

From (8) and (9), it follows that the longitudinal vibrations of a rod fixed at one end and free at the other represent the Fredholm integral equation (8) with Schmidt kernel.

Ex. 22. The resolvent kernel for the integral equation $u(x) = F(x) + \int_{\log_2}^x e^{t-x} u(t) dt$ is

- (a) $\cos(x - t)$ (b) 1 (c) e^{t-x} (d) $e^{2(t-x)}$

[GATE, 2009]

Sol. Ans. (d). Proceed as in Ex. 2, Page 5.38.

Ex. 23. For a continuous function $f(t)$, $0 \leq t \leq 1$, the integral equation

$$y(t) = f(t) + 3 \int_0^1 t s y(s) ds \text{ has}$$

- (a) Unique solution if $\int_0^1 s f(s) ds \neq 0$ (b) no solution if $\int_0^1 s f(s) ds = 0$
 (c) Infinitely many solutions if $\int_0^1 s f(s) ds = 0$
 (d) Infinitely many solutions if $\int_0^1 s f(s) ds \neq 0$

[GATE 2010]

Sol. Ans. (c) Given $y(t) = f(t) + 3 \int_0^1 t s y(s) ds$

or $y(t) = f(t) + 3t \int_0^1 s y(s) ds \quad \dots(1)$

Let $C = \int_0^1 s y(s) ds \quad \dots(2)$

Using (2), (1) yields $y(t) = f(t) + 3 C t \quad \dots(3)$

From (3), $y(s) = f(s) + 3 C s \quad \dots(4)$

Using (4), (2) yields $C = \int_0^1 s \{f(s) + 3 C s\} ds = \int_0^1 s f(s) ds + 3 C \left[s^3 / 3 \right]_0^1$

Thus, we have $C = \int_0^1 s f(s) ds + C$

which is satisfied by infinitely many values of C if $\int_0^1 s f(s) ds = 0$. Hence, the given integral equation (1) has infinitely many solutions if $\int_0^1 s f(s) ds = 0$.

Ex. 24. Common data for the following two questions (i) and (ii):

Consider the Fredholm integral equation $u(x) = x + \lambda \int_0^1 x e^t u(t) dt$

(i). The resolvent kernel $R(x, t; \lambda)$ for this integral equation is

- (a) $(x e^t) / (1 - \lambda)$ (b) $(\lambda x e^t) / (1 + \lambda)$ (c) $(x e^t) / (1 + \lambda^2)$ (d) $(x e^t) / (1 - \lambda^2)$ [GATE 2012]

(ii). The solution of this equation is

- (a) $(x + 1) / (1 - \lambda)$ (b) $x^2 / (1 - \lambda^2)$ (c) $x / (1 + \lambda^2)$ (d) $x / (1 - \lambda)$ [GATE 2012]

Sol. (i) Ans. (a). Given $u(x) = x + \lambda \int_0^1 x e^t u(t) dt \quad \dots(1)$

Comparing (1) with $u(x) = f(x) + \lambda \int_0^1 K(x, t) u(t) dt, \quad \dots(2)$

we have $u(x) = x, \quad \lambda = \lambda \quad \text{and} \quad K(x, t) = x e^t \quad \dots(3)$

Let $K_m(x, t)$ be the m th iterated kernel. Then, we have

$$K_1(x, t) = K(x, t) = x e^t, \text{ using (3)}$$

and
$$K_m(x, t) = \int_0^1 K(x, z) K_{m-1}(z, t) dz, \text{ where } m = 2, 3, 4, \dots (4)$$

Putting $m = 2$ in (5) and using (4), we have

$$K_2(x, t) = \int_0^1 K(x, z) K_1(z, t) dz = \int_0^1 x e^z z e^t dz = x e^t \int_0^1 z e^z dz$$

or
$$K_2(x, t) = x e^t [(z)(e^z) - (1)(e^z)]_0^1, \text{ on using chain rule of integration by parts}$$

Thus,
$$K_2(x, t) = x e^t [e - e - (0 - 1)] = x e^t \dots (6)$$

Next, putting $m = 3$ in (5) and using (6), we have

$$K_3(x, t) = \int_0^1 K(x, z) K_2(z, t) dz = \int_0^1 x e^z z e^t dz = x e^t, \text{ as before} \dots (7)$$

and so on. Hence, using Mathematical induction, we have

$$K_m(x, t) = x e^t, \text{ for } m = 1, 2, 3 \dots (8)$$

By definition, the required resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) = \sum_{m=1}^{\infty} \lambda^{m-1} x e^t, \text{ using (8)}$$

Thus,
$$R(x, t; \lambda) = x e^t (1 + \lambda + \lambda^2 + \dots) = (x e^t) / (1 - \lambda) \dots (9)$$

(ii). **Ans. (d).** As usual, the required solution of (1) is given by

$$u(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t) dt = x + \lambda \int_0^1 \frac{x e^t}{1 - \lambda} t dt, \text{ since here } f(x) = x \Rightarrow f(t) = t$$

or
$$u(x) = x + \frac{\lambda x}{1 - \lambda} \int_0^1 t e^t dt = x + \frac{\lambda x}{1 - \lambda} [(t)(e^t) - (1)(e^t)]_0^1, \text{ as before}$$

or
$$u(x) = x + \{(\lambda x) / (1 - \lambda)\} \times 1 \quad \text{or} \quad u(x) = x / (1 - \lambda).$$

Ex. 25 The function $\phi(x)$ satisfying the integral equation $\int_0^x e^{x-\xi} \phi(\xi) d\xi = x^2/2$ is

(a) $x^2/2$ (b) $x + (x^2/2)$ (c) $x - (x^2/2)$ (d) $1 + (x^2/2)$

[GATE 2012]

Sol. Ans. (c). Proceed as explained in Art. 5.17 and examples based on it given on page 5.65.

Given
$$\int_0^x e^{x-\xi} \phi(\xi) d\xi = x^2/2 \dots (1)$$

Differentiating both sides of (1) w.r.t. 'x' and using Leibnitz's rule of differentiation under the integral sign, we have

$$\int_0^x \frac{\partial(e^{x-\xi})}{\partial x} \phi(\xi) d\xi + e^{x-x} \phi(x) \frac{dx}{dx} - e^{x-0} \phi(0) \frac{d0}{dx} \quad \text{or} \quad \phi(x) = x - \int_0^x e^{x-\xi} \phi(\xi) d\xi, \dots (2)$$

which is a Volterra integral equation of the second kind.

Comparing (2) with $\phi(x) = f(x) + \lambda \int_0^x K(x, \xi) \phi(\xi) d\xi$,
$$\dots (3)$$

we have
$$f(x) = x, \quad \lambda = -1 \quad \text{and} \quad K(x, \xi) = e^{x-\xi} \dots (4)$$

Let $K_m(x, \xi)$ be the m th iterated kernel. Then, we have

$$K_1(x, \xi) = K(x, \xi) = e^{x-\xi}, \text{ using (3)} \quad \dots(5)$$

and

$$K_m(x, \xi) = \int_{\xi}^x K(x, z) K_{m-1}(z, \xi) dz, \text{ where } m = 2, 3, 4, \dots(6)$$

Putting $m = 2$ in (6) and using (5), we have

$$K_2(x, \xi) = \int_{\xi}^x K(x, z) K_1(z, \xi) dz = \int_{\xi}^x e^{x-z} e^{z-\xi} dz = e^{x-\xi} \int_{\xi}^x dz$$

$$\text{Thus, } K_2(x, \xi) = e^{x-\xi} [z]_{\xi}^x = (x - \xi) e^{x-\xi} \quad \dots(7)$$

Putting $m = 3$ in (6) and using (7), we have

$$K_3(x, \xi) = \int_{\xi}^x K(x, z) K_2(z, \xi) dz = \int_{\xi}^x e^{x-z} e^{z-\xi} (z - \xi) dz = e^{x-\xi} \int_{\xi}^x (z - \xi) dz$$

$$\text{Thus, } K_3(x, \xi) = e^{x-\xi} \left[\frac{(z-\xi)^2}{2} \right]_{\xi}^x = e^{x-\xi} \frac{(x-\xi)^2}{2!} \quad \dots(8)$$

Next, putting $m = 3$ in (6) and using (8), we have

$$K_4(x, \xi) = \int_{\xi}^x K(x, z) K_3(z, \xi) dz = \int_{\xi}^x e^{x-z} e^{z-\xi} \frac{(z-\xi)^2}{2} dz = \frac{e^{x-\xi}}{2} \int_{\xi}^x (z-\xi)^2 dz$$

$$\text{Thus, } K_4(x, \xi) = \frac{e^{x-\xi}}{2} \left[\frac{(z-\xi)^3}{3} \right]_{\xi}^x = e^{x-\xi} \frac{(x-\xi)^3}{3!}, \quad \dots(9)$$

and so on, Hence, using Mathematical induction, we have

$$K_m(x, \xi) = e^{x-\xi} \frac{(x-\xi)^{m-1}}{(m-1)!}, \text{ where } m = 1, 2, 3, \dots \quad \dots(10)$$

Hence, by definition, the resolvent kernel $R(x, t; \lambda)$ of (2) is given by

$$\begin{aligned} R(x, \xi; \lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, \xi) = \sum_{m=1}^{\infty} (-1)^{m-1} K_m(x, \xi), \text{ since from (3); } \lambda = -1 \\ &= K_1(x, \xi) - K_2(x, \xi) + K_3(x, \xi) - K_4(x, \xi) + \dots + \text{ad inf} \\ &= e^{x-\xi} \left\{ 1 - \frac{(x-\xi)}{1!} + \frac{(x-\xi)^2}{2!} - \frac{(x-\xi)^3}{3!} + \dots + \text{ad inf} \right\} = e^{x-\xi} e^{-(x-\xi)} = 1 \end{aligned}$$

Hence, as usual, the required function $\phi(x)$ satisfying (1) is given by

$$\phi(x) = f(x) + \lambda \int_0^x R(x, \xi; \lambda) f(\xi) d\xi = x - \int_0^x \xi d\xi = x - (x^2/2)$$

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